ACTA ARITHMETICA
151.3 (2012)

Arithmetic properties of overpartition pairs

by

William Y. C. Chen and Bernard L. S. Lin (Tianjin)

1. Introduction. A *partition* of a positive integer n is a non-increasing sequence of positive integers whose sum is n. An *overpartition* λ of n is a partition of n for which the first occurrence of a number may be overlined. Let $\overline{\mathrm{p}}(n)$ denote the number of overpartitions of n. Congruence properties for $\overline{\mathrm{p}}(n)$ have been extensively studied; see, for example, Fortin, Jacob and Mathieu [6], Hirschhorn and Sellers [9], Kim [11], Lovejoy and Osburn [13], and Mahlburg [14]. In this paper, we study arithmetic properties of the number of overpartition pairs of n. An *overpartition pair* π of n is a pair of overpartitions (λ, μ) such that the sum of all of the parts is n. Note that we allow λ and μ to be the overpartition of zero. Let $\overline{\mathrm{pp}}(n)$ denote the number of overpartition pairs of n. Then the generating function for $\overline{\mathrm{pp}}(n)$ is

$$\sum_{n=0}^{\infty} \overline{\mathrm{pp}}(n) q^n = \frac{(-q;q)_\infty^2}{(q;q)_\infty^2}. \tag{1.1}$$

Here, we adopt the following standard q-series notation:

$$(a;q)_\infty = \prod_{k=1}^{\infty} (1 - aq^{k-1}).$$

Throughout this paper, we assume that $|q| < 1$.

Bringmann and Lovejoy [4] defined a rank for overpartition pairs to investigate congruence properties of $\overline{\mathrm{pp}}(n)$. Let $\overline{\mathrm{NN}}(m,n)$ denote the number of overpartition pairs of n with rank m, and let $\overline{\mathrm{NN}}(r,t,n)$ denote the number of overpartition pairs of n with rank congruent to r modulo t. The authors of [4] obtained a bivariate generating function for $\overline{\mathrm{NN}}(m,n)$ from which they derived the following relation for $0 \le r \le 2$:

$$\overline{\mathrm{NN}}(r,3,3n+2) = \frac{\overline{\mathrm{pp}}(3n+2)}{3}.$$

2010 *Mathematics Subject Classification*: 05A17, 11P83.
Key words and phrases: overpartition pairs, rank of overpartition pairs, congruence.

DOI: 10.4064/aa151-3-3

This leads to the following Ramanujan-type congruence in the spirit of Ramanujan's congruences on the partition function $p(n)$ modulo 5 and 7 (see, e.g., Berndt [3, Chapter 2]):

$$\overline{\mathrm{pp}}(3n+2) \equiv 0 \pmod{3}. \tag{1.2}$$

Furthermore, by using the theory of Klein forms, Bringmann and Lovejoy [4] proved that there exist infinitely many Ramanujan-type congruences for $\overline{\mathrm{pp}}(n)$. Let l be an odd prime and let t be an odd number which is a power of l or is relatively prime to l. Then for any positive integer j, there are infinitely many non-nested arithmetic progressions $An+B$ such that

$$\overline{\mathrm{NN}}(r,t,An+B) \equiv 0 \pmod{l^j} \tag{1.3}$$

for any $0 \le r \le t-1$. Hence there are infinitely many non-nested arithmetic progressions $An+B$ satisfying

$$\overline{\mathrm{pp}}(An+B) \equiv 0 \pmod{l^j} \tag{1.4}$$

for any odd prime l and any positive integer j. For $l=2$, using the theory of modular forms, it is shown in [4] that (1.4) holds for any positive integer j.

However, the theory of Klein forms used to derive the congruence relation (1.4) is not constructive and it does not give explicit arithmetic progressions $An+B$ in the statement. So it is still desirable to find explicit congruences for $\overline{\mathrm{pp}}(n)$. In this paper, we obtain some such congruences modulo 3 and 5.

For the case of modulo 3, we obtain a Ramanujan-type identity

$$\sum_{n=0}^{\infty} \overline{\mathrm{pp}}(3n+2)q^n = 12\frac{(q^2;q^2)_\infty^6 (q^3;q^3)_\infty^6}{(q;q)_\infty^{14}}, \tag{1.5}$$

which is analogous to Ramanujan's identity (see, e.g., Berndt [3, Theorem 2.3.4])

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6}. \tag{1.6}$$

Furthermore, we show that there are infinite families of congruences modulo 3 satisfied by $\overline{\mathrm{pp}}(n)$. For example, for any $\alpha \ge 1$ and $n \ge 0$,

$$\overline{\mathrm{pp}}(9^\alpha(3n+1)) \equiv \overline{\mathrm{pp}}(9^\alpha(3n+2)) \equiv 0 \pmod{3}. \tag{1.7}$$

For the case of modulo 5, we obtain three Ramanujan-type congruences

$$\overline{\mathrm{pp}}(20n+11) \equiv \overline{\mathrm{pp}}(20n+15) \equiv \overline{\mathrm{pp}}(20n+19) \equiv 0 \pmod{5}, \tag{1.8}$$

for any $n \ge 0$. We also find infinite families of congruences modulo 5. For example, for any $\alpha \ge 1$ and $n \ge 0$,

$$\overline{\mathrm{pp}}(5^\alpha(5n+2)) \equiv \overline{\mathrm{pp}}(5^\alpha(5n+3)) \equiv 0 \pmod{5}. \tag{1.9}$$

