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New properties for the Ramanujan–Göllnitz–Gordon continued fraction

by

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1. Introduction. The Rogers–Ramanujan continued fraction is defined by ([2, p. 9], [13, p. xxviii])

$$R(q) := \frac{q^{1/5}}{1} \, \frac{q}{+\,1} \, \frac{q^2}{+\,1} \, \frac{q^3}{+\,1} + \cdots, \qquad |q| < 1.$$

As is customary, throughout this paper we assume that $|q| < 1$ and use the standard notation

$$(a;q)_n := \prod_{k=0}^{n-1} (1 - aq^k) \quad \text{and} \quad (a;q)_\infty := \prod_{n=0}^{\infty} (1 - aq^n).$$

The famous Rogers–Ramanujan functions $G(q)$ and $H(q)$ are defined by

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}, \tag{1.1}$$

$$H(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}, \tag{1.2}$$

where the two equalities on the right sides of (1.1) and (1.2) are the celebrated Rogers–Ramanujan identities [2, p. 87], [15, p. 347]. Using (1.1) and (1.2), Rogers [16] proved that

$$R(q) = q^{1/5} \frac{H(q)}{G(q)}.$$

In his notebooks, Ramanujan recorded many identities involving $R(q)$ which can be found in [2, 4, 14, 15]. For example, two of the most important

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formulas for $R(q)$ [2, p. 11], [15, pp. 135, 238] are

$$\frac{1}{R(q)} - 1 - R(q) = \frac{(q^{1/5};q^{1/5})_\infty}{q^{1/5}(q^5;q^5)_\infty}, \tag{1.3}$$

$$\frac{1}{R^5(q)} - 11 - R^5(q) = \frac{(q;q)_\infty^6}{q(q^5;q^5)_\infty^6}. \tag{1.4}$$

Furthermore, he established factorizations of (1.3) and (1.4) [2, pp. 21–22], [5], [15, p. 206],

$$\begin{aligned}
\frac{1}{\sqrt{R(q)}} - \gamma\sqrt{R(q)} &= \frac{1}{q^{1/10}}\sqrt{\frac{(q;q)_\infty}{(q^5;q^5)_\infty}}\prod_{n=1}^{\infty}\frac{1}{1+\gamma q^{n/5}+q^{2n/5}},\\
\frac{1}{\sqrt{R(q)}} - \delta\sqrt{R(q)} &= \frac{1}{q^{1/10}}\sqrt{\frac{(q;q)_\infty}{(q^5;q^5)_\infty}}\prod_{n=1}^{\infty}\frac{1}{1+\delta q^{n/5}+q^{2n/5}},\\
\left(\frac{1}{\sqrt{R(q)}}\right)^5 - \left(\gamma\sqrt{R(q)}\right)^5 &= \frac{1}{q^{1/2}}\sqrt{\frac{(q;q)_\infty}{(q^5;q^5)_\infty}}\prod_{n=1}^{\infty}\frac{1}{(1+\gamma q^{n/5}+q^{2n/5})^5},\\
\left(\frac{1}{\sqrt{R(q)}}\right)^5 - \left(\delta\sqrt{R(q)}\right)^5 &= \frac{1}{q^{1/2}}\sqrt{\frac{(q;q)_\infty}{(q^5;q^5)_\infty}}\prod_{n=1}^{\infty}\frac{1}{(1+\delta q^{n/5}+q^{2n/5})^5},
\end{aligned} \tag{1.5}$$

where $\gamma = (1-\sqrt{5})/2$ and $\delta = (1+\sqrt{5})/2$.

On page 229 of his second notebook [14], [4, p. 221], Ramanujan defined another continued fraction $v(q)$ that has properties similar to those of $R(q)$, namely,

$$v(q) := \frac{q^{1/2}}{1+q}\,{}_+\,\frac{q^2}{1+q^3}\,{}_+\,\frac{q^4}{1+q^5}\,{}_+\,\frac{q^5}{1+q^7}\,{}_+\cdots, \tag{1.6}$$

later called the Ramanujan–Göllnitz–Gordon continued fraction. In addition, Ramanujan recorded an identity for $v(q)$ [4, p. 221],

$$v(q) = q^{1/2}\frac{(q;q^8)_\infty(q^7;q^8)_\infty}{(q^3;q^8)_\infty(q^5;q^8)_\infty}. \tag{1.7}$$

H. Göllnitz [9] and B. Gordon [10] rediscovered and proved (1.7) independently. A proof of (1.7) can also be found in [4, p. 221]. Moreover, Ramanujan established two further identities for $v(q)$ [4, p. 221], [14, p. 229], namely,

$$\frac{1}{v(q)} - v(q) = \frac{(-q^2;q^4)_\infty^2(q^4;q^4)_\infty(q^4;q^8)_\infty}{q^{1/2}(q^8;q^8)_\infty}, \tag{1.8}$$

$$\frac{1}{v(q)} + v(q) = \frac{(-q;q^2)_\infty^2(q^2;q^2)_\infty(q^4;q^8)_\infty}{q^{1/2}(q^8;q^8)_\infty}. \tag{1.9}$$

The Göllnitz–Gordon functions $S(q)$ and $T(q)$ are defined as

$$(1.10)\qquad S(q) := \sum_{n=0}^{\infty} \frac{(-q;q^2)_n}{(q^2;q^2)_n} q^{n^2} \quad \text{and} \quad T(q) := \sum_{n=0}^{\infty} \frac{(-q;q^2)_n}{(q^2;q^2)_n} q^{n^2+2n}.$$

Then the Göllnitz–Gordon identities [9, 10] are given by

$$(1.11)\qquad S(q) = \frac{1}{(q;q^8)_\infty (q^4;q^8)_\infty (q^7;q^8)_\infty},$$

$$(1.12)\qquad T(q) = \frac{1}{(q^3;q^8)_\infty (q^4;q^8)_\infty (q^5;q^8)_\infty}.$$

We see that

$$v(q) = q^{1/2} \frac{T(q)}{S(q)}.$$

Recently, several authors, including N. D. Baruah and N. Saikia [3], H.-H. Chan and S.-S. Huang [6], S.-D. Chen and S.-S. Huang [7], S. Cooper [8] and M. D. Hirschhorn [11], have found additional identities and properties for $v(q)$. In this paper, we will establish several new identities and properties for $v(q)$ motivated by identities involving the Rogers–Ramanujan continued fraction.

