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WHITE NOISE DISTRIBUTION THEORY AND ITS APPLICATION

YOSHIHITO SHIMADA

Graduate School of Mathematics, Kyushu University
1-10-6 Hakozaki, Fukuoka 812-8581, Japan
E-mail: Shimada@math.kyushu-u.ac.jp

Abstract. The paper gives a new application of the white noise distribution theory via a proof of irreducibility of the energy representation of a group of C^∞-maps from a compact Riemann manifold to a semi-simple compact Lie group.

1. Introduction. This article is an exposition of the paper [12]. The purpose of this paper is to illustrate the usefulness of the white noise distribution theory via an application. Our white noise distribution theory is the theory for functionals $\mathcal{E}$ and distributions $\mathcal{E}^*$ on the white noise space, and (continuous) linear operators from $\mathcal{E}$ to $\mathcal{E}^*$.

From the viewpoint of quantum physics, the gauge groups and their unitary representations on the Fock space are very important objects. Here, the gauge group is a group of all C^∞-maps from a compact Riemann manifold M to a semi-simple compact Lie group G. For a 1-dimensional manifold M, for example $M = \mathbf{T}^1$, we already know some unitary representations defined on the Fock space. The first example is given by implementers of Bogoliubov automorphisms. The highest weight representation and the our object called the energy (or basic) representation are also known as examples of such representations. However, if $\dim M \geq 2$, we do not yet know unitary representations defined on the Fock space except for the energy representation. Our interest in this paper is to find whether the energy representation is irreducible or not. We solve this problem by using the white noise distribution theory.

This paper is organized as follows. In Section 2, we summarize only necessary points of the white noise distribution theory. In Section 3, the gauge group and the energy representation are defined and irreducibility of the energy representation is stated. In Section 4, we prove our main theorem by using the tool introduced in Section 2.

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2. The white noise theory. We give an outline of the white noise distribution theory. The details are in [9] and [10]. First of all, let us introduce the test function space E and the white noise space E^*.

DEFINITION 2.1. Let H be a complex Hilbert space with an inner product $\langle\cdot,\cdot\rangle_0$. Let A be a self-adjoint operator such that $A^{-\alpha}$ is a Hilbert-Schmidt class operator for some $\alpha > 0$.

Let $\langle x, y\rangle_p := \langle A^p x, A^p y\rangle_0$ for $p \in \mathbf{R}$. $E_p := (\mathrm{Dom}(A^p), |\cdot|_p)$ is a Sobolev space for each $p > 0$. On the other hand, if $p < 0$, we put $E_p := (\overline{H}, |\cdot|_p)$, here $\overline{H}$ is the closure of H with respect to the norm $|\cdot|_p$. Next,

$$E := \lim_{\leftarrow} E_p = \bigcap_{p\geq 0} E_p, \quad E^* = \lim_{\rightarrow} E_{-p} = \bigcup_{p\geq 0} E_{-p}.$$

Then we call E a *test function space*. E becomes a nuclear space from the Hilbert-Schmidt condition of $A^{-\alpha}$. We call the triple $E \subset H \subset E^*$ a *Gelfand triple*.

EXAMPLE: If $H = L^2(\mathbf{R}, dx)$ and $A = -(d/dx)^2 + x^2 + 1$, then E coincides with all rapidly decreasing functions on $\mathbf{R}$ as a topological vector space. Obviously E^* is the topological vector space of all tempered distributions on $\mathbf{R}$. Here is another example: If $H = L^2(\mathbf{T}^1, dx)$ and $A = -(d/dx)^2 + 2$, then E coincides with $C^\infty(\mathbf{T}^1, \mathbf{C}^1)$ as a topological vector space.

We denote the canonical bilinear form on $E^* \times E$ by $\langle\cdot,\cdot\rangle$. Then we have the following natural relation between the canonical bilinear form on $E^* \times E$ and the inner product on H:

$$\langle f, g\rangle = \langle \overline{f}, g\rangle_0$$

for all $f \in H$ and $g \in E$. $\bar{f}$ stands for the complex conjugate of f.

Next we define the Boson Fock space and the second quantization of a linear operator.

DEFINITION 2.2. For $g_1, \dots, g_n \in H$, we denote the symmetrization of $g_1 \otimes \dots \otimes g_n \in H^{\otimes n}$ by

$$g_1 \widehat{\otimes} \dots \widehat{\otimes} g_n := \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} g_{\sigma(1)} \otimes \dots \otimes g_{\sigma(n)},$$

where $\mathfrak{S}_n$ is the set of all permutations of $\{1, 2, \dots, n\}$.

Let H be a Hilbert space. The Hilbert space

$$\Gamma_{\mathrm{b}}(H) := \Big\{ \sum_{n=0}^{\infty} f_n \,|\, f_n \in H^{\widehat{\otimes} n}, \Big\| \sum_{n=0}^{\infty} f_n \Big\|_0 < +\infty \Big\},$$

$$\Big\langle\!\Big\langle \sum_{n=0}^{\infty} f_n, \sum_{n=0}^{\infty} g_n \Big\rangle\!\Big\rangle_0 := \sum_{n \in \mathbf{Z}_{\geq 0}} n! \langle f_n, g_n\rangle_0.$$

is called a *Boson Fock space*.

We define a useful tool for an analysis of a Boson Fock space.

Definition 2.3. For $f \in H$,

$$\exp(f) := \sum_{n=0}^{\infty} \frac{1}{n!} f^{\otimes n} \in \Gamma_{\mathrm{b}}(H)$$

is called an *exponential* (or *coherent*) *vector*.

The following lemma is well-known.

Lemma 2.4. $\{\exp(f) \,|\, f \in H\}$ *spans a dense subspace of* $\Gamma_{\mathrm{b}}(H)$.

Definition 2.5. Let X, Y be locally convex spaces. $\mathcal{L}(X, Y)$ is the set of all continuous linear operators from X to Y.

