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FREE CUMULANTS OF SOME PROBABILITY MEASURES

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1. Preliminaries. Let X be a finite, linearly ordered set. By a *noncrossing partition* of X we will mean a collection π of nonempty, pairwise disjoint subsets (called *blocks of* π) such that $\bigcup\pi = X$, which satisfies the following condition: if $x_1 < x_2 < x_3 < x_4$, with $x_1, x_3 \in U_1 \in \pi$ and $x_2, x_4 \in U_2 \in \pi$, then $U_1 = U_2$. The class of all noncrossing partitions of X will be denoted by $\mathrm{NC}(X)$. We also define $\mathrm{NC}_{1,2}(X)$ as the family of those $\sigma \in \mathrm{NC}(X)$ for which every block has at most 2 elements. We will write $\mathrm{NC}(m)$ and $\mathrm{NC}_{1,2}(m)$ instead of $\mathrm{NC}(\{1,2,\ldots,m\})$ and $\mathrm{NC}_{1,2}(\{1,2,\ldots,m\})$.

Every $\pi \in \mathrm{NC}(X)$ admits a natural partial order. Namely, for $U, V \in \pi$ we write $U \prec V$ whenever there are $r, s \in V$ such that $r < k < s$ holds for every $k \in U$. We define the *depth* of a block as $d(U,\pi) := |\{V \in \pi : U \prec V\}|$. A block is called *outer* if $d(U,\pi) = 0$, otherwise it is called *inner*. Note that for every inner block $U \in \pi$ there is a unique block in π, denoted by U', such that $U \prec U'$ and $d(U,\pi) = d(U',\pi) + 1$. We also define derivatives of higher orders by $V^{(0)} := V$ and $V^{(k)} := \left(V^{(k-1)}\right)'$.

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Let μ be a compactly supported probability measure on the real line, with the moment sequence

$$s_m(\mu) := \int_{t\in\mathbb{R}} t^m d\mu(t). \tag{1}$$

Then there is a unique sequence $\{P_m(x)\}_{m=0}^{\infty}$ of monic polynomials, with $\deg P_m = m$, which are orthogonal with respect to μ. It is known that they satisfy the recurrence relation: $P_0(x) = 1$ and for $m \geq 0$

$$xP_m(x) = P_{m+1}(x) + \beta_m P_m(x) + \gamma_{m-1}P_{m-1}(x), \tag{2}$$

under convention that $P_{-1} = 0$, where the *Jacobi coefficients* satisfy: $\beta_m \in \mathbb{R}$, $\gamma_m \geq 0$ and if $\gamma_m = 0$ for some m then $\gamma_n = \beta_n = 0$ for every $n > m$ (see [Ch]). These coefficients show up in the continued fraction expansion of the Cauchy transform of μ, namely:

$$G_\mu(z) := \int_{t\in\mathbb{R}} \frac{d\mu(t)}{t-z} = \cfrac{1}{z-\beta_0-\cfrac{\gamma_0}{z-\beta_1-\cfrac{\gamma_1}{z-\beta_2-\cfrac{\gamma_2}{z-\beta_3-\cfrac{\gamma_3}{\ddots}}}}}. \tag{3}$$

There is a combinatorial formula, due to Accardi and Bożejko, connecting moments and the Jacobi coefficients of μ, namely

$$s_m(\mu) = \sum_{\sigma\in\mathrm{NC}_{1,2}(m)} \prod_{\substack{V\in\sigma\\ |V|=1}} \beta_{d(V,\sigma)} \prod_{\substack{V\in\sigma\\ |V|=2}} \gamma_{d(V,\sigma)}. \tag{4}$$