In [12], T. Horie and N. Kanou found that

$$(1.13)\qquad \frac{1}{v^2(q)} - 6 + v^2(q) = \frac{(q;q)_\infty^4 (q^4;q^4)_\infty^2}{q(q^2;q^2)_\infty^2 (q^8;q^8)_\infty^4},$$

by employing modular forms (see also [8]). Here in Section 3, we will present factorizations for $v(q)$ in Theorems 3.1 and 3.2 which are similar to (1.5). We prove that

$$\frac{1}{v} + 2 - v = \frac{(-q^{1/2};q)_\infty^2 (q;q)_\infty (q^4;q^8)_\infty}{q^{1/2}(q^8;q^8)_\infty},$$
$$\frac{1}{v} - 2 - v = \frac{(q^{1/2};q)_\infty^2 (q;q)_\infty (q^4;q^8)_\infty}{q^{1/2}(q^8;q^8)_\infty},$$

which is employed to establish a shorter proof of (1.13).

Hirschhorn [11] showed that when infinite products associated with $v(q)$ are expanded as power series, the sign of the coefficients is periodic with period 8, and he also derived some identities for such coefficients. In Section 4, we will provide a new proof of Hirschhorn's results. The forms given in Theorems 4.3 and 4.4 are the same as the ones given by Hirschhorn, but are given here in a more compact form. Both Hirschhorn's form as well as the formulas in Theorems 4.3 and 4.4 immediately imply the sign of the coefficients. In the same section, some new identities for $v(q)$ are found here, namely Theorem 4.6.

2. Preliminary results. For $|ab| < 1$, Ramanujan's general theta-function $f(a,b)$ is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}. \tag{2.1}$$

Jacobi's triple product identity [4, p. 35] is given by

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{2.2}$$

The three most important special cases of $f(a,b)$ [4, p. 36] are

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty, \tag{2.3}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{2.4}$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty, \tag{2.5}$$

where the product representations in (2.3)–(2.5) follow from (2.2).

With the notations (2.1)–(2.5), we can rewrite (1.7)–(1.9), respectively, as

$$v(q) = q^{1/2}\frac{f(-q,-q^7)}{f(-q^3,-q^5)},$$

$$\frac{1}{v(q)} - v(q) = \frac{\varphi(q^2)}{q^{1/2}\psi(q^4)}, \tag{2.6}$$

$$\frac{1}{v(q)} + v(q) = \frac{\varphi(q)}{q^{1/2}\psi(q^4)}. \tag{2.7}$$

After Ramanujan, we define

$$\chi(q) := (-q;q^2)_\infty. \tag{2.8}$$

Ramanujan recorded several identities for $f(a,b)$, $\varphi(q)$, $\psi(q)$, $f(-q)$, and $\chi(q)$. The following lemma provides such identities.

LEMMA 2.1 ([4, p. 48]). *Let*

$$U_n = a^{n(n+1)/2} b^{n(n-1)/2} \quad \textit{and} \quad V_n = a^{n(n-1)/2} b^{n(n+1)/2}$$

for each integer n. Then

$$f(U_1,V_1) = \sum_{r=0}^{k-1} U_r f\left(\frac{U_{k+r}}{U_r}, \frac{V_{k-r}}{U_r}\right), \tag{2.9}$$

for every positive integer k.

LEMMA 2.2 ([4, p. 34]). *We have*

$$f(a,b) = f(b,a) \tag{2.10}$$

and if n is an integer, then

$$f(a,b) = a^{n(n+1)/2} b^{n(n-1)/2} f(a(ab)^n, b(ab)^{-n}). \tag{2.11}$$

LEMMA 2.3 ([4, pp. 39–40]). *We have*

$$\varphi(q)\psi(-q) = f(q)f(-q^2), \tag{2.12}$$

$$\varphi(q)\varphi(-q) = \varphi^2(-q^2), \tag{2.13}$$

$$\psi(q)\psi(-q) = \psi(q^2)\varphi(-q^2), \tag{2.14}$$

$$\varphi(q)\psi(q^2) = \psi^2(q), \tag{2.15}$$

$$\varphi(q) + \varphi(-q) = 2\varphi(q^4), \tag{2.16}$$

$$\varphi(q) - \varphi(-q) = 4q\psi(q^8). \tag{2.17}$$

3. New identities. In this section, we shall provide identities for $v(q)$ which resemble (1.3) and (1.5). Define

$$v := q^{1/2} \frac{f(-q,-q^7)}{f(-q^3,-q^5)}. \tag{3.1}$$

THEOREM 3.1. *We have*

$$\frac{1}{\sqrt{v(q)}} + i\sqrt{v(q)} = \frac{f^2(q)}{q^{1/4} f(-iq^{1/2}) f(-q^8)\sqrt{\chi(-q)}}, \tag{3.2}$$

$$\frac{1}{\sqrt{v(q)}} - i\sqrt{v(q)} = \frac{f^2(q)}{q^{1/4} f(iq^{1/2}) f(-q^8)\sqrt{\chi(-q)}}, \tag{3.3}$$

$$\frac{1}{\sqrt{v(q)}} + \sqrt{v(q)} = \frac{\chi(q^2)\chi(-q^4)}{q^{1/4}\sqrt{\chi(-q)}} \prod_{n=1}^{\infty}\left(1 + \left(\frac{n}{2}\right) q^{n/2}\right), \tag{3.4}$$

$$\frac{1}{\sqrt{v(q)}} - \sqrt{v(q)} = \frac{\chi(q^2)\chi(-q^4)}{q^{1/4}\sqrt{\chi(-q)}} \prod_{n=1}^{\infty}\left(1 - \left(\frac{n}{2}\right) q^{n/2}\right), \tag{3.5}$$

where $\left(\frac{n}{2}\right)$ is the Kronecker symbol.