Definition 2.6. Let A be a linear operator on H (or E). Then

$$\Gamma_{\mathrm{b}}(A) := \sum_{n=0}^{\infty} A^{\otimes n}$$

is called the *second quantization* of A. Moreover, put

$$d\Gamma^{(n)}(A) := \sum_{j=1}^{n} \mathrm{id}^{\otimes(j-1)} \otimes A \otimes \mathrm{id}^{\otimes(n-j)}, \qquad d\Gamma_{\mathrm{b}}(A) := \sum_{n=0}^{\infty} d\Gamma^{(n)}(A).$$

Then $d\Gamma_{\mathrm{b}}(A)$ is called the *differential second quantization* of A.

Definition 2.7. Let H be a complex Hilbert space and A be a self-adjoint operator on H given in Definition 2.1. Then we can obtain a Gelfand triple

$$\mathcal{E} \subset \Gamma_{\mathrm{b}}(H) \subset \mathcal{E}^*$$

constructed from $(\Gamma_{\mathrm{b}}(H), \Gamma_{\mathrm{b}}(A))$. We call $\mathcal{E}$ the space of *white noise functionals* and $\mathcal{E}^*$ the space of *generalized white noise functionals*.

Corollary 2.8.

(1) Let $\phi := \sum_{n=0}^{\infty} f_n \in \Gamma_{\mathrm{b}}(H)$, $f_n \in H^{\widehat{\otimes} n}$. *Then* $\phi \in \mathcal{E}$ *if and only if* $f_n \in E^{\widehat{\otimes} n}$ *for all* $n \geq 0$ *and* $\|\phi\|_p < +\infty$ *for all* $p \geq 0$.

(2) Moreover, $\{\exp(f) \,|\, f \in E\}$ *spans a dense subspace of* $\mathcal{E}$.

The following operators are also well-known.

Definition 2.9. Let $f_i \in E$ and $y \in E^*$.

(1) Let

$$a(y)(f_1 \widehat{\otimes} \dots \widehat{\otimes} f_n) := n \sum_{j=1}^{n} \langle y, f_j \rangle f_1 \widehat{\otimes} \dots \widehat{\otimes} f_{j-1} \widehat{\otimes} f_{j+1} \widehat{\otimes} \dots \widehat{\otimes} f_n.$$

$a(y) \in \mathcal{L}(\mathcal{E}, \mathcal{E})$ is called an *annihilation operator*.

(2) Let

$$a^{\dagger}(y)(f_1 \widehat{\otimes} \dots \widehat{\otimes} f_n) := y \widehat{\otimes} f_1 \widehat{\otimes} \dots \widehat{\otimes} f_n.$$

$a^{\dagger}(y)$ is called a *creation operator*. $a^{\dagger}(y)$ is in $\mathcal{L}(\mathcal{E}, \mathcal{E})$ if $y \in E$, and $a^{\dagger}(y)$ is in $\mathcal{L}(\mathcal{E}^*, \mathcal{E}^*)$ if $y \in E^*$.

To define an integral kernel operator, we need a contraction of tensor products.

DEFINITION 2.10. Let H be a complex Hilbert space and A be a self-adjoint operator on H given in Definition 2.1. Let $\{e_j\}_{j=1}^{\infty}$ be a C.O.N.S. of H consisting of normalized eigenvectors of A. Let

$$e(\mathbf{i}) := e_{i_1} \otimes \ldots \otimes e_{i_l}, \quad \mathbf{i} := (i_1, \ldots, i_l) \in \mathbf{N}^l.$$

(1) For $F \in (E^{\otimes(l+m)})^*$, let

$$|F|_{l,m;p,q}^2 := \sum_{\mathbf{i},\mathbf{j}} |\langle F, e(\mathbf{i}) \otimes e(\mathbf{j})\rangle|^2 \, |e(\mathbf{i})|_p^2 \, |e(\mathbf{j})|_q^2$$

where $\mathbf{i}$ and $\mathbf{j}$ run over the whole $\mathbf{N}^l$ and $\mathbf{N}^m$ respectively.

(2) For $F \in \left(E^{\otimes(l+m)}\right)^*$ and $g \in E^{\otimes(l+n)}$, we define a contraction $F \otimes_l g \in \left(E^{m+n}\right)^*$ of F and g as follows:

$$F \otimes_l g := \sum_{\mathbf{j},\mathbf{k}} \Big(\sum_{\mathbf{i}} \langle F, e(\mathbf{j}) \otimes e(\mathbf{i})\rangle \, \langle g, e(\mathbf{k}) \otimes e(\mathbf{i})\rangle \Big) e(\mathbf{j}) \otimes e(\mathbf{k})$$

where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ run over the whole $\mathbf{N}^l$, $\mathbf{N}^m$, and $\mathbf{N}^n$ respectively.

DEFINITION 2.11. Let $\kappa \in (E^{\otimes(l+m)})^*$ and

$$\Xi_{l,m}(\kappa)\phi := \sum_{n=0}^{\infty} \frac{(n+m)!}{n!} s_{l+n}(\kappa \otimes_m f_{m+n})$$

for $\phi := \sum_{n=0}^{\infty} f_n \in \mathcal{E}$, $f_n \in E^{\widehat{\otimes} n}$. Then $\Xi_{l,m}(\kappa) \in \mathcal{L}(\mathcal{E}, \mathcal{E}^*)$. We call $\Xi_{l,m}(\kappa)$ an *integral kernel operator* with a kernel distribution κ.

REMARK 2.12. We list some properties of integral kernel operators.

(1) If $H = L^2(\mathbf{R})$, $\Xi_{l,m}(\kappa)$ has the following formal expression:

$$\Xi_{l,m}(\kappa) = \int_{s_i, t_j \in \mathbf{R}} \kappa(s_1, \ldots, s_l, t_1 \ldots, t_m) \\ a^\dagger(\delta_{s_1}) \ldots a^\dagger(\delta_{s_l}) a(\delta_{t_1}) \ldots a(\delta_{t_m}) ds_1 \ldots ds_l dt_1 \ldots dt_m,$$

where δ_s is the delta function for $s \in \mathbf{R}$. From this expression, it is entirely fair to call $\Xi_{l,m}(\kappa)$ an integral kernel operator.