Motivated by the work of Paule and Radu [17] on some strange congruences, we obtain similar congruences for $\overline{\mathrm{pp}}(n)$. For example, for any $k \geq 0$,

$$\overline{\mathrm{pp}}(5 \cdot 29^k) \equiv 3(k+1) \pmod 5, \tag{1.10}$$

$$\overline{\mathrm{pp}}(2 \cdot 13^k) \equiv 3(k+1) \pmod 9. \tag{1.11}$$

To give combinatorial interpretations of the fact that $\overline{\mathrm{pp}}(3n+2)$ is divisible by 3, we find three ranks of overpartition pairs that serve this purpose.

This paper is organized as follows. In Section 2, we obtain two Ramanujan-type identities and some Ramanujan-type congruences modulo 5 and 64. In Section 3, we give three combinatorial interpretations for the congruence (1.2). Section 4 gives infinite families of congruences modulo 3 and 5. In Section 5, we obtain congruences modulo 9 which are similar to the congruences of Paule and Radu for the number of broken 2-diamond partitions.

2. Ramanujan-type identities and congruences. In this section, we establish two Ramanujan-type identities and derive some congruence relations modulo 5 and 64.

Theorem 2.1. *We have*

$$\sum_{n=0}^{\infty} \overline{\mathrm{pp}}(3n+2)q^n = 12\frac{(q^2;q^2)_\infty^6 (q^3;q^3)_\infty^6}{(q;q)_\infty^{14}}, \tag{2.1}$$

$$\sum_{n=0}^{\infty} \overline{\mathrm{pp}}(4n+3)q^n = 32\frac{(q^2;q^2)_\infty^{20}}{(q;q)_\infty^{22}}. \tag{2.2}$$

To prove the above identities, we recall two Ramanujan's theta functions:

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}, \qquad \psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2}.$$

The following two identities are due to Gauss (see, e.g., Berndt [3, p. 11]):

$$\varphi(-q) = \frac{(q;q)_\infty^2}{(q^2;q^2)_\infty}, \qquad \psi(q) = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}.$$

As shown by Hirschhorn and Sellers [8], the generating function of $\overline{\mathrm{p}}(n)$ is

$$\sum_{n=0}^{\infty} \overline{\mathrm{p}}(n)q^n = \frac{1}{\varphi(-q)}.$$

This implies that the generating function of $\overline{\mathrm{pp}}(n)$ equals

$$\sum_{n=0}^{\infty} \overline{\mathrm{pp}}(n)q^n = \frac{1}{\varphi(-q)^2}. \tag{2.3}$$

The following dissection formula of Hirschhorn and Sellers [8] plays a key role in the proof of Theorem 2.1.

Lemma 2.1. *Let*

$$A(q) = \frac{(q;q)_\infty (q^6;q^6)_\infty^2}{(q^2;q^2)_\infty (q^3;q^3)_\infty}.$$

Then

$$\begin{aligned}\frac{1}{\varphi(-q)} &= \frac{\varphi(-q^9)}{\varphi(-q^3)^4}(\varphi(-q^9)^2 + 2q\varphi(-q^9)A(q^3) + 4q^2A(q^3)^2) \\ &= \frac{1}{\varphi(-q^4)^4}(\varphi(q^4)^3 + 2q\varphi(q^4)^2\psi(q^8) \\ &\quad + 4q^2\varphi(q^4)\psi(q^8)^2 + 8q^3\psi(q^8)^3).\end{aligned} \tag{2.4}$$

Proof of Theorem 2.1. Substituting the 3-dissection formula (2.4) into (2.3), we see that

$$\sum_{n=0}^\infty \overline{\mathrm{pp}}(n)q^n = \frac{\varphi(-q^9)^2}{\varphi(-q^3)^8}(\varphi(-q^9)^2 + 2q\varphi(-q^9)A(q^3) + 4q^2A(q^3)^2)^2. \tag{2.5}$$

Choosing those terms for which the powers of q are of the form $3n+2$, we find that

$$\begin{aligned}\sum_{n=0}^\infty \overline{\mathrm{pp}}(3n+2)q^{3n+2} &= \frac{\varphi(-q^9)^2}{\varphi(-q^3)^8}(8q^2\varphi(-q^9)^2A(q^3)^2 + 4q^2\varphi(-q^9)^2A(q^3)^2) \\ &= 12q^2A(q^3)^2\frac{\varphi(-q^9)^4}{\varphi(-q^3)^8}.\end{aligned}$$

Dividing both sides by q^2 and replacing q^3 by q, we obtain

$$\sum_{n=0}^\infty \overline{\mathrm{pp}}(3n+2)q^n = 12A(q)^2\frac{\varphi(-q^3)^4}{\varphi(-q)^8}.$$

This yields (2.1). Similarly,

$$\begin{aligned}\sum_{n=0}^\infty \overline{\mathrm{pp}}(n)q^n = \frac{1}{\varphi(-q^4)^8}(\varphi(q^4)^3 &+ 2q\varphi(q^4)^2\psi(q^8) \\ &+ 4q^2\varphi(q^4)\psi(q^8)^2 + 8q^3\psi(q^8)^3)^2.\end{aligned}$$