Proof. By (3.1),

$$\frac{1}{\sqrt{v}} + i\sqrt{v} = \frac{f(-q^3,-q^5) + iq^{1/2} f(-q,-q^7)}{q^{1/4}\sqrt{f(-q,-q^7)f(-q^3,-q^5)}}. \tag{3.6}$$

By Jacobi's triple product identity (2.2),

$$\begin{aligned} &f(-q,-q^7)f(-q^3,-q^5) \\ &\quad = (q;q^8)_\infty (q^7;q^8)_\infty (q^3;q^8)_\infty (q^5;q^8)_\infty (q^8;q^8)_\infty^2 \\ &\quad = (q;q^2)_\infty (q^8;q^8)_\infty^2 = \chi(-q) f^2(-q^8). \end{aligned} \tag{3.7}$$

Take $k=2$, $a=iq^{1/2}$, and $b=-iq^{3/2}$ in (2.9) to obtain

$$f(iq^{1/2},-iq^{3/2})=f(-q^3,-q^5)+iq^{1/2}f(-q,-q^7).$$

By (2.2),

$$\begin{aligned}f(iq^{1/2},-iq^{3/2})&=(-iq^{1/2};q^2)_\infty(iq^{3/2};q^2)_\infty(q^2;q^2)_\infty\\&=\frac{f^2(q)}{f(-iq^{1/2})}.\end{aligned}\tag{3.8}$$

Substituting (3.7) and (3.8) in (3.6), we deduce that

$$\frac{1}{\sqrt{v}}+i\sqrt{v}=\frac{f^2(q)}{q^{1/4}f(-iq^{1/2})f(-q^8)\sqrt{\chi(-q)}}.$$

To prove (3.3), take $k=2$, $a=-iq^{1/2}$, and $b=iq^{3/2}$ in (2.9).

The proof of (3.4) uses $k=2$, $a=q^{1/2}$, and $b=-q^{3/2}$ in (2.9).

Employ (2.9) with $k=2$, $a=-q^{1/2}$, and $b=q^{3/2}$ to prove (3.5). ∎

We notice that Theorem 3.1 is a factorization of (1.8) and (1.9). Before proceeding further, we define $\alpha:=\sqrt{2}-1$, $\beta:=\sqrt{2}+1$ and $\zeta:=e^{\pi i/4}$.

THEOREM 3.2. *We have*

$$\frac{1}{\sqrt{v}}-\alpha\sqrt{v}=\frac{\prod_{n=1}^\infty(1+\alpha(-1)^nq^{n/2}-\alpha q^n-(-1)^nq^{3n/2})}{q^{1/4}f(-q^8)\sqrt{\chi(-q)}},\tag{3.9}$$

$$\frac{1}{\sqrt{v}}+\alpha\sqrt{v}=\frac{\prod_{n=1}^\infty(1+\alpha q^{n/2}-\alpha q^n-q^{3n/2})}{q^{1/4}f(-q^8)\sqrt{\chi(-q)}},\tag{3.10}$$

$$\frac{1}{\sqrt{v}}-\beta\sqrt{v}=\frac{\prod_{n=1}^\infty(1-\beta q^{n/2}+\beta q^n-q^{3n/2})}{q^{1/4}f(-q^8)\sqrt{\chi(-q)}},\tag{3.11}$$

$$\frac{1}{\sqrt{v}}+\beta\sqrt{v}=\frac{\prod_{n=1}^\infty(1-\beta(-1)^nq^{n/2}+\beta q^n-(-1)^nq^{3n/2})}{q^{1/4}f(-q^8)\sqrt{\chi(-q)}}.\tag{3.12}$$

Proof. By (3.1),

$$\frac{1}{\sqrt{v}}-\alpha\sqrt{v}=\frac{f(-q^3,-q^5)-\alpha q^{1/2}f(-q,-q^7)}{q^{1/4}\sqrt{f(-q,-q^7)f(-q^3,-q^5)}}.\tag{3.13}$$

Take $k=4$, $a=\zeta$, and $b=\zeta^3q^{1/2}$ in (2.9) to obtain

$$\begin{aligned}f(\zeta,\zeta^3q^{1/2})&=f(-q^3,-q^5)+\zeta f(-q^5,-q^3)\\&\quad+\zeta^6q^{1/2}f(-q^7,-q)+\zeta^7q^{3/2}f(-q^9,-q^{-1}).\end{aligned}\tag{3.14}$$

Set $n=1$, $a=-q^{-1}$, and $b=-q^9$ in (2.11). This yields

$$f(-q^9,-q^{-1})=-q^{-1}f(-q^7,-q).\tag{3.15}$$

Substituting (3.15) in (3.14), we obtain

$$\begin{aligned} f(\zeta, \zeta^3 q^{1/2}) &= (1+\zeta) f(-q^3, -q^5) + \zeta^6 q^{1/2} f(-q^7, -q) - \zeta^7 q^{1/2} f(-q^7, -q) \\ &= (1+\zeta) f(-q^3, -q^5) + (\zeta^6 - \zeta^7) q^{1/2} f(-q, -q^7), \end{aligned}$$

by (2.10).

Note that $\alpha = \zeta + \zeta^7 - 1$, so $\zeta^6 - \zeta^7 = -\alpha(1+\zeta)$. It follows that

$$\frac{f(\zeta, \zeta^3 q^{1/2})}{1+\zeta} = f(-q^3, -q^5) - \alpha q^{1/2} f(-q, -q^7). \tag{3.16}$$

Substituting (3.7) and (3.16) in (3.13), we deduce that

$$\frac{1}{\sqrt{v}} - \alpha \sqrt{v} = \frac{f(\zeta, \zeta^3 q^{1/2})}{(1+\zeta)\, q^{1/4} \sqrt{\chi(-q) f^2(-q^8)}}. \tag{3.17}$$

By Jacobi's triple product identity (2.2),

$$\begin{aligned} \frac{f(\zeta, \zeta^3 q^{1/2})}{1+\zeta} &= \frac{(-\zeta; -q^{1/2})_\infty (-\zeta^3 q^{1/2}; -q^{1/2})_\infty (-q^{1/2}; -q^{1/2})_\infty}{1+\zeta} \\ &= (\zeta q^{1/2}; -q^{1/2})_\infty (-\zeta^3 q^{1/2}; -q^{1/2})_\infty (-q^{1/2}; -q^{1/2})_\infty \\ &= \prod_{n=1}^{\infty} (1 + \zeta(-q^{1/2})^n)(1 - \zeta^3 (-q^{1/2})^n)(1 - (-q^{1/2})^n). \end{aligned}$$

Note that $\zeta - \zeta^3 = \sqrt{2}$ and $\zeta^4 = -1$. Then

$$\begin{aligned} \frac{f(\zeta, \zeta^3 q^{1/2})}{1+\zeta} &= \prod_{n=1}^{\infty} (1 + \sqrt{2}(-q^{1/2})^n + (-q^{1/2})^{2n})(1 - (-q^{1/2})^n) \\ &= \prod_{n=1}^{\infty} (1 + (\sqrt{2}-1)(-1)^n q^{n/2} - (\sqrt{2}-1) q^n - (-1)^n q^{3n/2}) \\ &= \prod_{n=1}^{\infty} (1 + \alpha(-1)^n q^{n/2} - \alpha q^n - (-1)^n q^{3n/2}). \end{aligned} \tag{3.18}$$

Substituting (3.18) in (3.17), we complete the proof of (3.9).