(2) In addition, we should not overlook uniqueness of integral kernel operators. From the formal expression given in (1), it is obvious that $\Xi_{l,m}(\kappa_{l,m}) = \Xi_{l,m}(s_{l,m}(\kappa_{l,m}))$ for $\kappa_{l,m} \in (E^{\otimes(l+m)})^*$. Here

$$s_{l,m}(f \otimes g) := \frac{1}{l!m!} \sum_{\sigma \in \mathfrak{S}_l, \tau \in \mathfrak{S}_m} \sigma(f) \otimes \tau(g), \quad f \in E^{\otimes l}, \; g \in E^{\otimes m}.$$

(3) For $\Xi_{l,m}(\kappa_{l,m}) \in \mathcal{L}(\mathcal{E}, \mathcal{E}^*)$,

$$\Xi_{l,m}(\kappa_{l,m})^* = \Xi_{m,l}(t_{m,l}(\kappa_{l,m})),$$

where the map $t_{m,l}$ is defined by

$$\langle t_{m,l}(\kappa_{l,m}), \eta \otimes \zeta\rangle := \langle \kappa_{l,m}, \zeta \otimes \eta\rangle, \quad \eta \in E^{\otimes m}, \, \xi \in E^{\otimes l}.$$

Moreover, we have

PROPOSITION 2.13. *For* $\Xi_{l,m}(\kappa) \in \mathcal{L}(\mathcal{E},\mathcal{E}^*)$ *and* $\Xi_{l',m'}(\lambda) \in \mathcal{L}(\mathcal{E},\mathcal{E})$,

$$\Xi_{l,m}(\kappa)\Xi_{l',m'}(\lambda) = \sum_{k=0}^{\min\{m,l'\}} k!\binom{m}{k}\binom{l'}{k}\Xi_{l+l'-k,m+m'-k}(S^{l}_{m-k}{}^{l'-k}_{m'}(\kappa \circ_k \lambda)).$$

Here we used

DEFINITION 2.14. Let $\kappa \in (E^{\otimes(l+m)})^*$, $\lambda \in E^{\otimes l'} \otimes (E^{\otimes m'})^*$. For $0 \le k \le \min\{m, l'\}$, we define $S^{l}_{m-k}{}^{l'-k}_{m'}(\kappa \circ_k \lambda) \in \otimes(E^{\otimes(l+l'+m+m'-2k)})^*$ as follows.

$$\begin{aligned} S^{l}_{m-k}{}^{l'-k}_{m'}(\kappa \circ_k \lambda) := \sum_{\mathbf{i},\mathbf{j},\mathbf{i}',\mathbf{j}'} \sum_{\mathbf{h}} &\langle \kappa, e(\mathbf{i}) \otimes e(\mathbf{j}) \otimes e(\mathbf{h})\rangle \\ &\times \langle \lambda, e(\mathbf{h}) \otimes e(\mathbf{i}') \otimes e(\mathbf{j}')\rangle\, e(\mathbf{i}) \otimes e(\mathbf{i}') \otimes e(\mathbf{j}) \otimes e(\mathbf{j}'), \end{aligned}$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{i}'$, $\mathbf{j}'$, and $\mathbf{h}$ run over the whole $\mathbf{N}^l$, $\mathbf{N}^{m-k}$, $\mathbf{N}^{l'-k}$, $\mathbf{N}^{m'}$, and $\mathbf{N}^k$ respectively.

The following criterion for continuous linear operators on (generalized) white noise functionals plays a crucial role in our work.

PROPOSITION 2.15 (Fock expansion). *For any* $\Xi \in \mathcal{L}(\mathcal{E},\mathcal{E}^*)$, *there exists a unique* $\{\kappa_{l,m}\}_{l,m=0}^{\infty}$, $\kappa_{l,m} \in (E^{\otimes(l+m)})^*_{\mathrm{sym}(l,m)}$ *such that*

$$\Xi\phi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m})\phi, \quad \phi \in \mathcal{E}, \tag{2.1}$$

where the sum on the right hand side of (2.1) *converges in* $\mathcal{E}^*$, *and*

$$(E^{\otimes(l+m)})^*_{\mathrm{sym}(l,m)} := \{\kappa \in (E^{\otimes(l+m)})^* \mid s_{l,m}(\kappa) = \kappa\}.$$

Moreover, if $\Xi \in \mathcal{L}(\mathcal{E},\mathcal{E})$, *then*

$$\kappa_{l,m} \in E^{\widehat{\otimes} l} \otimes (E^{\widehat{\otimes} m})^*, \quad l, m \ge 0$$

and the sum on the right hand side of (2.1) *converges in* $\mathcal{E}$.

Roughly, the Fock expansion is the Taylor expansion for continuous linear operators on (generalized) white noise functionals. In fact, to prove the Fock expansion, we use the Taylor expansion for complex analytic functions characterizing continuous linear operators.

3. Representations of the gauge group. Let G be a Lie group and $\mathfrak{g}$ be the Lie algebra of G, and $\mathfrak{g}^{\mathrm{c}}$ be the complexification of $\mathfrak{g}$. Let Ad be the adjoint representation of G. For the complexification of Ad, we use the same notation.

Now let G be a semi-simple compact Lie group. Let $(\cdot,\cdot)_{\mathfrak{g}}$ be the inner product on $\mathfrak{g}^{\mathrm{c}}$ determined by the Killing form of $\mathfrak{g}$. Then the representation $(\mathfrak{g}^{\mathrm{c}}, \mathrm{Ad})$ of G is a unitary representation with respect to the inner product $(\cdot,\cdot)_{\mathfrak{g}}$.

For X, $Y \in \mathfrak{g}$, let

$$\mathrm{ad}(X)Y := [X, Y]$$

where $[X, Y]$ be the Lie bracket of $\mathfrak{g}$. Then $\mathrm{ad}(X)$ is a representation of the Lie algebra $\mathfrak{g}$ on the vector space $\mathfrak{g}$.

Next, we define a gauge group and its representation. Let M be a compact Riemann manifold and $(\cdot,\cdot)_x$ be the inner product on T_x^*M determined by the Riemannian structure of M.

Let $C^\infty(M,G)$ be the set of all C^∞-maps from M to G. We call $C^\infty(M,G)$ a gauge group. Let $C^\infty(M,\mathfrak{g})$ be the set of all C^∞-maps from M to $\mathfrak{g}$. This is the "Lie algebra" of $C^\infty(M,G)$.