Choosing the terms for which the powers of q are of the form $4n+3$, we find that

$$\begin{aligned}\sum_{n=0}^\infty \overline{\mathrm{pp}}(4n+3)q^{4n+3} &= \frac{1}{\varphi(-q^4)^8}(16q^3\varphi(q^4)^3\psi(q^8)^3 + 16q^3\varphi(q^4)^3\psi(q^8)^3) \\ &= 32q^3\frac{\varphi(q^4)^3\psi(q^8)^3}{\varphi(-q^4)^8}.\end{aligned}$$

Dividing both sides by q^3 and replacing q^4 by q, we deduce that

$$\sum_{n=0}^{\infty} \overline{\mathrm{pp}}(4n+3)q^n = 32\frac{\varphi(q)^3\psi(q^2)^3}{\varphi(-q)^8}, \tag{2.6}$$

which is equivalent to (2.2). This completes the proof. ∎

In view of Theorem 2.1, it can be seen that $\overline{\mathrm{pp}}(3n+2)$ and $\overline{\mathrm{pp}}(4n+3)$ are divisible by 4. In fact, for all $n \geq 1$, $\overline{\mathrm{pp}}(n)$ is divisible by 4, since

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{\mathrm{pp}}(n)q^n &\equiv \Big(1+2\sum_{n=0}^{\infty}(-q)^{n^2}\Big)^2 \sum_{n=0}^{\infty} \overline{\mathrm{pp}}(n)q^n \pmod 4\\ &= \varphi(-q)^2\frac{1}{\varphi(-q)^2} = 1.\end{aligned}$$

In fact, Keister, Sellers and Vary [10] have shown that, for $n \geq 1$,

$$\overline{\mathrm{pp}}(n) \equiv \begin{cases} 4 \pmod 8 & \text{if } n \text{ is a square or twice a square,}\\ 0 \pmod 8 & \text{otherwise.}\end{cases}$$

Recently, Kim [12] has given a combinatorial proof of the above fact and studied arithmetic properties of $\overline{\mathrm{pp}}(n)$ modulo powers of 2.

With the aid of (2.2) and the following relation for any prime p:

$$(q;q)_\infty^p \equiv (q^p;q^p)_\infty \pmod p, \tag{2.7}$$

we are led to the following congruence relations modulo 5 and 64.

COROLLARY 2.1. *For any non-negative integer n,*

$$\overline{\mathrm{pp}}(8n+7) \equiv 0 \pmod{64}, \tag{2.8}$$

$$\overline{\mathrm{pp}}(20n+11) \equiv 0 \pmod 5, \tag{2.9}$$

$$\overline{\mathrm{pp}}(20n+15) \equiv 0 \pmod 5, \tag{2.10}$$

$$\overline{\mathrm{pp}}(20n+19) \equiv 0 \pmod 5. \tag{2.11}$$

Proof. From (2.2) and (2.7) with $p = 2$, we have

$$\sum_{n=0}^{\infty} \frac{\overline{\mathrm{pp}}(4n+3)}{32}q^n \equiv \frac{(q^2;q^2)_\infty^{20}}{(q^2;q^2)_\infty^{11}} \equiv (q^2;q^2)_\infty^9 \pmod 2.$$

This yields congruence (2.8) by equating the coefficients of q^{2n+1} for $n \geq 0$. Again by (2.2) and (2.7) with $p = 5$, we see that

$$\sum_{n=0}^{\infty} \overline{\mathrm{pp}}(4n+3)q^n \equiv 2\frac{(q^{10};q^{10})_\infty^4}{(q^5;q^5)_\infty^4}\cdot\frac{1}{(q;q)_\infty^2} \pmod 5. \tag{2.12}$$

Let $p_{-2}(n)$ be defined by

$$\sum_{n=0}^{\infty} p_{-2}(n)q^n = \frac{1}{(q;q)_\infty^2}.$$

It has been shown by Ramanathan [18] that for $n \geq 0$,

$$p_{-2}(5n+2) \equiv p_{-2}(5n+3) \equiv p_{-2}(5n+4) \equiv 0 \pmod 5.$$

Combining (2.12) and the above three congruences, we obtain the congruence relations (2.9), (2.10) and (2.11). This completes the proof. ∎

3. Three ranks for overpartition pairs. In this section, we give three combinatorial interpretations for the fact that $\overline{\mathrm{pp}}(3n+2)$ is divisible by 3.