Use $k = 4$, $a = -\zeta^3$, and $b = -\zeta^5 q^{1/2}$ in (2.9), and use the fact that $\zeta^6 + \zeta = \alpha(1 - \zeta^3)$ to establish (3.10).

To prove (3.11), employ $\zeta^6 - \zeta = -\beta(1+\zeta^3)$ and $k = 4$, $a = \zeta^3$, and $b = \zeta^5 q^{1/2}$ in (2.9).

The proof of (3.12) uses $\zeta^6+\zeta^7=\beta(1-\zeta)$ and $k=4$, $a=-\zeta$, and $b=-\zeta^3q^{1/2}$ in (2.9). ∎

Corollary 3.3. *We have*

$$\frac{1}{v}+2-v=\frac{\varphi(q^{1/2})}{q^{1/2}\psi(q^4)}, \tag{3.19}$$

$$\frac{1}{v}-2-v=\frac{\varphi(-q^{1/2})}{q^{1/2}\psi(q^4)}. \tag{3.20}$$

Proof. By (3.9) and (3.12), we observe that

$$\begin{aligned}\frac{1}{v}+2-v &= \left(\frac{1}{\sqrt{v}}-\alpha\sqrt{v}\right)\left(\frac{1}{\sqrt{v}}+\beta\sqrt{v}\right) \\ &= \frac{\begin{aligned}\textstyle\prod_{n=1}^{\infty}&(1+\alpha(-1)^nq^{n/2}-\alpha q^n-(-1)^nq^{3n/2}) \\ &\times(1-\beta(-1)^nq^{n/2}+\beta q^n-(-1)^nq^{3n/2})\end{aligned}}{q^{1/2}\chi(-q)f^2(-q^8)}.\end{aligned} \tag{3.21}$$

We see that

$$\begin{aligned}&(1+\alpha(-1)^nq^{n/2}-\alpha q^n-(-1)^nq^{3n/2})(1-\beta(-1)^nq^{n/2}+\beta q^n-(-1)^nq^{3n/2}) \\ &\quad= 1-2(-1)^nq^{n/2}+q^n+q^{2n}-2(-1)^nq^{5n/2}+q^{3n} \\ &\quad= (1+q^{2n})(1-2(-1)^nq^{n/2}+q^n) \\ &\quad= \begin{cases}(1+q^{2n})(1-2q^k+q^{2k}) & \text{if } n=2k,\\ (1+q^{2n})(1+2q^{k-1/2}+q^{2k-1}) & \text{if } n=2k-1.\end{cases}\end{aligned} \tag{3.22}$$

Substituting (3.22) in (3.21), we deduce that

$$\begin{aligned}\frac{1}{v}+2-v &= \frac{\prod_{n=1}^{\infty}(1+q^{2n})(1-q^n)^2(1+q^{n-1/2})^2}{q^{1/2}\chi(-q)f^2(-q^8)} \\ &= \frac{(-q^2;q^2)_\infty(q;q)_\infty^2(-q^{1/2};q)_\infty^2}{q^{1/2}(q;q^2)_\infty(q^8;q^8)_\infty^2} \\ &= \frac{(-q^2;q^2)_\infty(q^2;q^2)_\infty(q;q)_\infty(-q^{1/2};q)_\infty^2}{q^{1/2}(q^8;q^8)_\infty^2} \\ &= \frac{(q^4;q^8)_\infty(q;q)_\infty(-q^{1/2};q)_\infty^2}{q^{1/2}(q^8;q^8)_\infty} = \frac{\varphi(q^{1/2})}{q^{1/2}\psi(q^4)},\end{aligned}$$

by (2.3) and (2.4), which completes the proof of (3.19).

It is easy to verify (3.20) in a similar manner from (3.10) and (3.11) or simply by replacing $q^{1/2}$ with $-q^{1/2}$. ∎

COROLLARY 3.4. *If $\zeta = e^{\pi i/4}$, then*

$$\frac{1}{v} + 2\sqrt{2} + v = \frac{f^2(-q)(\zeta^3 q^{1/2};q)_\infty^2(\zeta^5 q^{1/2};q)_\infty^2}{q^{1/2}f(-q^4)f(-q^8)},$$

$$\frac{1}{v} - 2\sqrt{2} + v = \frac{f^2(-q)(\zeta q^{1/2};q)_\infty^2(\zeta^7 q^{1/2};q)_\infty^2}{q^{1/2}f(-q^4)f(-q^8)},$$

$$\frac{1}{v} - \alpha^2 v = \frac{f^2(-q)\chi(q^2)(\zeta^3 q;q)_\infty^2(\zeta^5 q;q)_\infty^2}{q^{1/2}f^2(-q^8)},$$

$$\frac{1}{v} - \beta^2 v = \frac{f^2(-q)\chi(q^2)(\zeta q;q)_\infty^2(\zeta^7 q;q)_\infty^2}{q^{1/2}f^2(-q^8)}.$$

Proof. The proof is similar to that of the previous corollary. ∎

The following corollary is (1.13), and originally was proved by using modular forms.

COROLLARY 3.5. *We have*

$$\frac{1}{v^2} - 6 + v^2 = \frac{\varphi^2(-q)}{q\psi^2(q^4)} = \frac{f^4(-q)f^2(-q^4)}{qf^2(-q^2)f^4(-q^8)}. \tag{3.23}$$

Proof. By (3.19) and (3.20), we deduce that

$$\frac{1}{v^2} - 6 + v^2 = \left(\frac{1}{v} + 2 - v\right)\left(\frac{1}{v} - 2 - v\right)$$
$$= \left(\frac{\varphi(q^{1/2})}{q^{1/2}\psi(q^4)}\right)\left(\frac{\varphi(-q^{1/2})}{q^{1/2}\psi(q^4)}\right) = \frac{\varphi^2(-q)}{q\psi^2(q^4)}.$$

The last equation is obtained by (2.13). By employing the product representations in (2.3)–(2.5), we easily obtain the last equality of (3.23). ∎

