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AN EXISTENCE THEOREM FOR SYSTEMS OF IMPLICIT DIFFERENTIAL EQUATIONS

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This note is based on the thesis [1] of the first author written under the guidance of the second author. The main technical input is Theorem 6 below. It will be proved in more generality in the subsequent paper [4].

Let $f_1, \dots, f_l$ be differential polynomials in one derivative and N variables with coefficients in $\mathbb{R}$. Suppose $I \subseteq \mathbb{R}$ is an open interval and $c : I \to \mathbb{R}^N$ is a C^∞-map with $f_1(c(t)) = \ldots = f_l(c(t)) = 0$ $(t \in I)$. Let $\mathfrak{a}$ be the differential ideal generated by $f_1, \dots, f_l$ in the differential polynomial ring $\mathbb{R}\{X_1, \dots, X_N\}$. Then $\mathfrak{a}$ is certainly a semireal ideal, i.e. for all $g_1, \dots, g_m \in \mathbb{R}\{X_1, \dots, X_N\}$ we have $1 + \sum_{j=1}^m g_j^2 \notin \mathfrak{a}$. This follows immediately from our assumption that c is a differential solution of the generators $f_1, \dots, f_l$ of $\mathfrak{a}$. We'll prove here the converse of this observation, in other words we'll prove

Theorem 1. *If $\mathfrak{a}$ is a differential ideal of $\mathbb{R}\{X_1, \dots, X_N\}$ and $\mathfrak{a}$ is semireal, then there is some nonempty open interval $I \subseteq \mathbb{R}$ and an analytic map $c : I \to \mathbb{R}^N$ with $f(c(t)) = 0$ $(f \in \mathfrak{a},\ t \in I)$.*

In order to find an analytic map $c = (c_1, \dots, c_N) : I \to \mathbb{R}^N$ solving each relation $f = 0$ with $f \in \mathfrak{a}$ it is enough to find a nonempty open interval I of $\mathbb{R}$ together with a differential homomorphism $\mathbb{R}\{X_1, \dots, X_N\}/\mathfrak{a} \to C^\omega(I)$ and then take $c_i :=$ the image of $X_i \bmod \mathfrak{a}$ under this map. We divide this problem into an algebraic part (Theorem 2) and an analytic part (Proposition 3).

Theorem 2. *Let F be a differential field and let A be a differentially finitely generated F-algebra. Suppose A is semireal, i.e. -1 is not a sum of squares in A. There is a real, differential F-algebra C, which is an integral domain and finitely generated as an F-algebra together with a differential F-algebra homomorphism $A \to C$.*

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Proposition 3. *Let C be a real, differential $\mathbb{R}$-algebra, which is an integral domain and finitely generated as an $\mathbb{R}$-algebra. Then there is a differential $\mathbb{R}$-algebra homomorphism $C \to C^{\omega}(I)$ for some open interval $I \subseteq \mathbb{R}$.*

Clearly 1 follows from 2 and 3 applied to $A = \mathbb{R}\{X_1, \ldots, X_N\}/\mathfrak{a}$. Before we prove Theorem 2 and Proposition 3 we need some real algebraic preparations.

Definition 1. A ring A is called *semireal* if -1 is not a sum of squares in A. A is called *real* if $a_1^2 + \ldots + a_n^2 = 0$ implies $a_1 = \ldots = a_n = 0$ for all $n \in \mathbb{N}$ and all $a_1, \ldots, a_n \in A$. An ideal $\mathfrak{a}$ of A is called (*semi*) *real* if the ring $A/\mathfrak{a}$ is (semi) real.

Definition 2. Let A be a differential ring in K derivatives and let $\mathfrak{a}$ be an ideal of A. We define

$$\mathfrak{a}^{\#} := \{a \in \mathfrak{a} \mid \text{ every derivative of } a \text{ is in } \mathfrak{a}\}.$$

The useful construction $\mathfrak{a}^{\#}$ was first introduced by Keigher in [2]. Clearly $\mathfrak{a}^{\#}$ is the largest differential ideal of A contained in $\mathfrak{a}$. Let $\sigma : A \to B$ a ring homomorphism into a ring B. Let $B[[T]]$ be the power series ring over B in one variable T. $B[[T]]$ is a differential ring with the standard derivative $\frac{d}{dT}$. We define the *Taylor morphism* $T_\sigma : A \to B[[T]]$ by

$$T_\sigma(a) := \sum_{n \geq 0} \frac{\sigma(d^n a)}{n!} T^n.$$

Here $d^n a$ denotes the n-th derivative of $a \in A$.

The Leibniz rule implies that T_σ is a differential homomorphism. If $\sigma : A \to A/\mathfrak{a}$ is the residue map corresponding to an ideal $\mathfrak{a}$ of A, then clearly $\mathfrak{a}^{\#}$ is the kernel of T_σ.