Let $\Omega^1(M)$ be the space of real-valued 1-forms on M and $\Omega^1(M,\mathfrak{g}) := \Omega^1(M)\otimes\mathfrak{g}^{\mathrm{c}}$. We can define a natural inner product on $\Omega^1(M,\mathfrak{g})$ as follows. First, let

$$(\omega_x\otimes X, \omega'_x\otimes X')_x := (\omega_x,\omega'_x)_x(X,X')_{\mathfrak{g}}$$

for all ω_x, $\omega'_x\in T_x^*M$ and X, $X'\in\mathfrak{g}^{\mathrm{c}}$. For each $x\in M$, $(\cdot,\cdot)_x$ is an inner product on $T_x^*M\otimes\mathfrak{g}^{\mathrm{c}}$. Then

$$\langle f,g\rangle_0 := \int_M (f(x),g(x))_x dv(x), \tag{3.1}$$

for f, $g\in\Omega^1(M,\mathfrak{g})$. Here dv stands for the volume measure on M. This is an inner product on $\Omega^1(M,\mathfrak{g})$. We denote the completion of $\Omega^1(M,\mathfrak{g})$ with respect to the inner product $\langle\cdot,\cdot\rangle_0$ by $H(M,\mathfrak{g})$.

Let

$$(V(\psi)f)(x) := [\mathrm{id}_{T_x^*M}\otimes\mathrm{Ad}(\psi(x))]f(x)$$

for all $\psi\in C^\infty(M,G)$ and $f\in H(M,\mathfrak{g})$. Then $V(\psi)$ is a unitary operator on the Hilbert space $H(M,\mathfrak{g})$. We call V the adjoint representation of the gauge group $C^\infty(M,G)$.

For $\psi\in C^\infty(M,G)$, we define the right logarithmic derivative $\beta(\psi)\in\Omega^1(M,\mathfrak{g})$ as follows:

$$(\beta(\psi))(x) := (d\psi)_x\psi(x)^{-1}.$$

$\beta(\psi)$ is called the Maurer-Cartan cocycle, and satisfies

$$\beta(\psi\cdot\varphi) = V(\psi)\beta(\varphi)+\beta(\psi), \tag{3.2}$$

where $\psi\cdot\varphi$ is defined by the pointwise multiplication.

Definition 3.1. Let $U(\psi)$ be the unitary operator on the Boson Fock space $\Gamma_{\mathrm{b}}(H(M,\mathfrak{g}))$ satisfying

$$U(\psi)\exp(f) := \exp\left(-\frac{1}{2}|\beta(\psi)|_0^2\right)\exp(-\langle\beta(\psi),V(\psi)f\rangle_0)\exp(V(\psi)f+\beta(\psi))$$

for $f\in H(M,\mathfrak{g})$ and $\psi\in C^\infty(M,G)$. We call $U(\cdot)$ the *energy* (or *basic*) *representation* of the gauge group $C^\infty(M,G)$.

Remark 3.2. Naturally, we require that our representation of $C^\infty(M,G)$ should depend on structures of both G and M. From this viewpoint, let us consider $\Gamma_{\mathrm{b}}(V(\cdot))$, where $\Gamma_{\mathrm{b}}(\cdot)$ is the second quantization of operators, and $V(\cdot)$ is the adjoint representation of $C^\infty(M,G)$. $\Gamma_{\mathrm{b}}(V(\cdot))$ is a unitary representation of $C^\infty(M,G)$, however $\Gamma_{\mathrm{b}}(V(\cdot))$ can be decomposed into the adjoint representation of G at points of M. This means that the structure of $\Gamma_{\mathrm{b}}(V(\cdot))$ is fully determined by the adjoint representation of G, and is independent of structures of M. Obviously $\Gamma_{\mathrm{b}}(V(\cdot))$ is not irreducible. On the contrary, the energy representation $U(\cdot)$ depends on the differential structure of M. Indeed, we use the Maurer-Cartan cocycle $\beta(\cdot)$ to construct $U(\cdot)$.

In addition, we will have more interest in $U(\cdot)$ after seeing the following main theorem:

THEOREM 3.3. *Let M be a compact Riemann manifold without boundary. Then the energy representation $\{U(\psi)|\psi \in C^\infty(M,G)\}$ is irreducible.*

Obviously, to prove Theorem 3.3 with the help of the white noise distribution theory, we have to construct a test function space E first of all.

Let Δ be the Bochner Laplacian on $\Omega^1(M)$ and $H(M)$ be the completion of $\Omega^1(M)$. Let A be the self-adjoint extension of $(\Delta+2)\otimes \mathrm{id}_{\mathfrak{g}}$. (For the general theory of Laplacian on a vector bundle, see Chapter 1 of [5].) Let E be the test function space constructed from $(H(M,\mathfrak{g}),A)$.

It is difficult to deal with the energy representation $U(\cdot)$ directly. Thus we treat not the representation of "Lie group" $C^\infty(M,G)$ but the representation of "Lie algebra" $C^\infty(M,\mathfrak{g})$.

Here we remark the following proposition on the differentiability of an operator $V(\psi)$ on the test function space E.

LEMMA 3.4. *Let $\psi_t(x) := \exp(t\,\Psi(x))$ for $\Psi \in C^\infty(M,\mathfrak{g})$, $x \in M$ and $t \in \mathbf{R}$. Then $\{V(\psi_t)\}_{t\in\mathbf{R}}$ is a regular one-parameter subgroup of $GL(E)$, that is, for any $p \geq 0$ there exists $q \geq 0$ such that*

$$\lim_{t\to 0} \sup_{f\in E;\,|f|_q\leq 1} \left| \frac{V(\psi_t)f - f}{t} - V(\Psi)f \right|_p = 0$$

where

$$(V(\Psi)f)(x) := [\mathrm{id}_{T_x^*M} \otimes \mathrm{ad}(\Psi(x))]f(x)$$

for all $f \in E$.