Another important numbers related to μ are the *free cumulants* $r_m(\mu)$, $m \geq 1$ (see [S1, S2]), which are defined by:

$$s_m(\mu) = \sum_{\pi\in\mathrm{NC}(m)} \prod_{U\in\pi} r_{|U|}(\mu). \tag{5}$$

Their generating function

$$R_\mu(z) := \sum_{k=0}^{\infty} r_{k+1}(\mu) z^k \tag{6}$$

is called the R-transform. These two functions are related by

$$\frac{1}{G_\mu(z)} = z - R_\mu(G_\mu(z)). \tag{7}$$

From now on we will confine ourselves to a special class of measures μ. For $a > 0$, $b \geq 0$ and $u, v \in \mathbb{R}$ we define $\mu(a, b, u, v)$ as the unique $\mu \in \mathcal{M}$ such that its Jacobi coefficients are:

$$\gamma_m = \begin{cases} a \text{ if } m = 0, \\ b \text{ if } m \geq 1, \end{cases} \tag{8}$$

$$\beta_m = \begin{cases} u \text{ if } m = 0, \\ v \text{ if } m = 1, \\ v \text{ if } m > 1 \text{ and } b > 0, \\ 0 \text{ if } m > 1 \text{ and } b = 0. \end{cases} \tag{9}$$

This class of measures was first studied by Cohen and Trenholme [CT] from the point of view of harmonic analysis. Then Saitoh and Yoshida studied them from the point of view of free probability. For $\mu = \mu(a, b, u, v)$ they calculated the Cauchy transform:

$$G_\mu(z) = \frac{2b(z-u) - a(z-v) - a\sqrt{(z-v)^2 - 4b}}{2b(z-u)^2 - 2a(z-u)(z-v) + 2a^2} \tag{10}$$

and the R-transform:

$$R_\mu(w) = u + \frac{aw}{1 + (u-v)w}. \tag{11}$$

if $a = b$ and

$$R_\mu(w) = u + \frac{a}{b-a}\frac{1 - (v-u)w + \sqrt{((v-u)w-1)^2 - 4(b-a)w^2}}{2w} \tag{12}$$

otherwise. For $a \le b$ the authors of [SY] found the Lévy-Khinchin formula:

$$R_\mu(z) = u + a\int_{\mathbb{R}} \frac{z}{1-tz}d\nu(t), \tag{13}$$

where if $a = b$ then $\nu = \delta_{v-u}$ and if $a < b$ then ν is absolutely continuous with density

$$\frac{1}{2\pi(b-a)}\sqrt{4(b-a) - \big(t-(v-u)\big)^2} \tag{14}$$

on the interval $\big(t-(v-u)\big)^2 \le 4(b-a)$. Basing on this they proved that $\mu(a, b, u, v)$ is infinitely divisible with respect to the free convolution if and only if $a \le b$. Bożejko and Bryc [BB] observed that one can use (13) to find the free cumulants of $\mu(a, b, u, v)$ when $a \le b$. Now, since every free cumulant $r_m(\mu(a, b, u, v))$ is a polynomial in a, b, u, v, the resulting formula holds for all $\mu(a, b, u, v)$. The aim of this paper is to find the free cumulants of $\mu(a, b, u, v)$ in a purely combinatorial way.

2. The result.

Now we are ready to state the result.

THEOREM. *For the free cumulants $r_m := r_m(\mu(a, b, u, v))$ we have: $r_1 = u$ and for $n \ge 1$*

$$r_{2n} = a\sum_{k=1}^{n} \frac{(2n-2)!}{(2k-2)!(n-k)!(n-k+1)!}(b-a)^{n-k}(v-u)^{2k-2}, \tag{15}$$

$$r_{2n+1} = a\sum_{k=1}^{n} \frac{(2n-1)!}{(2k-1)!(n-k)!(n-k+1)!}(b-a)^{n-k}(v-u)^{2k-1}. \tag{16}$$

Proof. Putting $s_m := s_m(\mu(a, b, u, v))$ we have from the Bożejko-Accardi formula:

$$s_m = \sum_{\sigma\in \mathrm{NC}_{1,2}(m)} a^{\mathrm{out}_2(\sigma)}u^{\mathrm{out}_1(\sigma)}b^{\mathrm{inn}_2(\sigma)}v^{\mathrm{inn}_1(\sigma)}, \tag{17}$$

where $\mathrm{out}_2(\sigma)$, $\mathrm{out}_1(\sigma)$, $\mathrm{inn}_2(\sigma)$, $\mathrm{inn}_1(\sigma)$ denotes the number of outer or inner blocks $V \in \sigma$, with $|V| = 2$ or $|V| = 1$, respectively. For fixed $\sigma \in \mathrm{NC}_{1,2}(m)$ the related summand can be written as

$$a^{\mathrm{out}_2(\sigma)}u^{\mathrm{out}_1(\sigma)}\big((b-a)+a\big)^{\mathrm{inn}_2(\sigma)}\big((v-u)+u\big)^{\mathrm{inn}_1(\sigma)}. \tag{18}$$