4. The expansion of the Ramanujan–Göllnitz–Gordon continued fraction. In [7], Chen and Huang established the 4-dissection of the Ramanujan–Göllnitz–Gordon continued fraction. In this section, we shall provide a new proof of the 8-dissection identities for this continued fraction found by Hirschhorn [11]. Define

$$D(q) := \frac{f(-q^3,-q^5)}{f(-q,-q^7)}.$$

By (2.6) and (2.7), we find that

$$D(q) = \frac{1}{2\psi(q^4)}(\varphi(q^2)+\varphi(q)) = \frac{1}{2\psi(q^4)}\Big(\sum_{n=-\infty}^{\infty} q^{2n^2} + \sum_{n=-\infty}^{\infty} q^{n^2}\Big) \tag{4.1}$$
$$=: \sum_{n=0}^{\infty} u_n q^n.$$

If $f(q)=\sum_{n=0}^{\infty}a_nq^n$, we define the operator U_8 operating on $f(q)$ by [1, p. 161]

$$U_8f(q):=\sum_{n=0}^{\infty}a_{8n}q^n=\frac{1}{8}\sum_{j=0}^{7}f(\zeta^jq^{1/8}),$$

where $\zeta=e^{\pi i/4}$. Then apply U_8 to (4.1) and deduce that

$$\begin{aligned}(4.2)\qquad \sum_{n=0}^{\infty}u_{8n+a}q^n&=U_8q^{-a}D(q)=\frac{1}{8}\sum_{j=0}^{7}\zeta^{-aj}q^{-a/8}D(\zeta^jq^{1/8})\\
&=\frac{1}{16}\sum_{j=0}^{7}\zeta^{-aj}q^{-a/8}\frac{1}{\psi(\zeta^{4j}q^{1/2})}\Big(\sum_{n=-\infty}^{\infty}\zeta^{2n^2j}q^{n^2/4}+\sum_{n=-\infty}^{\infty}\zeta^{n^2j}q^{n^2/8}\Big)\\
&=\frac{1}{16}\sum_{j=0}^{7}\frac{1}{\psi((-1)^jq^{1/2})}\Big(\sum_{n=-\infty}^{\infty}\zeta^{(2n^2-a)j}q^{n^2/4-a/8}+\sum_{n=-\infty}^{\infty}\zeta^{(n^2-a)j}q^{n^2/8-a/8}\Big)\\
&=\frac{1}{16\psi(q^{1/2})}\sum_{j=0}^{3}\Big(\sum_{n=-\infty}^{\infty}\zeta^{(2n^2-a)2j}q^{n^2/4-a/8}+\sum_{n=-\infty}^{\infty}\zeta^{(n^2-a)2j}q^{n^2/8-a/8}\Big)\\
&\quad+\frac{1}{16\psi(-q^{1/2})}\sum_{j=0}^{3}\Big(\sum_{n=-\infty}^{\infty}\zeta^{(2n^2-a)(2j+1)}q^{n^2/4-a/8}\\
&\qquad\qquad+\sum_{n=-\infty}^{\infty}\zeta^{(n^2-a)(2j+1)}q^{n^2/8-a/8}\Big).\end{aligned}$$

LEMMA 4.1. *If $\zeta=e^{\pi i/4}$, then*

$$(4.3)\qquad \sum_{j=0}^{3}\zeta^{r(2j)}=\begin{cases}4 & \text{if } r\equiv 0 \pmod 4,\\ 0 & \text{otherwise,}\end{cases}$$

$$(4.4)\qquad \sum_{j=0}^{3}\zeta^{r(2j+1)}=\begin{cases}4 & \text{if } r\equiv 0 \pmod 8,\\ -4 & \text{if } r\equiv 4 \pmod 8,\\ 0 & \text{otherwise.}\end{cases}$$

LEMMA 4.2. *We have*

$$(4.5)\qquad \psi(q)\psi(-q)=f(-q^2)f(-q^4),$$

$$(4.6)\qquad \psi(q)+\psi(-q)=2f(q^6,q^{10}),$$

$$(4.7)\qquad \psi(q)-\psi(-q)=2qf(q^2,q^{14}).$$

Proof. The identity (4.5) follows immediately from (2.14) and (2.12).

To prove (4.6), by (2.4), we have

$$\begin{aligned}
\psi(q)+\psi(-q) &= \sum_{n=0}^{\infty} q^{n(n+1)/2}+\sum_{n=0}^{\infty}(-q)^{n(n+1)/2}\\
&= \sum_{n=0}^{\infty} q^{n(2n+1)}+\sum_{n=0}^{\infty} q^{(n+1)(2n+1)}+\sum_{n=0}^{\infty}(-1)^n q^{n(2n+1)}+\sum_{n=0}^{\infty}(-1)^n q^{(n+1)(2n+1)}\\
&= 2\sum_{n=0}^{\infty} q^{2n(4n+1)}+2\sum_{n=0}^{\infty} q^{(2n+2)(4n+3)} = 2\sum_{n=-\infty}^{\infty} q^{2n(4n+1)}.
\end{aligned}$$

By (2.1) and (2.10), we obtain (4.6).

Similarly, by (2.4), we deduce that

$$\begin{aligned}
\psi(q)-\psi(-q) &= \sum_{n=0}^{\infty} q^{n(n+1)/2}-\sum_{n=0}^{\infty}(-q)^{n(n+1)/2}\\
&= \sum_{n=0}^{\infty} q^{n(2n+1)}+\sum_{n=0}^{\infty} q^{(n+1)(2n+1)}\\
&\quad -\sum_{n=0}^{\infty}(-1)^n q^{n(2n+1)}-\sum_{n=0}^{\infty}(-1)^n q^{(n+1)(2n+1)}\\
&= 2\sum_{n=0}^{\infty} q^{(2n+1)(4n+3)}+2\sum_{n=0}^{\infty} q^{(2n+1)(4n+1)}\\
&= 2\sum_{n=-\infty}^{\infty} q^{(2n+1)(4n+1)},
\end{aligned}$$

so the proof of (4.7) is complete by employing (2.1) and (2.10). ■

THEOREM 4.3. *We have*

$$\begin{aligned}
\sum_{n=0}^{\infty} u_{8n}q^n &= \frac{\varphi(q^4)f(q^3,q^5)}{f(-q)f(-q^2)},\\
\sum_{n=0}^{\infty} u_{8n+1}q^n &= \frac{\psi(q)f(q^3,q^5)}{f(-q)f(-q^2)},\\
\sum_{n=0}^{\infty} u_{8n+2}q^n &= \frac{\psi(q^2)f(q^3,q^5)}{f(-q)f(-q^2)},\\
\sum_{n=0}^{\infty} u_{8n+3}q^n &= 0,
\end{aligned}$$

$$\sum_{n=0}^{\infty} u_{8n+4}q^n = -\frac{2q\psi(q^8)f(q,q^7)}{f(-q)f(-q^2)},$$

$$\sum_{n=0}^{\infty} u_{8n+5}q^n = -\frac{\psi(q)f(q,q^7)}{f(-q)f(-q^2)},$$

$$\sum_{n=0}^{\infty} u_{8n+6}q^n = -\frac{\psi(q^2)f(q,q^7)}{f(-q)f(-q^2)},$$

$$\sum_{n=0}^{\infty} u_{8n+7}q^n = 0.$$

Proof. It is obvious from (4.2)–(4.4) that the fourth and the last identities hold.