From Theorem 5.4.5, Theorem 5.7.9 of [10] and Lemma 3.4, we can show

LEMMA 3.5. *Let $\psi_t(x) := \exp(t\Psi(x))$ for $\Psi \in C^\infty(M,\mathfrak{g})$ and $t \in \mathbf{R}$. Then $\{U(\psi_t)\}_{t\in\mathbf{R}}$ is a regular one-parameter subgroup of $GL(\mathcal{E})$ with infinitesimal generator*

$$\pi(\Psi) := d\Gamma_{\mathrm{b}}(V(\Psi)) + a^\dagger(d\Psi) - a(d\Psi) \in \mathcal{L}(\mathcal{E},\mathcal{E}).$$

Note that $\pi(\Psi)$ has the following expression:

$$\pi(\Psi) = \Xi_{1,1}(\lambda_{1,1}) + \Xi_{1,0}(\lambda_{1,0}) - \Xi_{0,1}(\lambda_{0,1}),$$

where $\lambda_{1,1} = (\mathrm{id}\otimes V(\Psi))^*\tau$ and $\lambda_{1,0} = \lambda_{0,1} = d\Psi$. Here $\tau \in (E\otimes E)^*$ is the trace, that is,

$$\langle \tau, f\otimes g\rangle := \langle f, g\rangle\,, \quad f, g \in E,$$

4. Proof of Theorem 3.3. To show irreducibility of the energy representation, we firstly assume that a bounded operator Ξ on $\Gamma_{\mathrm{b}}(H(M,\mathfrak{g}))$ satisfies

$$U(\exp(t\Psi))\Xi = \Xi U(\exp(t\Psi)) \tag{4.1}$$

for all $\Psi \in C^\infty(M,\mathfrak{g})$ and $t \in \mathbf{R}$. Then we can easily see that (4.1) implies

$$\widetilde{\pi}(\Psi)\Xi = \Xi\pi(\Psi) \tag{4.2}$$

for all $\Psi \in C^\infty(M, \mathfrak{g})$ as a continuous linear operator from $\mathcal{E}$ to $\mathcal{E}^*$. Here $\widetilde{\pi}(\Psi) := -\pi(\Psi)^*$. $\widetilde{\pi}$ satisfies $\widetilde{\pi}(\Psi) \in \mathcal{L}(\mathcal{E}^*, \mathcal{E}^*)$ and $\widetilde{\pi}(\Psi)|_{\mathcal{E}} = \pi(\Psi)$. Therefore our problem is to find $\Xi \in \mathcal{L}(\mathcal{E}, \mathcal{E}^*)$ satisfying (4.2).

We have

$$\begin{aligned}
&\sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m})\pi(\Psi)\\
&\quad = \Xi_{0,0}(\kappa_{0,1} \circ_1 \lambda_{1,0})\\
&\quad + \sum_{l=1}^{\infty} \Xi_{l,0}(S_0^{l-1}{}_0^1(\kappa_{l-1,0} \circ \lambda_{1,0} + S_0^l{}_0^0(\kappa_{l,1} \circ_1 \lambda_{1,0}))\\
&\quad + \sum_{m=1}^{\infty} \Xi_{0,m}(m\, S^0_{m-1}{}^0_1(\kappa_{0,m} \circ_1 \lambda_{1,1})\\
&\qquad + (m+1)\, S^0_m{}^0_0(\kappa_{0,m+1} \circ_1 \lambda_{1,0}) - S^0_{m-1}{}^0_1(\kappa_{0,m-1} \circ \lambda_{0,1}))\\
&\quad + \sum_{l,m=1}^{\infty} \Xi_{l,m}(S^{l-1}_{m-1}{}^1_1(\kappa_{l-1,m-1} \circ \lambda_{1,1})\\
&\qquad + mS^l_{m-1}{}^0_1(\kappa_{l,m} \circ_1 \lambda_{1,1}) + S^{l-1}_m{}^1_0(\kappa_{l-1,m} \circ \lambda_{1,0})\\
&\qquad + (m+1)S^l_m{}^0_0(\kappa_{l,m+1} \circ_1 \lambda_{1,0}) - S^l_{m-1}{}^0_1(\kappa_{l,m-1} \circ \lambda_{0,1})).
\end{aligned}$$

On the other hand, note that

$$\widetilde{\pi}(\Psi)\Big(\sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m})\Big) = -\Big\{\Big(\sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m})^*\Big)\pi(\Psi)\Big\}^* \tag{4.3}$$

as a continuous linear operator from $\mathcal{E}$ to $\mathcal{E}^*$.

Thus (4.2) implies the following three relations:

$$d\Gamma_{\mathrm{b}}(V(\Psi)^*)^{(l)}\kappa_{l,0} = s_{l,0}((l+1)S_0^0{}_0^l(\lambda_{0,1} \circ_1 \kappa_{l+1,0}) + S_0^l{}_0^0(\kappa_{l,1} \circ \lambda_{1,0})), \tag{4.4}$$

$$-d\Gamma_{\mathrm{b}}(V(\Psi)^*)^{(m)}\kappa_{0,m} = s_{0,m}((m+1)S^0_m{}^0_0(\kappa_{0,m+1} \circ_1 \lambda_{1,0}) + S^0_0{}^0_m(\lambda_{0,1} \circ_1 \kappa_{1,m})), \tag{4.5}$$

$$\begin{aligned}
&(d\Gamma_{\mathrm{b}}(V(\Psi)^*)^{(l)} \otimes \mathrm{id}^{\otimes m} + \mathrm{id}^{\otimes l} \otimes d\Gamma_{\mathrm{b}}(V(\Psi)^*)^{(m)})\kappa_{l,m}\\
&\quad = s_{l,m}((l+1)S^0_0{}^l_m(\lambda_{0,1} \circ_1 \kappa_{l+1,m}) + (m+1)S^l_m{}^0_0(\kappa_{l,m+1} \circ_1 \lambda_{1,0}))
\end{aligned} \tag{4.6}$$

for all $\Psi \in C^\infty(M, \mathfrak{g})$.

In particular, (4.4)–(4.6) become

$$d\Gamma_{\mathrm{b}}(V(\Psi)^*)^{(l)}\kappa_{l,0} = 0, \tag{4.7}$$

$$d\Gamma_{\mathrm{b}}(V(\Psi)^*)^{(m)}\kappa_{0,m} = 0, \tag{4.8}$$

$$\{(d\Gamma_{\mathrm{b}}(V(\Psi)^*)^{(l)} \otimes \mathrm{id}^{\otimes m} + \mathrm{id}^{\otimes l} \otimes d\Gamma_{\mathrm{b}}(V(\Psi)^*)^{(m)}\}\kappa_{l,m} = 0. \tag{4.9}$$

for constant maps $\Psi \in C^\infty(M, \mathfrak{g})$. (Remark that $C^\infty(M, \mathfrak{g})$ contains constant maps because of compactness of M.) To compute (4.7)–(4.9), we have to find a C.O.N.S. of $H(M, \mathfrak{g})^{\widehat{\otimes} n}$ $(n \geq 1)$.