Proposition 4. *Let $\mathfrak{a}$ be an ideal in the differential ring A. If $\mathfrak{a}$ is prime, semireal, real respectively, then $\mathfrak{a}^{\#}$ is prime, semireal, real respectively.*

Proof. If $\mathfrak{a}$ is prime, semireal, real respectively, then $A/\mathfrak{a}$ is a domain, semireal, real respectively. Hence the power series ring $A/\mathfrak{a}[[T]]$ is a domain, semireal, real respectively, and so $\mathfrak{a}^{\#} = \mathrm{Ker}(T_{A \to A/\mathfrak{a}})$ is prime, semireal, real respectively. ∎

Proposition 5. *Let A be a differential ring and let $\mathfrak{p} \subseteq A$ be a differential ideal. Then $\mathfrak{p}$ is maximal among the proper, semireal and differential ideals of A if and only if $\mathfrak{p}$ is maximal among the proper, real and differential ideals of A. In this case $\mathfrak{p}$ is prime.*

Proof. Let $\mathfrak{p}$ be maximal among all proper, semireal and differential ideals of A. The Proposition is proved if we can show that $\mathfrak{p}$ is real and prime. By classical real algebra (cf. [3], III, §3, Satz 2), there is a real prime ideal $\mathfrak{q}$ of A containing $\mathfrak{p}$. By Proposition 4, $\mathfrak{q}^{\#}$ is a real, differential prime ideal of A. Since $\mathfrak{q}^{\#}$ contains $\mathfrak{p}$, the maximality of $\mathfrak{p}$ implies $\mathfrak{p} = \mathfrak{q}^{\#}$, thus $\mathfrak{p}$ is real and prime. ∎

Finally we use a structure theorem for differential algebras (in one derivative), as explained in [4].

Theorem 6. *Let $S = (S, d)$ be a differential domain in one derivative, containing $\mathbb{Z}$, and let $R = (R, d) \subseteq (S, d)$ be a differential subring such that S is differentially finitely generated over R. Then there are R-subalgebras B and U of S and an element $h \in B$, $h \neq 0$ such that:*

(a) B is a finitely generated R-algebra and B_h is a finitely presented R-algebra.
(b) $S_h = (B{\cdot}U)_h$ is a differentially finitely presented R-algebra.
(c) The homomorphism $B \otimes_R U \to B{\cdot}U$ induced by multiplication is an isomorphism of R-algebras.
(d) U is a differential polynomial ring over R in finitely many variables.

Proof. This is Theorem 1 in [4] for the case of one derivative. Take $U := P_{\{d\}}$ and replace B by $B{\cdot}P_\emptyset$ in [4], Theorem 1. ∎

Proof of Theorem 2. Since A is semireal, A contains an ideal $\mathfrak{p}$, which is maximal among all proper, semireal and differential ideals of A. By Proposition 5, $\mathfrak{p}$ is a real, differential prime ideal. Let S be the differential F-algebra $S := A/\mathfrak{p}$. Take F-subalgebras B, U of S and an element $h \in B$, $h \neq 0$ as in Theorem 6. Since S is real, B and B_h are real, too. It is enough to show that $U = F$, then the differential map $A \to A/\mathfrak{p} = S \hookrightarrow S_h = B_h =: C$ has the required properties. Suppose $U \neq F$. Since B_h is a finitely generated, real F-algebra, Tarski's principle gives a homomorphism $\varphi : B_h \to \overline{F}$ into a real closed field $\overline{F}$ containing F. Since $U \neq F$ is a differential polynomial ring, there is a differential F-algebra homomorphism $\tau : U \to F$ with nontrivial kernel. By Theorem 6, there is an F-algebra homomorphism $\sigma : S \to \overline{F}$, extending $\varphi|_B$ and τ. Thus $\mathfrak{q} := \operatorname{Ker}\sigma$ is a real ideal of S containing $\operatorname{Ker}\tau$. By Proposition 4, $\mathfrak{q}^\#$ is a real, differential ideal of S. Since τ is a differential homomorphism, $\mathfrak{q}^\#$ contains $\operatorname{Ker}\tau$, hence $\mathfrak{q}^\#$ is a nontrivial, real, differential ideal of S, which contradicts the maximality of $\mathfrak{p}$. ∎

Proof of Proposition 3. Let $C = \mathbb{R}[a_1, \ldots, a_n]$ and let $g_i \in \mathbb{R}[X_1, \ldots, X_n]$ such that $g_i(a)$ is the derivative of a_i in C. We consider the ring $\mathbb{R}[X_1, \ldots, X_n]$ as a differential ring with derivation $d : \mathbb{R}[X_1, \ldots, X_n] \to \mathbb{R}[X_1, \ldots, X_n]$ defined by $dX_i = g_i$. Then the homomorphism $\lambda : \mathbb{R}[X_1, \ldots, X_n] \to C$ sending X_i to a_i is differential. Since C is a real, finitely generated $\mathbb{R}$-algebra, there is an $\mathbb{R}$-algebra homomorphism $\varepsilon : C \to \mathbb{R}$. The fundamental theorem on ordinary differential equations gives an open interval I of $\mathbb{R}$ containing 0 and analytic maps $c_i : I \to \mathbb{R}$ $(1 \leq i \leq n)$ such that $c_i(0) = \varepsilon(a_i)$ and

$$c_i'(t) = g_i(c_1(t), \ldots, c_n(t)) \ (1 \leq i \leq n).$$

Now a straightforward computation shows that the Taylor morphism T_ε of $\varepsilon : C \to \mathbb{R}$ maps a_i to the Taylor expansion of c_i at 0. By shrinking I if necessary, we get that T_ε has values in $C^\omega(I)$, which proves Proposition 3. ∎

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