The *first rank* of an overpartition pair $\pi = (\lambda, \mu)$, denoted $r_1(\pi)$, is defined to be $n_1(\lambda) - n_1(\mu)$, where $n_1(\lambda)$ denotes the number of parts of an overpartition λ. As usual, let $R_1(m,n)$ denote the number of overpartition pairs of n with $r_1(\pi) = m$ and let $R_1(s,t,n)$ denote the number of overpartition pairs of n with $r_1(\pi) \equiv s \pmod t$. By symmetry, we see that $R_1(m,n) = R_1(-m,n)$, and so $R_1(s,t,n) = R_1(t-s,t,n)$. It is easy to derive the bivariate generating function for $R_1(m,n)$, that is,

$$\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} R_1(m,n)z^m q^n = \frac{(-qz;q)_\infty}{(qz;q)_\infty} \cdot \frac{(-q/z;q)_\infty}{(q/z;q)_\infty}. \tag{3.1}$$

Here we adopt the convention that the overpartition pair of 0 has rank zero. This convention is also valid for the other two ranks that will be introduced in this section. The following theorem shows that the rank $r_1(\pi)$ leads to a classification of overpartition pairs of $3n+2$ into three equinumerous sets.

Theorem 3.1. *For $0 \leq s \leq 2$, we have*

$$R_1(s,3,3n+2) = \overline{\mathrm{pp}}(3n+2)/3. \tag{3.2}$$

Proof. Substituting $z = \xi = e^{2\pi i/3}$ into (3.1) and using the symmetry relation $R_1(1,3,n) = R_1(2,3,n)$, we find that

$$\begin{aligned}
&\sum_{n=0}^{\infty}(R_1(0,3,n) - R_1(1,3,n))q^n \\
&\qquad = \frac{(-q\xi;q)_\infty(-q\xi^2;q)_\infty}{(q\xi;q)_\infty(q\xi^2;q)_\infty} = \frac{(-q^3;q^3)_\infty}{(q^3;q^3)_\infty} \cdot \frac{(q;q)_\infty}{(-q;q)_\infty} \\
&\qquad = \frac{(-q^3;q^3)_\infty}{(q^3;q^3)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}.
\end{aligned} \tag{3.3}$$

Here the second equality follows from the identity

$$(1-x^3) = (1-x)(1-x\xi)(1-x\xi^2).$$

Equating the coefficients of q^{3n+2} on both sides of (3.3), and observing that there are no squares congruent to 2 modulo 3, we conclude that

$$R_1(0,3,3n+2) = R_1(1,3,3n+2),$$

and so

$$R_1(0,3,3n+2) = R_1(1,3,3n+2) = R_1(2,3,3n+2) = \overline{\mathrm{pp}}(3n+2)/3. \blacksquare$$

We now introduce the *second rank* r_2. Let $\pi = (\lambda, \mu)$ be an overpartition pair. Define

$$r_2(\pi) = n_2(\lambda) - n_2(\mu), \tag{3.4}$$

where $n_2(\lambda)$ denotes the number of overlined parts of an overpartition λ. Similarly, let $R_2(m,n)$ denote the number of overpartition pairs of n with $r_2(\pi) = m$ and let $R_2(s,t,n)$ denote the number of overpartition pairs of n with $r_2(\pi) \equiv s \pmod{t}$. Then we have the following relation.

THEOREM 3.2. *For $n \geq 0$, we have*

$$R_2(0,3,3n+2) \equiv R_2(1,3,3n+2) \equiv R_2(2,3,3n+2) \pmod{3}. \tag{3.5}$$

Proof. It is routine to check that

$$\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} R_2(m,n) z^m q^n = \frac{(-qz;q)_\infty}{(q;q)_\infty} \cdot \frac{(-q/z;q)_\infty}{(q;q)_\infty}. \tag{3.6}$$

Using the fact that $R_2(1,3,n) = R_2(2,3,n)$ and setting $z = \xi = e^{2\pi i/3}$ in (3.6), we find

$$\begin{aligned} \sum_{n=0}^{\infty}&(R_2(0,3,n) - R_2(1,3,n))q^n \\ &= \frac{(-q\xi;q)_\infty(-q\xi^2;q)_\infty}{(q;q)_\infty^2} = \frac{(-q^3;q^3)_\infty}{(q;q)_\infty(q^2;q^2)_\infty} \\ &= \frac{(-q^3;q^3)_\infty}{(q;q)_\infty^3}\sum_{n=-\infty}^{\infty}(-q)^{n^2} \equiv \frac{(-q^3;q^3)_\infty}{(q^3;q^3)_\infty}\sum_{n=-\infty}^{\infty}(-q)^{n^2} \pmod{3}. \end{aligned} \tag{3.7}$$

Since there are no squares congruent to 2 modulo 3, we see that

$$R_2(0,3,3n+2) - R_2(1,3,3n+2) \equiv 0 \pmod{3},$$

and hence the proof is complete. $\blacksquare$

It is worth mentioning that Andrews, Lewis and Lovejoy [1] investigated the arithmetic properties of the number $\mathrm{PD}(n)$ of partitions of n with designated summands, whose generating function is given by (3.7), that is,

$$\sum_{n=0}^{\infty}\mathrm{PD}(n)q^n = \frac{(q^6;q^6)_\infty}{(q;q)_\infty(q^2;q^2)_\infty(q^3;q^3)_\infty}.$$

For example, it has been shown that $\mathrm{PD}(3n+2)$ is divisible by 3. It should also be mentioned that Chan [5] studied the number $a(n)$ given by

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{1}{(q;q)_\infty (q^2;q^2)_\infty},$$

and derived a Ramanujan-type identity for $a(3n+2)$, that is,

$$\sum_{n=0}^{\infty} a(3n+2)q^n = 3\frac{(q^3;q^3)_\infty^3 (q^6;q^6)_\infty^3}{(q;q)_\infty^4 (q^2;q^2)_\infty^4}. \tag{3.8}$$

From (3.7) and (3.8), we get the following formula.