To prove the first equality, by (4.2) with $a = 0$, (4.3) and (4.4), we have

$$\begin{aligned}(4.8)\qquad \sum_{n=0}^{\infty} u_{8n}q^n &= \frac{1}{4\psi(q^{1/2})}\Big(\sum_{n=-\infty}^{\infty} q^{n^2} + \sum_{n=-\infty}^{\infty} q^{n^2/2}\Big) \\ &\quad + \frac{1}{4\psi(-q^{1/2})}\Big(\sum_{n=-\infty}^{\infty} q^{n^2} + \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2/2}\Big) \\ &= \frac{1}{4\psi(q^{1/2})}(\varphi(q)+\varphi(q^{1/2})) + \frac{1}{4\psi(-q^{1/2})}(\varphi(q)+\varphi(-q^{1/2})) \\ &= \frac{1}{4}\bigg(\varphi(q)\bigg(\frac{1}{\psi(q^{1/2})}+\frac{1}{\psi(-q^{1/2})}\bigg) + \bigg(\frac{\varphi(q^{1/2})}{\psi(q^{1/2})}+\frac{\varphi(-q^{1/2})}{\psi(-q^{1/2})}\bigg)\bigg).\end{aligned}$$

Employing (2.15) and (2.14), respectively, we see that

$$(4.9)\qquad \frac{\varphi(q^{1/2})}{\psi(q^{1/2})} = \frac{\psi(q^{1/2})}{\psi(q)} = \frac{\varphi(-q)}{\psi(-q^{1/2})}.$$

Similarly,

$$(4.10)\qquad \frac{\varphi(-q^{1/2})}{\psi(-q^{1/2})} = \frac{\psi(-q^{1/2})}{\psi(q)} = \frac{\varphi(-q)}{\psi(q^{1/2})}.$$

Putting (4.9) and (4.10) in (4.8) yields

$$\sum_{n=0}^{\infty} u_{8n}q^n = \frac{1}{4}(\varphi(q)+\varphi(-q))\bigg(\frac{1}{\psi(q^{1/2})}+\frac{1}{\psi(-q^{1/2})}\bigg).$$

By (2.16), (4.5) and (4.6),

$$\sum_{n=0}^{\infty} u_{8n}q^n = \frac{\varphi(q^4)f(q^3,q^5)}{f(-q)f(-q^2)},$$

which completes the proof of the first identity.

Now by (4.2) with $a = 1$, (4.3) and (4.4), we find that

$$\begin{aligned}(4.11)\qquad \sum_{n=0}^{\infty} u_{8n+1}q^n &= \frac{1}{4\psi(q^{1/2})}\sum_{n=-\infty}^{\infty} q^{n(n+1)/2} + \frac{1}{4\psi(-q^{1/2})}\sum_{n=-\infty}^{\infty} q^{n(n+1)/2} \\ &= \frac{1}{2}\left(\frac{1}{\psi(q^{1/2})} + \frac{1}{\psi(-q^{1/2})}\right)\psi(q).\end{aligned}$$

Utilize (4.5) and (4.6) to deduce that

$$\sum_{n=0}^{\infty} u_{8n+1}q^n = \frac{\psi(q)f(q^3,q^5)}{f(-q)f(-q^2)}.$$

This completes the proof of the second identity.

Again by (4.2) with $a = 2$, (4.3) and (4.4), we have

$$\begin{aligned}(4.12)\qquad \sum_{n=0}^{\infty} u_{8n+2}q^n &= \frac{1}{4\psi(q^{1/2})}\sum_{n=-\infty}^{\infty} q^{n(n+1)} + \frac{1}{4\psi(-q^{1/2})}\sum_{n=-\infty}^{\infty} q^{n(n+1)} \\ &= \frac{1}{2}\left(\frac{1}{\psi(q^{1/2})} + \frac{1}{\psi(-q^{1/2})}\right)\psi(q^2),\end{aligned}$$

so the proof of the third identity is complete after employing (4.5) and (4.6) in (4.12).

As before, by (4.2) with $a = 4$, (4.3) and (4.4), we have

$$\begin{aligned}(4.13)\qquad \sum_{n=0}^{\infty} u_{8n+4}q^n &= \frac{1}{4\psi(q^{1/2})}\Big(\sum_{n=-\infty}^{\infty} q^{n^2-1/2} + \sum_{n=-\infty}^{\infty} q^{(n^2-1)/2}\Big) \\ &\quad - \frac{1}{4\psi(-q^{1/2})}\Big(\sum_{n=-\infty}^{\infty} q^{n^2-1/2} + \sum_{n=-\infty}^{\infty} (-1)^n q^{(n^2-1)/2}\Big) \\ &= \frac{q^{-1/2}}{4\psi(q^{1/2})}(\varphi(q) + \varphi(q^{1/2})) - \frac{q^{-1/2}}{4\psi(-q^{1/2})}(\varphi(q) + \varphi(-q^{1/2})) \\ &= \frac{q^{-1/2}}{4}\left(\varphi(q)\left(\frac{1}{\psi(q^{1/2})} - \frac{1}{\psi(-q^{1/2})}\right) + \left(\frac{\varphi(q^{1/2})}{\psi(q^{1/2})} - \frac{\varphi(-q^{1/2})}{\psi(-q^{1/2})}\right)\right).\end{aligned}$$

Putting (4.9) and (4.10) in (4.13) yields

$$\sum_{n=0}^{\infty} u_{8n+4}q^n = \frac{q^{-1/2}}{4}(\varphi(q) - \varphi(-q))\left(\frac{1}{\psi(q^{1/2})} - \frac{1}{\psi(-q^{1/2})}\right).$$

By (2.17), (4.5) and (4.7),

$$\sum_{n=0}^{\infty} u_{8n+4}q^n = -\frac{2q\psi(q^8)f(q,q^7)}{f(-q)f(-q^2)},$$

and the proof of the fifth identity is complete.