Let us recall the well-known C.O.N.S. of $\mathfrak{g}^{\mathrm{c}}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and $\{H_1, \ldots, H_{\dim \mathfrak{h}}\}$ be a C.O.N.S. of $\mathfrak{h}$. Let Δ' be a positive root system of $\mathfrak{g}^{\mathrm{c}}$ and X_α, $\alpha \in \Delta' \cup (-\Delta')$ be normalized elements of $\mathfrak{g}^{\mathrm{c}}$ such that $[H, X_\alpha] = \alpha(H)X_\alpha$ for all $H \in \mathfrak{h}$.

Then

$$\{H_1, \ldots, H_{\dim \mathfrak{h}}, X_\alpha, X_{-\alpha} \mid \alpha \in \Delta'\} \tag{4.10}$$

is a C.O.N.S. of the complex vector space $\mathfrak{g}^c$ with respect to the inner product on $\mathfrak{g}^c$.

We put $\dim \mathfrak{g} = N_0$, $\dim \mathfrak{h} = N_1$, and

$$u_j := \begin{cases} H_j, & \text{if } 1 \leq j \leq N_1, \\ X_{\alpha_{j-N_1}}, & \text{if } N_1 + 1 \leq j \leq N_1 + N_2, \\ X_{-\alpha_{j-(N_1+N_2)}}, & \text{if } N_1 + N_2 + 1 \leq j \leq N_1 + 2N_2 = N_0, \end{cases}$$

and let $\{e_i\}_{i \in \mathbf{N}}$ be a C.O.N.S. of $H(M)$. Then

$$\{e(i,j) := e_i \otimes u_j \mid i \in \mathbf{N},\ j \in \{1, 2, \ldots, N_0\}\}$$

is a C.O.N.S. of $H(M, \mathfrak{g})$.

Fix $d \in \{1, 2, \ldots, n\}$ and

$$\mathbf{i} := (\overbrace{i_1, \ldots, i_1}^{N(1)\,times}, \overbrace{i_2, \ldots, i_2}^{N(2)\,times}, \ldots, \overbrace{i_d, \ldots, i_d}^{N(d)\,times}) \in \mathbf{N}^n,$$
$$N(1) + N(2) + \ldots + N(d) = n, \quad i_1 < i_2 < \ldots < i_d.$$

For this $\mathbf{i} \in \mathbf{N}^n$, we define $\mathbf{j} \in \{1, 2, \ldots, N_0\}^n$ as follows:

$$\mathbf{j} := (j(i_1, 1), \ldots, j(i_1, N(1)), j(i_2, 1), \ldots, j(i_2, N(2)), \ldots, j(i_d, 1), \ldots, j(i_d, N(d))).$$

Here $j(i,k) \in \{1, 2, \ldots, N_0\}$ satisfies $j(i, k_1) \leq j(i, k_2)$ for each $i \in \mathbf{N}$ and $k_1 < k_2$.

Now we denote the set of all such pairs $(\mathbf{i}, \mathbf{j})$ by $\Lambda(n)$. $\Lambda(n)$ is a subset of $\mathbf{N}^n \times \{1, 2, \ldots, N_0\}^n$. For $(\mathbf{i}, \mathbf{j}) \in \Lambda(n)$, we put

$$\begin{aligned} e(\mathbf{i}, \mathbf{j}) := & e(i_1, j(i_1, 1)) \widehat{\otimes} \ldots \widehat{\otimes} e(i_1, j(i_1, N(1))) \\ & \widehat{\otimes} e(i_2, j(i_2, 1)) \widehat{\otimes} \ldots \widehat{\otimes} e(i_2, j(i_2, N(2))) \\ & \ldots \widehat{\otimes} e(i_d, j(i_d, 1)) \widehat{\otimes} \ldots \widehat{\otimes} e(i_d, j(i_d, N(d))), \end{aligned}$$

then $\{e(\mathbf{i}, \mathbf{j}) \mid (\mathbf{i}, \mathbf{j}) \in \Lambda(n)\}$ is a C.O.N.S. of $H(M, \mathfrak{g})^{\widehat{\otimes} n}$.

Let $(\mathbf{i}, \mathbf{j}) \in \Lambda(l)$ and $\mathbf{j} = (j_1, \ldots, j_l) \in \{1, 2, \ldots, N_0\}^l$. For $1 \leq p \leq N_2$, let

$$n_{p,+}(\mathbf{j}) := \#\{q \in \{1, 2, \ldots, l\} \mid j_q = N_1 + p\},$$
$$n_{p,-}(\mathbf{j}) := \#\{q \in \{1, 2, \ldots, l\} \mid j_q = N_1 + N_2 + p\}.$$

Since

$$d\Gamma_{\mathrm{b}}(V(H)^{(l)}) e(\mathbf{i}, \mathbf{j}) = \sum_{1 \leq p \leq N_2} \alpha_p(H)(n_{p,+}(\mathbf{j}) - n_{p,-}(\mathbf{j})) e(\mathbf{i}, \mathbf{j})$$

for $H \in \mathfrak{h}$, (4.9) implies

$$\kappa_{l,m} = \sum \langle \kappa_{l,m}, e(\mathbf{i}, \mathbf{j}) \otimes e(\mathbf{i}', \mathbf{j}') \rangle \, e(\mathbf{i}, \mathbf{j}) \otimes e(\mathbf{i}', \mathbf{j}') \tag{4.11}$$

where the sum is over all $(\mathbf{i}, \mathbf{j}) \in \Lambda(l)$ and $(\mathbf{i}', \mathbf{j}') \in \Lambda(m)$ satisfying

$$\sum_{1 \leq p \leq N_2} \alpha_p(H)(n_{p,+}(\mathbf{j}) - n_{p,-}(\mathbf{j}) + n_{p,+}(\mathbf{j}') - n_{p,-}(\mathbf{j}')) = 0$$

for all $H \in \mathfrak{h}$.