Corollary 3.1. *We have*

$$\sum_{n=0}^{\infty} (R_2(0,3,3n+2) - R_2(1,3,3n+2))q^n = 3\frac{(q^3;q^3)_\infty^3 (q^6;q^6)_\infty^3}{(q;q)_\infty^5 (q^2;q^2)_\infty^3}. \tag{3.9}$$

Finally, we turn to the *third rank* r_3 of an overpartition pair $\pi = (\lambda, \mu)$, which is defined by

$$r_3(\pi) = n_3(\lambda) - n_3(\mu), \tag{3.10}$$

where $n_3(\lambda)$ denotes the number of non-overlined parts of an overpartition λ. Similarly, let $R_3(m,n)$ denote the number of overpartition pairs of n with $r_3(\pi) = m$ and let $R_3(s,t,n)$ denote the number of overpartition pairs of n with $r_3(\pi) \equiv s \pmod{t}$. Then we have the following relation.

Theorem 3.3. *For $0 \le s \le 2$, we have*

$$R_3(s,3,3n+2) = \overline{\mathrm{pp}}(3n+2)/3. \tag{3.11}$$

Proof. It is easy to derive that

$$\sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} R_3(m,n) z^m q^n = \frac{(-q;q)_\infty^2}{(qz;q)_\infty (q/z;q)_\infty}. \tag{3.12}$$

Using the fact that $R_3(1,3,n) = R_3(2,3,n)$ and setting $z = \xi = e^{2\pi i/3}$ in (3.12), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} (R_3(0,3,n) - R_3(1,3,n))q^n &= \frac{(-q;q)_\infty^2}{(q\xi;q)_\infty (q/\xi;q)_\infty} \\ &= \frac{(-q;q)_\infty^2 (q;q)_\infty}{(q^3;q^3)_\infty} = \frac{1}{(q^3;q^3)_\infty} \sum_{n=0}^{\infty} q^{n(n+1)/2}.\end{aligned}$$

Note that there are no triangular numbers that are congruent to 2 modulo 3. It follows that

$$R_3(0,3,3n+2) = R_3(1,3,3n+2).$$

Since $R_3(1,3,3n+2) = R_3(2,3,3n+2)$, the proof is complete. ∎

To conclude this section, we have the following theorem.

THEOREM 3.4. *Let l be an odd prime, and let t be an odd number which is a power of l or is relatively prime to l. Then for any positive integer j, there are infinitely many non-nested arithmetic progressions $An + B$ such that*

$$R_3(r, t, An + B) \equiv 0 \pmod{l^j} \tag{3.13}$$

for any $0 \leq r \leq t - 1$.

Proof. Note that the generating function for $R_3(s, t, n)$ can be written as a linear combination of certain modular forms similar to the case for $\overline{\mathrm{NN}}(r, t, n)$. Suppose that t is an odd integer and $0 \leq s < t$. Let $\zeta_t = e^{2\pi i/t}$ and define the rank of the overpartition pair of 0 to be 0. Then

$$\sum_{n=0}^{\infty} R_3(s, t, n)q^n = \frac{1}{t}\sum_{k=0}^{t-1} \zeta_t^{-ks} R_3(\zeta_t^k; q),$$

where

$$R_3(z; q) = \frac{(-q; q)_\infty^2}{(qz; q)_\infty (q/z; q)_\infty}.$$

Observe that $R_3(\zeta_t^k; q)$ differs from $R(\zeta_t^k; q)$ (see Bringmann and Lovejoy [4, Proposition 2.4]) only by a factor $\frac{4}{(1+\zeta_t^k)(1+\zeta_t^{-k})}$. Hence the argument of Bringmann and Lovejoy for (1.3) can be carried over to deduce (3.13). ∎

4. Infinite families of congruences modulo 3 and 5. In this section, we obtain a formula for $\overline{\mathrm{pp}}(3n)$ modulo 3 based on the number of representations of n as a sum of two squares. We further derive a formula for $\overline{\mathrm{pp}}(5n)$ modulo 5 in connection with the number of representations of n in the form $x^2 + 5y^2$. As consequences, we give infinite families of congruences modulo 3 and 5.

THEOREM 4.1. *If the prime factorization of n is given by*

$$n = 2^a \prod_{i=1}^{r} p_i^{v_i} \prod_{j=1}^{s} q_j^{w_j}, \tag{4.1}$$

where $p_i \equiv 1 \pmod 4$ and $q_j \equiv 3 \pmod 4$, then

$$\overline{\mathrm{pp}}(3n) \equiv (-1)^n \prod_{i=1}^{r}(1 + v_i) \prod_{j=1}^{s} \frac{1 + (-1)^{w_j}}{2} \pmod 3. \tag{4.2}$$