By (4.2) with $a = 5$, (4.3) and (4.4), we have

$$\sum_{n=0}^{\infty} u_{8n+5} q^n = \frac{1}{4\psi(q^{1/2})} \sum_{n=-\infty}^{\infty} q^{(n^2+n-1)/2} - \frac{1}{4\psi(-q^{1/2})} \sum_{n=-\infty}^{\infty} q^{(n^2+n-1)/2}$$
$$= \frac{q^{-1/2}}{2}\left(\frac{1}{\psi(q^{1/2})} - \frac{1}{\psi(-q^{1/2})}\right)\psi(q).$$

Using (4.5) and (4.7), we find that the sixth identity holds.

Again by (4.2) with $a = 6$, (4.3) and (4.4), we see that

$$\sum_{n=0}^{\infty} u_{8n+6} q^n = \frac{1}{4\psi(q^{1/2})} \sum_{n=-\infty}^{\infty} q^{n(n+1)-1/2} - \frac{1}{4\psi(-q^{1/2})} \sum_{n=-\infty}^{\infty} q^{n(n+1)-1/2}$$
$$= \frac{q^{-1/2}}{2}\left(\frac{1}{\psi(q^{1/2})} - \frac{1}{\psi(-q^{1/2})}\right)\psi(q^2).$$

After utilizing (4.5) and (4.7), the proof of the seventh identity is complete. ■

By (2.6) and (2.7), we see that

$$\frac{1}{D(q)} = \frac{1}{2q\psi(q^4)}(\varphi(q) - \varphi(q^2)) \tag{4.14}$$
$$= \frac{1}{2\psi(q^4)}\Big(\sum_{n=-\infty}^{\infty} q^{n^2-1} - \sum_{n=-\infty}^{\infty} q^{2n^2-1}\Big) =: \sum_{n=0}^{\infty} v_n q^n.$$

Then apply U_8 to (4.14) and deduce that

$$\sum_{n=0}^{\infty} v_{8n+a} q^n = U_8 q^{-a}\frac{1}{D(q)} = \frac{1}{8}\sum_{j=0}^{7} \zeta^{-aj} q^{-a/8} \frac{1}{D(\zeta^j q^{1/8})} \tag{4.15}$$

$$= \frac{1}{16}\sum_{j=0}^{7} \zeta^{-aj} q^{-a/8} \frac{1}{\psi(\zeta^{4j} q^{1/2})}\Big(\sum_{n=-\infty}^{\infty} \zeta^{(n^2-1)j} q^{(n^2-1)/8} - \sum_{n=-\infty}^{\infty} \zeta^{(2n^2-1)j} q^{(2n^2-1)/8}\Big)$$
$$= \frac{1}{16}\sum_{j=0}^{7} \frac{1}{\psi((-1)^j q^{1/2})}\Big(\sum_{n=-\infty}^{\infty} \zeta^{(n^2-a-1)j} q^{(n^2-a-1)/8} - \sum_{n=-\infty}^{\infty} \zeta^{(2n^2-a-1)j} q^{(2n^2-a-1)/8}\Big)$$
$$= \frac{1}{16\psi(q^{1/2})}\sum_{j=0}^{3}\Big(\sum_{n=-\infty}^{\infty} \zeta^{(n^2-a-1)j} q^{(n^2-a-1)/8} - \sum_{n=-\infty}^{\infty} \zeta^{(2n^2-a-1)j} q^{(2n^2-a-1)/8}\Big)$$
$$+ \frac{1}{16\psi(-q^{1/2})}\sum_{j=0}^{3}\Big(\sum_{n=-\infty}^{\infty} \zeta^{(n^2-a-1)j} q^{(n^2-a-1)/8} - \sum_{n=-\infty}^{\infty} \zeta^{(2n^2-a-1)j} q^{(2n^2-a-1)/8}\Big).$$

THEOREM 4.4. *We have*

$$\sum_{n=0}^{\infty} v_{8n}q^n = \frac{\psi(q)f(q^3,q^5)}{f(-q)f(-q^2)},$$

$$\sum_{n=0}^{\infty} v_{8n+1}q^n = -\frac{\psi(q^2)f(q^3,q^5)}{f(-q)f(-q^2)},$$

$$\sum_{n=0}^{\infty} v_{8n+2}q^n = 0,$$

$$\sum_{n=0}^{\infty} v_{8n+3}q^n = \frac{\varphi(q^4)f(q,q^7)}{f(-q)f(-q^2)},$$

$$\sum_{n=0}^{\infty} v_{8n+4}q^n = -\frac{\psi(q)f(q,q^7)}{f(-q)f(-q^2)},$$

$$\sum_{n=0}^{\infty} v_{8n+5}q^n = \frac{\psi(q^2)f(q,q^7)}{f(-q)f(-q^2)},$$

$$\sum_{n=0}^{\infty} v_{8n+6}q^n = 0,$$

$$\sum_{n=0}^{\infty} v_{8n+7}q^n = -\frac{2\psi(q^8)f(q^3,q^5)}{f(-q)f(-q^2)}.$$

Proof. We observe that by (4.2) and (4.15),

$$\sum_{n=0}^{\infty} v_{8n+a}q^n = \sum_{n=0}^{\infty} u_{8n+a+1}q^n \qquad \text{for } a = 0,2,5,6,$$

$$\sum_{n=0}^{\infty} v_{8n+a}q^n = -\sum_{n=0}^{\infty} u_{8n+a+1}q^n \qquad \text{for } a = 1,4.$$

Hence we can deduce all identities except the fourth and the last one.

To prove the fourth equality, by (4.15) with $a = 3$, (4.3) and (4.4), we have

$$\sum_{n=0}^{\infty} v_{8n+3}q^n = \frac{1}{4\psi(q^{1/2})}\Big(\sum_{n=-\infty}^{\infty} q^{(n^2-1)/2} - \sum_{n=-\infty}^{\infty} q^{n^2-1/2}\Big) + \frac{1}{4\psi(-q^{1/2})}\Big(\sum_{n=-\infty}^{\infty} (-1)^{n+1}q^{(n^2-1)/2} + \sum_{n=-\infty}^{\infty} q^{n^2-1/2}\Big) \tag{4.16}$$

$$= \frac{q^{-1/2}}{4\psi(q^{1/2})}(\varphi(q^{1/2}) - \varphi(q)) + \frac{q^{-1/2}}{4\psi(-q^{1/2})}(\varphi(q) - \varphi(-q^{1/2}))$$
$$= \frac{q^{-1/2}}{4}\left(\varphi(q)\left(\frac{1}{\psi(-q^{1/2})} - \frac{1}{\psi(q^{1/2})}\right) + \left(\frac{\varphi(q^{1/2})}{\psi(q^{1/2})} - \frac{\varphi(-q^{1/2})}{\psi(-q^{1/2})}\right)\right).$$

Putting (4.9) and (4.10) in (4.16) yields

$$\sum_{n=0}^{\infty} v_{8n+3}q^n = \frac{q^{-1/2}}{4}(\varphi(q) + \varphi(-q))\left(\frac{1}{\psi(-q^{1/2})} - \frac{1}{\psi(q^{1/2})}\right).$$

Using (2.16), (4.5) and (4.7), we complete the proof of the fourth identity.