After expanding we get a sum of products of factors of the form $a, u, (b-a), (v-u)$, which can be described in terms of *signed noncrossing partitions*. By a *signing* of $\sigma \in \mathrm{NC}_{1,2}(m)$ we will mean a function $\epsilon : \sigma \to \{0,1\}$ such that $\epsilon(V) = 0$ whenever $V \in \sigma$ is an outer block. We will denote by $\mathrm{Sign}(\sigma)$ the family of all signings of σ. Now we define the *weight* of a signed block:

$$w(V,\epsilon) := \begin{cases} u & \text{if } |V| = 1 \text{ and } \epsilon(V) = 0,\\ v-u & \text{if } |V| = 1 \text{ and } \epsilon(V) = 1,\\ a & \text{if } |V| = 2 \text{ and } \epsilon(V) = 0,\\ b-a & \text{if } |V| = 2 \text{ and } \epsilon(V) = 1, \end{cases} \tag{19}$$

and the weight of a signed partition:

$$w(\sigma,\epsilon) := \prod_{V\in\sigma} w(V,\epsilon). \tag{20}$$

Then the expansion of the product (18) can be written as

$$\sum_{\epsilon\in\mathrm{Sign}(\sigma)} w(\sigma,\epsilon). \tag{21}$$

Now, for a fixed pair (σ,ϵ), with $\sigma \in \mathrm{NC}_{1,2}(m)$, $\epsilon \in \mathrm{Sign}(\sigma)$ we define a partition $\Pi(\sigma,\epsilon)$ by gluing a block V with V' whenever $\epsilon(V) = 1$. More precisely, define a relation $\mathcal{R}$ on σ: $U\mathcal{R}V$ iff $V = U'$ and $\epsilon(U) = 1$. Let $\sim$ be the smallest equivalence relation on σ containing $\mathcal{R}$. Then we define a partition $\Pi(\sigma,\epsilon)$ of $\{1,2,\ldots,m\}$ whose blocks are of the form $\bigcup\mathcal{C}$, with $\mathcal{C} \in \sigma/\sim$. This means that k and l are in the same block of $\Pi(\sigma,\epsilon)$ if and only if there are blocks $U, V \in \sigma$, with $k \in U$, $l \in V$, and numbers $r, s \geq 0$ such that

$$\epsilon(U) = \epsilon(U') = \cdots = \epsilon(U^{(r-1)}) = 1,$$
$$\epsilon(V) = \epsilon(V') = \cdots = \epsilon(V^{(s-1)}) = 1$$

and $U^{(r)} = V^{(s)}$. It is easy to see that $\Pi(\sigma,\epsilon)$ is noncrossing.

On the other hand, for fixed $\pi \in \mathrm{NC}(m)$, we have $\Pi(\sigma,\epsilon) = \pi$ if and only if σ is finer than π (i.e. for every $V \in \sigma$ there is $U \in \pi$ such that $V \subseteq U$) and for every $U \in \pi$, if $U = \{k_1, k_2, \ldots, k_r\}$, $k_1 < k_2 < \cdots < k_r$, then we have $\{k_1, k_r\} \in \sigma$, with sign 0, and all the blocks of σ which are contained in $\{k_2, \ldots, k_{r-1}\}$ have sign 1. In particular, every one-element block $U \in \pi$ must be a block of σ with sign 0. Therefore the sum of weights $w(\sigma,\epsilon)$ with $\Pi(\sigma,\epsilon) = \pi$ is equal to

$$\sum_{\substack{\sigma\in\mathrm{NC}_{1,2}(m),\\ \epsilon\in\mathrm{Sign}(\sigma)\\ \Pi(\sigma,\epsilon)=\pi}} w(\sigma,\epsilon) = u^{N_1(\pi)} \prod_{\substack{U\in\pi\\ |U|>1}} \sum_{\sigma_U\in\mathrm{NC}_{1,2}(|U|-2)} a(v-u)^{N_1(\sigma_U)}(b-a)^{N_2(\sigma_U)}, \tag{22}$$