LEMMA 4.1. $\kappa_{1,0} = 0$ *and* $\kappa_{0,1} = 0$.

Proof. In this proof, we regard $u_j \in \mathfrak{g}^{\mathrm{c}}$ as a constant map in $C^\infty(M, \mathfrak{g})$. (4.11) implies

$$\kappa_{1,0} = \sum_{(i,j)\in\Lambda(1);1\le j\le N_1} \langle \kappa_{1,0}, e(i,j)\rangle\, e(i,j).$$

On the other hand,

$$V(u_{k+N_1})^* e(i,j) = -\alpha_k(u_j) e(i, k+N_1+N_2)$$

is obtained directly from $V(u_{k+N_1})^*|_E = -V(u_{k+N_1+N_2})$ for $1 \le k \le N_2$. Thus

$$0 = d\Gamma_{\mathrm{b}}(V(u_{k+N_1})^*)\kappa_{1,0} = -\sum_{i=1}^{\infty} \sum_{1\le j\le N_1} \alpha_k(u_j)\, \langle \kappa_{1,0}, e(i,j)\rangle\, e(i, k+N_1+N_2),$$

that is,

$$\sum_{1\le j\le N_1} \langle \kappa_{1,0}, e(i,j)\rangle\, \alpha_k(u_j) = 0$$

for all $i \in \mathbf{N}$, $k \in \{1, 2, \ldots, N_2\}$. Since $\mathfrak{h}^*$ is generated by the linear combination of $\{\alpha_k\}_{k=1}^{N_2}$, we can choose a basis $\{\alpha_{k_1}, \ldots, \alpha_{k_{N_1}}\}$ of $\mathfrak{h}^*$. Then the matrix $(\alpha_{k_i}(u_j))_{1\le i,j\le N_1}$ $\in \mathrm{Mat}(N_1, \mathbf{C})$ is invertible. Therefore

$$\langle \kappa_{1,0}, e(i,j)\rangle = 0$$

for all $i \in \mathbf{N}$ and $j \in \{1, 2, \ldots, N_1\}$, i.e. $\kappa_{1,0} = 0$. In the same manner, $\kappa_{0,1} = 0$. ∎

LEMMA 4.2. *$\kappa_{l,1} = 0$ and $\kappa_{1,m} = 0$.*

Proof. Let $\Psi(x) := u_{k+N_1}$, $x \in M$. Then, from (4.9), we can obtain

$$\sum_{1\le j\le N_1} \langle \kappa_{l,1}, e(\mathbf{i},\mathbf{j}) \otimes e(i,j)\rangle\, \alpha_k(u_j) = 0 \tag{4.12}$$

for all $i \in \mathbf{N}$, $1 \le k \le N_2$, and all $(\mathbf{i},\mathbf{j}) \in \Lambda(l)$ satisfying

$$\sum_{1\le p\le N_2} \alpha_p(H)(n_{p,+}(\mathbf{j}) - n_{p,-}(\mathbf{j})) = 0. \tag{4.13}$$

Now we can select k_1, k_2,…, $k_{N_1} \in \{1, 2, \ldots, N_2\}$ such that the matrix $(\alpha_{k_i}(u_j))_{1\le i,j\le N_1}$ in $\mathrm{Mat}(N_1, \mathbf{C})$ is invertible. Therefore we obtain

$$\langle \kappa_{l,1}, e(\mathbf{i},\mathbf{j}) \otimes e(i,j)\rangle = 0$$

for all $i \in \mathbf{N}$, $1 \le j \le N_1$ and all $(\mathbf{i},\mathbf{j}) \in \Lambda(l)$ satisfying (4.13), i.e. $\kappa_{l,1} = 0$.

In the same manner, $\kappa_{1,m} = 0$ holds. ∎

Note that we used only constant maps in $C^\infty(M, \mathfrak{g})$ to obtain $\kappa_{l,1} = 0$ and $\kappa_{1,m} = 0$ for all l, $m \ge 0$. However we have to use non-constant maps to show the following lemma.

LEMMA 4.3. *$\kappa_{l,0} = 0$ and $\kappa_{0,m} = 0$ for all $l, m \ge 1$.*

Proof. We only show $\kappa_{l,0}=0$. We prove that by induction. We have already proved $\kappa_{1,0}=0$. Let $\kappa_{l,0}=0$. Then

$$\begin{aligned}0&=s_{l,0}((l+1)S^{0\,l}_{0\,0}(\lambda_{0,1}\circ_1\kappa_{l+1,0}))\\&=\sum_{(\mathbf{i},\mathbf{j})\in\Lambda(l)}\sum_{(i,j)\in\Lambda(1)}\langle\lambda_{0,1},e(i,j)\rangle\langle\kappa_{l+1,0},e(i,j)\otimes e(\mathbf{i},\mathbf{j})\rangle e(\mathbf{i},\mathbf{j})\\&=\sum_{(\mathbf{i},\mathbf{j})\in\Lambda(l)}\langle\kappa_{l+1,0},\lambda_{0,1}\otimes e(\mathbf{i},\mathbf{j})\rangle e(\mathbf{i},\mathbf{j})\end{aligned}$$

by $\kappa_{l,1}=0$ and (4.6). Hence

$$\langle\kappa_{l+1,0},\lambda_{0,1}\otimes e(\mathbf{i},\mathbf{j})\rangle=0 \tag{4.14}$$

for all $(\mathbf{i},\mathbf{j})\in\Lambda(l)$. This implies

$$\langle\kappa_{l+1,0},d\Psi\otimes e(\mathbf{i},\mathbf{j})\rangle=0, \tag{4.15}$$

$$\langle\kappa_{l+1,0},V(\Psi)d\Psi'\otimes e(\mathbf{i},\mathbf{j})\rangle=0 \tag{4.16}$$

for all $(\mathbf{i},\mathbf{j})\in\Lambda(l)$, and Ψ, $\Psi'\in C^\infty(M,\mathfrak{g})$. (4.15) is obvious. We show (4.16).