Proof. First, it is easy to see that

$$\varphi(-q)^3 \equiv \varphi(-q^3) \pmod 3 \quad \text{and} \quad \varphi(-q) = \varphi(-q^9) + qB(q^3),$$

where $B(q)$ is an infinite series in q with integer coefficients. Hence,

$$\sum_{n=0}^{\infty} \overline{\mathrm{pp}}(n)q^n = \frac{\varphi(-q)}{\varphi(-q)^3} \equiv \frac{\varphi(-q)}{\varphi(-q^3)} \pmod 3 = \frac{\varphi(-q^9) + qB(q^3)}{\varphi(-q^3)}.$$

Extracting the terms q^{3n} for $n \geq 0$, and replacing q^3 by q, we find that

$$\sum_{n=0}^{\infty} \overline{\mathrm{pp}}(3n)q^n \equiv \frac{\varphi(-q^3)}{\varphi(-q)} \equiv \varphi(-q)^2 \pmod 3. \tag{4.3}$$

Let $r_2(n)$ denote the number of representations of n as a sum of two squares. So we have

$$\varphi(-q)^2 = \sum_{n=0}^{\infty} (-1)^n r_2(n) q^n. \tag{4.4}$$

From (4.3) and (4.4) it follows that

$$\overline{\mathrm{pp}}(3n) \equiv (-1)^n r_2(n) \pmod 3. \tag{4.5}$$

Given the prime factorization of n in the form of (4.1), it is well known that (see, e.g., Berndt [3] or Grosswald [7])

$$r_2(n) = 4 \prod_{i=1}^{r} (1 + v_i) \prod_{j=1}^{s} \frac{1 + (-1)^{w_j}}{2}. \tag{4.6}$$

Combining (4.5) and (4.6), we get (4.2). ∎

THEOREM 4.2. *Assume that p is prime with $p \equiv 3 \pmod 4$, and s is an integer with $1 \leq s < p$. Then for any $\alpha \geq 0$ and $n \geq 0$, we have*

$$\overline{\mathrm{pp}}(3p^{2\alpha+1}(pn + s)) \equiv 0 \pmod 3. \tag{4.7}$$

In particular, setting $p = 3$ we have, for any $\alpha \geq 1$ and $n \geq 0$,

$$\overline{\mathrm{pp}}(9^\alpha(3n + 1)) \equiv 0 \pmod 3, \tag{4.8}$$

$$\overline{\mathrm{pp}}(9^\alpha(3n + 2)) \equiv 0 \pmod 3. \tag{4.9}$$

Proof. Recall that $r_2(n) = 0$ if and only if there exists a prime congruent to 3 modulo 4 that has an odd exponent in the canonical factorization of n. It can be seen that

$$r_2(p^{2\alpha+1}(pn + s)) = 0,$$

since p is not a factor of $pn + s$. By (4.5) we obtain (4.7). ∎

THEOREM 4.3. *Let $R(n, x^2 + 5y^2)$ denote the number of representations of n by the quadratic form $x^2 + 5y^2$. Then for any $n \geq 0$,*

$$\overline{\mathrm{pp}}(5n) \equiv (-1)^n R(n, x^2 + 5y^2) \pmod 5. \tag{4.10}$$

Proof. It is easy to see that $\varphi(-q)^8$ is a modular form of weight 4 on $\Gamma_0(2)$, where

$$\Gamma_0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod 2 \right\}.$$

For the background on modular forms, see Ono [16]. Now $\varphi(-q)^8 | T_5$ is also a modular form of weight 4 on $\Gamma_0(2)$. Here T_5 is the Hecke operator which acts on

$$\varphi(-q)^8 := \sum_{n=0}^{s} r(n) q^n$$

defined by

$$\varphi(-q)^8 | T_5 = \sum_{n=0}^{\infty} r(5n) q^n + \sum_{n=0}^{\infty} 125 r(n) q^{5n}.$$

By Sturm's theorem (see [16, p. 40]), we have

$$\varphi(-q)^8 | T_5 \equiv \varphi(-q)^8 \pmod 5,$$

and so

$$\sum_{n=0}^{\infty} r(5n) q^n \equiv \varphi(-q)^8 \pmod 5. \tag{4.11}$$

On the other hand,

$$\varphi(-q)^8 = \varphi(-q)^{10} \cdot \frac{1}{\varphi(-q)^2} \equiv \varphi(-q^5)^2 \sum_{n=0}^{\infty} \overline{\mathrm{pp}}(n) q^n \pmod 5.$$

Considering the terms for which the powers of q are multiples of 5, and replacing q^5 by q, we deduce that

$$\sum_{n=0}^{\infty} r(5n) q^n \equiv \varphi(-q)^2 \sum_{n=0}^{\infty} \overline{\mathrm{pp}}(5n) q^n \pmod 5. \tag{4.12}$$

Combining (4.11) and (4.12), we deduce that

$$\sum_{n=0}^{\infty} \overline{\mathrm{pp}}(5n) q^n \equiv \varphi(-q)^6 \equiv \varphi(-q)\varphi(-q^5) \pmod 5. \qquad \blacksquare$$

The formula for $R(n, x^2 + 5y^2)$ due to Berkovich and Yesilyurt [2] leads to the following formula for $\overline{\mathrm{pp}}(5n)$ modulo 5.