To prove the last equality, by (4.15) with $a = 7$, (4.3) and (4.4), we have

$$\sum_{n=0}^{\infty} v_{8n+7}q^n = \frac{1}{4\psi(q^{1/2})}\left(\sum_{n=-\infty}^{\infty} q^{n^2/2-1} - \sum_{n=-\infty}^{\infty} q^{n^2-1}\right) \tag{4.17}$$
$$+ \frac{1}{4\psi(-q^{1/2})}\left(\sum_{n=-\infty}^{\infty} (-1)^{n+1}q^{n^2/2-1} - \sum_{n=-\infty}^{\infty} q^{n^2-1}\right)$$
$$= \frac{q^{-1}}{4\psi(q^{1/2})}(\varphi(q^{1/2}) - \varphi(q)) + \frac{q^{-1}}{4\psi(-q^{1/2})}(\varphi(-q^{1/2}) - \varphi(q))$$
$$= \frac{1}{4q}\left(\left(\frac{\varphi(q^{1/2})}{\psi(q^{1/2})} + \frac{\varphi(-q^{1/2})}{\psi(-q^{1/2})}\right) - \varphi(q)\left(\frac{1}{\psi(q^{1/2})} + \frac{1}{\psi(-q^{1/2})}\right)\right).$$

Putting (4.9) and (4.10) into (4.17) yields

$$\sum_{n=0}^{\infty} v_{8n+7}q^n = \frac{-1}{4q}(\varphi(q) - \varphi(-q))\left(\frac{1}{\psi(q^{1/2})} + \frac{1}{\psi(-q^{1/2})}\right),$$

so the proof is complete after employing (2.17), (4.5) and (4.6). ∎

Corollary 4.5. *With u_n and v_n defined by* (4.1) *and* (4.14), *we have, for $n \geq 0$,*

$$\begin{array}{lllll} u_{8n} > 0, & u_{8n+1} > 0, & u_{8n+2} > 0, & u_{8n+3} = 0, & \\ u_{8n+12} < 0, & u_{8n+5} < 0, & u_{8n+6} < 0, & u_{8n+7} = 0, & u_4 = 0, \\ v_{8n} > 0, & v_{8n+1} < 0, & v_{8n+2} = 0, & v_{8n+3} > 0, & \\ v_{8n+4} < 0, & v_{8n+5} > 0, & v_{8n+6} = 0, & v_{8n+7} < 0. & \end{array}$$

Proof. These results follow immediately from Theorems 4.3 and 4.4. ∎

The last corollary was proved by Chen and Huang [7] and Hirschhorn [11] by different methods. Moreover, the formulas of Theorems 4.3 and 4.4 are in more compact forms than those of Hirschhorn.

We observe that by the two identities (1.11) and (1.12),

$$D(q) = \frac{S(q)}{T(q)}. \tag{4.18}$$

Chen and Huang [7] showed that, with u_n and v_n defined by (4.1) and (4.14),

$$\sum_{n=0}^{\infty} u_{2n} q^n = D(q) \frac{\psi(q^4)}{\psi(q^2)}, \tag{4.19}$$

$$\sum_{n=0}^{\infty} v_{2n+1} q^n = -\frac{1}{D(q)} \frac{\psi(q^4)}{\psi(q^2)}, \tag{4.20}$$

$$\sum_{n=0}^{\infty} u_{2n+1} q^n = \sum_{n=0}^{\infty} v_{2n} q^n = \frac{\psi(q^4)}{\psi(q^2)}. \tag{4.21}$$

THEOREM 4.6. *We have*

$$D(q) = \frac{1}{\psi(q^4)}(q\psi(q^8) + q^2\psi(q^{16}) + q^4\psi(q^{32}) + q^8\psi(q^{64}) + \cdots),$$
$$\frac{1}{D(q)} = \frac{1}{\psi(q^4)}(\psi(q^8) - q\psi(q^{16}) + q^2\psi(q^{32}) - q^4\psi(q^{64}) + \cdots).$$

Proof. By (4.19) and (4.21), it follows that

$$\begin{aligned} D(q) &= \sum_{n=0}^{\infty} u_{2n+1} q^{2n+1} + \sum_{n=0}^{\infty} u_{2n} q^{2n} = q\frac{\psi(q^8)}{\psi(q^4)} + D(q^2)\frac{\psi(q^8)}{\psi(q^4)} \\ &= q\frac{\psi(q^8)}{\psi(q^4)} + q^2\frac{\psi(q^{16})}{\psi(q^4)} + q^4\frac{\psi(q^{32})}{\psi(q^4)} + q^8\frac{\psi(q^{64})}{\psi(q^4)} + \cdots \\ &= \frac{1}{\psi(q^4)}(q\psi(q^8) + q^2\psi(q^{16}) + q^4\psi(q^{32}) + q^8\psi(q^{64}) + \cdots), \end{aligned}$$

which completes the proof of the first result.

Similarly, by (4.20) and (4.21), we find that

$$\begin{aligned} \frac{1}{D(q)} &= \sum_{n=0}^{\infty} v_{2n} q^{2n} + \sum_{n=0}^{\infty} v_{2n+1} q^{2n+1} = \frac{\psi(q^8)}{\psi(q^4)} - q\frac{\psi(q^8)}{\psi(q^4)}\frac{1}{D(q^2)} \\ &= \frac{\psi(q^8)}{\psi(q^4)} - q\frac{\psi(q^{16})}{\psi(q^4)} + q^2\frac{\psi(q^{32})}{\psi(q^4)} - q^4\frac{\psi(q^{64})}{\psi(q^4)} + \cdots \\ &= \frac{1}{\psi(q^4)}(\psi(q^8) - q\psi(q^{16}) + q^2\psi(q^{32}) - q^4\psi(q^{64}) + \cdots), \end{aligned}$$

as desired. ∎

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