For each Ψ, $\Psi'\in C^\infty(M,\mathfrak{g})$ and $|s|,|t|\ll 1$, there exists a unique $\Phi_{s,t}\in C^\infty(M,\mathfrak{g})$ such that

$$\exp(t\Psi)\exp(s\Psi')=\exp(\Phi_{s,t}).$$

Since

$$d\Phi_{s,t}=\beta(\exp(\Phi_{s,t}))=sV(\exp(t\Psi))d\Psi'+td\Psi$$

and (4.15), we have

$$\begin{aligned}0&=\langle\kappa_{l+1,0},d\Phi_{s,t}\otimes e(\mathbf{i},\mathbf{j})\rangle\\&=s\langle\kappa_{l+1,0},V(\exp(t\Psi))d\Psi'\otimes e(\mathbf{i},\mathbf{j})\rangle+t\langle\kappa_{l+1,0},d\Psi\otimes e(\mathbf{i},\mathbf{j})\rangle\\&=s\langle\kappa_{l+1,0},V(\exp(t\Psi))d\Psi'\otimes e(\mathbf{i},\mathbf{j})\rangle\end{aligned}$$

Hence $\kappa_{l+1,0}$ satisfies (4.16) by considering the differential of the above equation at $t\in\mathbf{R}$.

Moreover, $H(M,\mathfrak{g})$ is generated by

$$\{d\Psi,\,V(\Psi)d\Psi'\,|\,\Psi,\Psi'\in C^\infty(M,\mathfrak{g})\}.$$

(See Lemma 3.5 of [2].) Thus, for each $(i,j)\in\Lambda(1)$ and $(\mathbf{i},\mathbf{j})\in\Lambda(l)$,

$$\langle\kappa_{l+1,0},e(i,j)\otimes e(\mathbf{i},\mathbf{j})\rangle=0$$

follows from (4.15) and (4.16). Therefore we obtain $\kappa_{l+1,0}=0$.

In the same manner, we can show $\kappa_{0,m}=0$ for all $m\geq 1$. ∎

LEMMA 4.4. $\kappa_{l,m}=0$ *for all* $(l,m)\in\mathbf{Z}^2_{\geq 0}\setminus\{(0,0)\}$.

Proof. We prove this statement by induction. We have already shown the case of $l=1$, i.e. $\kappa_{1,m}=0$ for all $m\geq 0$. Let $\kappa_{l,m}=0$ for all $m\geq 0$. Then we show $\kappa_{l+1,m}=0$ for all $m\geq 0$. Fix $m\geq 0$. Since $\kappa_{l,m}=0$ and $\kappa_{l,m+1}=0$ and (4.6), we have

$$\begin{aligned}0&=s_{l,m}((l+1)S^{0\,l}_{0\,m}(\lambda_{0,1}\circ_1\kappa_{l+1,m}))\\&=(l+1)\sum_{(\mathbf{i},\mathbf{j})\in\Lambda(l),\,(\mathbf{i}',\mathbf{j}')\in\Lambda(m)}\left\langle\kappa_{l+1,m},\lambda_{0,1}\widehat{\otimes}e(\mathbf{i},\mathbf{j})\otimes e(\mathbf{i}',\mathbf{j}')\right\rangle e(\mathbf{i},\mathbf{j})\otimes e(\mathbf{i}',\mathbf{j}'),\end{aligned}$$

that is, $\langle \kappa_{l+1,m}, \lambda_{0,1} \widehat{\otimes} e(\mathbf{i},\mathbf{j}) \otimes e(\mathbf{i}',\mathbf{j}') \rangle = 0$ for all $(\mathbf{i},\mathbf{j}) \in \Lambda(l)$, $(\mathbf{i}',\mathbf{j}') \in \Lambda(m)$. This implies

$$\langle \kappa_{l+1,m}, e(i,j) \widehat{\otimes} e(\mathbf{i},\mathbf{j}) \otimes e(\mathbf{i}',\mathbf{j}') \rangle = 0$$

for all $(i,j) \in \Lambda(1)$, $(\mathbf{i},\mathbf{j}) \in \Lambda(l)$, and $(\mathbf{i}',\mathbf{j}') \in \Lambda(m)$. Therefore $\kappa_{l+1,m} = 0$. Since $m \geq 0$ is arbitrary, the proof has been completed. ∎

5. Remarks. In this paper, since the Riemann manifold M is compact, the gauge group is given by $C^\infty(M,G)$. However, more generally, the term "gauge group" is used to refer to $C_{\mathrm{c}}^\infty(M,G)$. Here $C_{\mathrm{c}}^\infty(M,G)$ stands for the set of all compactly supported C^∞-maps from M to G. The energy representation can be also defined for $C_{\mathrm{c}}^\infty(M,G)$. However, in case of non-compact Riemann manifold M, it is not easy to show irreducibility of the energy representation by our tool (= white noise distribution theory, in particular, the Fock expansion), moreover, unfortunately, we know that the energy representation of $C_{\mathrm{c}}^\infty(\mathbf{R},G)$ is reducible. (See [2] and [3].) Thus, some remarks are in order.

(1) Recall Definition 2.1. The test function space E is constructed from a pair (H,A). If M is compact, the operator A is easily given, for example, we can take the Bochner Laplacian etc. However, if M is non-compact, such a suitable self-adjoint operator A is not known except for $M = \mathbf{R}^n$. For example, if $M = \mathbf{R}$, as we have already seen in Example 2, we can obtain E from $H = L^2(\mathbf{R})$ and $A := -(d/dx)^2 + x^2 + 1$.

(2) In the proof of irreducibility of the energy representation, compactness of M is also used effectively. Recall the proof of Lemma 4.2. We use the fact that $C^\infty(M,\mathfrak{g})$ contains constant maps. However, if M is non-compact, $C_{\mathrm{c}}^\infty(M,\mathfrak{g})$ does not contain constant maps and this suggests that we may need another proof of Lemma 4.2.

(3) We have an unsettled question. From [2] and [3], reducibility of the energy representation of $C_{\mathrm{c}}^\infty(\mathbf{R},G)$ should be proved by our method, and also, for a non-compact manifold M with $\dim M \geq 2$, we left the problem of irreducibility of the energy representation of $C_{\mathrm{c}}^\infty(M,G)$ by our method. It seems that these problems cannot be solved easily because of (1) and (2).

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