THEOREM 4.4. *If the prime factorization of n is given by*

$$n = 2^a 5^b \prod_{i=1}^{r} p_i^{v_i} \prod_{j=1}^{s} q_j^{w_j}, \tag{4.13}$$

where $p_i \equiv 1, 3, 7,$ or $9 \pmod{20}$ and $q_j \equiv 11, 13, 17,$ or $19 \pmod{20}$, then

$$\overline{\mathrm{pp}}(5n) \equiv (-1)^n (1 + (-1)^{a+t}) \prod_{i=1}^{r} (1 + v_i) \prod_{j=1}^{s} \frac{1 + (-1)^{w_j}}{2} \pmod 5, \tag{4.14}$$

where t is the number of prime factors of n, counting multiplicity, that are congruent to 3 or 7 modulo 20.

Proof. Given the prime factorization of n in the form of (4.13), it is known that (see Berkovich and Yesilyurt [2, Corollary 3.3])

$$R(n, x^2+5y^2) = (1+(-1)^{a+t})\prod_{i=1}^{r}(1+v_i)\prod_{j=1}^{s}\frac{1+(-1)^{w_j}}{2}. \tag{4.15}$$

Combining (4.10) and (4.15), we get (4.14). ∎

As consequences, we have the following congruences.

Corollary 4.1. *Let p be a prime with $p \equiv 11, 13, 17,$ or $19 \pmod{20}$. Then for any odd positive integer t and any positive integer n that is not divisible by p,*

$$\overline{\mathrm{pp}}(5p^t n) \equiv 0 \pmod 5. \tag{4.16}$$

Corollary 4.2. *Let p be a prime with $p \equiv 1$ or $9 \pmod{20}$. Then for any positive integer k,*

$$\overline{\mathrm{pp}}(5p^k) \equiv 3(k+1) \pmod 5. \tag{4.17}$$

Based on Theorem 4.4, we get two infinite families of congruences modulo 5.

Theorem 4.5. *For any $\alpha \geq 1$ and $n \geq 0$,*

$$\overline{\mathrm{pp}}(5^\alpha(5n+2)) \equiv 0 \pmod 5, \tag{4.18}$$

$$\overline{\mathrm{pp}}(5^\alpha(5n+3)) \equiv 0 \pmod 5. \tag{4.19}$$

Proof. Considering the possible residues of x^2+5y^2 modulo 5, we find that

$$R(5n+2, x^2+5y^2) = R(5n+3, x^2+5y^2) = 0.$$

In light of (4.10), we deduce that

$$\overline{\mathrm{pp}}(25n+10) \equiv (-1)^{5n+2}R(5n+2, x^2+5y^2) \equiv 0 \pmod 5, \tag{4.20}$$

$$\overline{\mathrm{pp}}(25n+15) \equiv (-1)^{5n+3}R(5n+3, x^2+5y^2) \equiv 0 \pmod 5. \tag{4.21}$$

Observe that formula (4.14) for $\overline{\mathrm{pp}}(5n)$ modulo 5 is independent of the exponent of 5 in the factorization of n. This means that, for $\alpha \geq 1$,

$$\overline{\mathrm{pp}}(5n) \equiv \overline{\mathrm{pp}}(5^\alpha n) \pmod 5. \tag{4.22}$$

Combining (4.20), (4.21) and (4.22), we obtain (4.18) and (4.19). ∎

5. Further congruences for overpartition pairs. In this section, we find some congruences for $\overline{\mathrm{pp}}(n)$ modulo 9 which are similar to the congruences for the number of broken 2-diamond partitions obtained by Paule and Radu [17]. Let us begin with the congruences modulo 9.

THEOREM 5.1. *For any prime p with $p \equiv 1 \pmod{12}$, we have*

$$\overline{\mathrm{pp}}((3n+2)p) \equiv \frac{\overline{\mathrm{pp}}(2p)}{3}\overline{\mathrm{pp}}(3n+2) \pmod 9 \tag{5.1}$$

for all positive integers n such that $3n+2 \not\equiv 0 \pmod p$.

To prove the above theorem, we need the following lemma which is a special case of Newman's [15, Theorem 3].

LEMMA 5.1. *For each prime p with $p \equiv 1 \pmod{12}$ and for all positive integers n,*

$$b\left(np + \frac{2p-2}{3}\right) + p^4 b\left(\frac{n}{p} - 2\frac{p-1}{3p}\right) = b\left(\frac{2p-2}{3}\right)b(n), \tag{5.2}$$

where $b(n)$ is defined by

$$\sum_{n=0}^{\infty} b(n)q^n = (q;q)_\infty^4 (q^2;q^2)_\infty^6.$$

Since the equality is derived by equating coefficients of series in q, it is safe to assume that $b(t) = 0$ if t is not a non-negative integer.