where $N_1(\sigma)$, $N_2(\sigma)$ denotes the number of one- and two-element blocks in σ. Hence

$$\begin{aligned} s_m &= \sum_{\sigma\in\mathrm{NC}_{1,2}(\mathrm{m})} a^{\mathrm{out}_2(\sigma)}u^{\mathrm{out}_1(\sigma)}b^{\mathrm{inn}_2(\sigma)}v^{\mathrm{inn}_1(\sigma)}\\ &= \sum_{\sigma\in\mathrm{NC}_{1,2}(\mathrm{m})} a^{\mathrm{out}_2(\sigma)}u^{\mathrm{out}_1(\sigma)}\big((b-a)+a\big)^{\mathrm{inn}_2(\sigma)}\big((v-u)+u\big)^{\mathrm{inn}_1(\sigma)} \end{aligned}$$

$$= \sum_{\pi\in \mathrm{NC(m)}} u^{N_1(\pi)} \prod_{\substack{U\in\pi\\|U|>1}} \sum_{\tau\in \mathrm{NC}_{1,2}(|U|-2)} a(v-u)^{N_1(\tau)}(b-a)^{N_2(\tau)}$$
$$= \sum_{\pi\in \mathrm{NC}(m)} \prod_{U\in\pi} c_{|U|},$$

where $c_1 = u$ and for $m \ge 2$

$$c_m = a \cdot \sum_{\tau\in \mathrm{NC}_{1,2}(m-2)} (v-u)^{N_1(\tau)}(b-a)^{N_2(\tau)}. \tag{23}$$

Since the numbers c_m satisfy the same recurrence relation (5) as r_m, we have $r_m = s_m$ for every $m \ge 1$. It is well known that the number of those $\pi \in \mathrm{NC}(2j)$ for which $|V| = 2$ for every $V \in \pi$ is equal to the Catalan number $\frac{1}{j+1}\binom{2j}{j}$. Hence the number of partitions $\sigma \in \mathrm{NC}_{1,2}(m)$ with i blocks of order one and j blocks of order two, $i + 2j = m$, is equal to $\binom{m}{i}\frac{1}{j+1}\binom{2j}{j}$, which leads to the coefficients in (10) and (11). ■

Observe that the subclass $\{\mu(a,0,u,v) : a > 0, u,v \in \mathbb{R}\}$ coincides with the family of two-point probability measures on $\mathbb{R}$, namely:

$$\mu(a,0,u,v) = p_-\delta_{x_-} + p_+\delta_{x_+}, \tag{24}$$

where

$$p_\pm = \frac{\sqrt{(u-v)^2+4a} \pm (u-v)}{2\sqrt{(u-v)^2+4a}}, \tag{25}$$
$$x_\pm = \frac{u+v \pm \sqrt{(u-v)^2+4a}}{2}. \tag{26}$$

and, on the other hand,

$$a = p_+p_-(x_+ - x_-)^2, \tag{27}$$
$$u = p_+x_+ + p_-x_-, \tag{28}$$
$$v = p_+x_- + p_-x_+. \tag{29}$$

Corollary. *For $a, b > 0$, $u, v \in \mathbb{R}$ and $t \ge 0$ the free power $\mu(a,b,u,v)^{\boxplus t}$ exists if and only if $b - a + ta \ge 0$ and then*

$$\mu(a,b,u,v)^{\boxplus t} = \mu(ta, b-a+ta, tu, v-u+tu). \tag{30}$$

In particular, $\mu(a,b,u,v)$ is infinitely divisible if and only if $a \le b$.

If $0 \le b < a$ then $\mu(a,b,u,v)$ is a free power of a two point measure:

$$\mu(a,b,u,v) = \mu\left(a-b, 0, \frac{a-b}{a}u, v-u+\frac{a-b}{a}u\right)^{\boxplus \frac{a}{a-b}}. \tag{31}$$

Proof. The first statement holds because one can see from the theorem that for every $m \ge 1$

$$r_m(\mu(ta, b-a+ta, tu, v-u+tu)) = t \cdot r_m(\mu(a,b,u,v)). \tag{32}$$

The rest is an obvious consequence. ■

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