Proof of Theorem 5.1. By (2.1), we see that

$$\sum_{n=0}^{\infty} \frac{\overline{\mathrm{pp}}(3n+2)}{3} q^n \equiv (q;q)_\infty^4 (q^2;q^2)_\infty^6 \pmod 3.$$

From the definition of $b(n)$, we deduce that, for $n \geq 0$,

$$\frac{\overline{\mathrm{pp}}(3n+2)}{3} \equiv b(n) \pmod 3. \tag{5.3}$$

On the other hand, for those prime p with $p \equiv 1 \pmod{12}$ and those n such that $3n+2$ is not a multiple of p, it follows that $b\big(\frac{n}{p} - 2\frac{p-1}{3p}\big) = 0$. Thus, by Lemma 5.1 we obtain

$$b\left(np + \frac{2p-2}{3}\right) = b\left(\frac{2p-2}{3}\right)b(n). \tag{5.4}$$

Substituting (5.3) into (5.4), we get

$$\frac{1}{3}\overline{\mathrm{pp}}(3np+2p) \equiv \frac{1}{9}\overline{\mathrm{pp}}(2p)\overline{\mathrm{pp}}(3n+2) \pmod 3,$$

as required. ∎

Next, we use Lemma 5.1 to obtain the following congruence in the spirit of Paule and Radu [17].

THEOREM 5.2. *For any $k \geq 0$, we have*

$$\overline{\mathrm{pp}}(2 \cdot 13^k) \equiv 3(k+1) \pmod 9. \tag{5.5}$$

Proof. Let p be a prime with $p \equiv 1 \pmod{12}$. Setting $n = 2(p^{k+1} - 1)/3$ in (5.2) and using (5.3), we get

$$\frac{1}{3}\overline{\mathrm{pp}}(2p^{k+2}) + \frac{1}{3}\overline{\mathrm{pp}}(2p^{k}) \equiv \frac{1}{9}\overline{\mathrm{pp}}(2p)\overline{\mathrm{pp}}(2p^{k+1}) \pmod{3}.$$

When $p = 13$, since $\overline{\mathrm{pp}}(26) \equiv 6 \pmod{9}$, we deduce that

$$\overline{\mathrm{pp}}(2 \cdot 13^{k+2}) + \overline{\mathrm{pp}}(2 \cdot 13^{k}) \equiv 2\overline{\mathrm{pp}}(2 \cdot 13^{k+1}) \pmod{9}. \tag{5.6}$$

Given the initial conditions $\overline{\mathrm{pp}}(2) \equiv 3 \pmod{9}$ and $\overline{\mathrm{pp}}(26) \equiv 6 \pmod{9}$, by iteration of (5.6), we reach (5.5). ∎

Acknowledgments. We wish to thank the referee for helpful suggestions. This work was supported by the 973 Project, the PCSIRT Project of the Ministry of Education, and the National Science Foundation of China.

References

[1] G. E. Andrews, R. P. Lewis and J. Lovejoy, *Partitions with designated summands*, Acta Arith. 105 (2002), 51–66.

[2] A. Berkovich and H. Yesilyurt, *Ramanujan's identities and representation of integers by certain binary and quaternary quadratic forms*, Ramanujan J. 20 (2009), 375–408.

[3] B. C. Berndt, *Number Theory in the Spirit of Ramanujan*, Amer. Math. Soc., Providence, RI, 2006.

[4] K. Bringmann and J. Lovejoy, *Rank and congruences for overpartition pairs*, Int. J. Number Theory 4 (2008), 303–322.

[5] H.-C. Chan, *Ramanujan's cubic continued fraction and an analog of his "most beautiful identity"*, ibid. 6 (2010), 673–680.

[6] J.-F. Fortin, P. Jacob and P. Mathieu, *Jagged partitions*, Ramanujan J. 10 (2005), 215–235.

[7] E. Grosswald, *Representations of Integers as Sums of Squares*, Springer, New York, 1985.

[8] M. D. Hirschhorn and J. A. Sellers, *Arithmetic relations for overpartitions*, J. Combin. Math. Combin. Comput. 53 (2005), 65–73.

[9] —, —, *An infinite family of overpartition congruences modulo 12*, Integers 5 (2005), A20.

[10] D. M. Keister, J. A. Sellers and R. G. Vary, *Some arithmetic properties of overpartition k-tuples*, ibid. 9 (2009), A17.

[11] B. Kim, *A short note on the overpartition function*, Discrete Math. 309 (2009), 2528–2532.

[12] —, *Overpartition pairs modulo powers of 2*, ibid. 311 (2011), 835–840.

[13] J. Lovejoy and R. Osburn, *Quadratic forms and four partition functions modulo 3*, Integers 11 (2011), A4.

[14] K. Mahlburg, *The overpartition function modulo small powers of 2*, Discrete Math. 286 (2004), 263–267.

[15] M. Newman, *Modular forms whose coefficients possess multiplicative properties*, Ann. of Math. 70 (1959), 478–489.

[16] K. Ono, *The Web of Modularity: Arithmetic of the Coefficients of Modular Forms and q-Series*, CBMS Reg. Conf. Ser. Math. 102, Amer. Math. Soc., Providence, RI, 2004.

[17] P. Paule and S. Radu, *Infinite families of strange partition congruences for broken 2-diamonds*, Ramanujan J. 23 (2010), 409–416.

[18] K. G. Ramanathan, *Identities and congruences of the Ramanujan type*, Canad. J. Math. 2 (1950), 168–178.

William Y. C. Chen, Bernard L. S. Lin
Center for Combinatorics, LPMC-TJKLC
Nankai University
Tianjin 300071, P.R. China
E-mail: chen@nankai.edu.cn
linlishuang@cfc.nankai.edu.cn

Received on 25.9.2010
and in revised form on 13.3.2011 (6502)