FIXED POINT THEORY AND ITS APPLICATIONS
BANACH CENTER PUBLICATIONS, VOLUME 77
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2007

NONLINEAR SYSTEMS WITH MEAN CURVATURE-LIKE OPERATORS

PIERLUIGI BENEVIERI

Dipartimento di Matematica Applicata "G. Sansone"
Via S. Marta 3, I-50139 Firenze, Italy
E-mail: pierluigi.benevieri@unifi.it

JOÃO MARCOS DO Ó and EVERALDO SOUTO DE MEDEIROS

Departamento de Matemática, Universidade Federal da Paraiba, CCEN
58059-900 João Pessoa, PB, Brazil
E-mail: jmbo@mat.ufpb.br, everaldo@mat.ufpb.br

Abstract. We give an existence result for a periodic boundary value problem involving mean curvature-like operators. Following a recent work of R. Manásevich and J. Mawhin, we use an approach based on the Leray–Schauder degree.

1. Introduction. In this paper we study the existence of periodic solutions of the nonlinear differential problem

$$\begin{cases} (\phi(u'))' = f(t,u,u') \\ u(0) = u(T) \\ u'(0) = u'(T), \end{cases} \tag{1.1}$$

where $f : [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function and ϕ is a homeomorphism between $\mathbb{R}^N$ and the open ball in $\mathbb{R}^N$ with center zero and radius 1, verifying the following condition:

(H1) *$\phi(x) = w(\|x\|)Ax$, where $w : [0,+\infty) \to [0,+\infty)$ is continuous and A is a linear isomorphism.*

Our purpose here is to enrich some recent results obtained in [1] and [2] about problem (1.1) in the more restrictive assumption that A is the identity.

2000 *Mathematics Subject Classification*: Primary 34B15; Secondary 47H11.

Key words and phrases: boundary value problem, mean curvature-like operators, periodic solutions, Leray–Schauder degree, homotopy invariance.

The paper is in final form and no version of it will be published elsewhere.

The class of nonlinear operators $u \mapsto (\phi(u'))'$ verifying (H1) is interesting since it includes the scalar version of the mean curvature operator

$$u \mapsto \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)$$

which is usually considered in the case when u is a scalar function defined on an open subset of $\mathbb{R}^N$.

The study of problem (1.1) is motivated by the attempt of applying in our context the topological approach followed by Manásevich and Mawhin in [8] (see also [9]), in which they proved an existence result for the periodic boundary value problem

$$\begin{cases} (\phi(u'))' = f(t,u,u') \\ \quad u(0) = u(T) \\ \quad u'(0) = u'(T), \end{cases} \tag{1.2}$$

where $f : [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is still a Carathéodory function, whereas $\phi : \mathbb{R}^N \to \mathbb{R}^N$ is a homeomorphism satisfying particular monotonicity conditions which include for instance p-Laplacian-like operators. Precisely, under further conditions on ϕ and f, they applied the Leray–Schauder degree to prove that (1.2) admits a solution ([8, Theorem 3.1]).

In [1] and [2] we proceeded in the general spirit of Manásevich–Mawhin's ideas and we proved, as said before, an existence result for (1.1), assuming that A is the identity. We still follow here the same approach: under suitable assumptions on f, which we specify in the sequel, we apply the Leray–Schauder degree and we show (Theorem 4.1 below) that (1.1) admits a solution.

We point out that similar results has been recently obtained, independently, by Bereanu and Mawhin (see [3], [4] and [5]). They study the problem

$$(\phi(u'))' = f(t,u,u'), \tag{1.3}$$

with Dirichlet, Neumann or periodic boundary conditions on u, where $\phi : \mathbb{R} \to (-a,a)$ is a homeomorphism such that $\phi(0) = 0$ and $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous. Bereanu and Mawhin follow a topological approach based on the Leray–Schauder degree (analogously to [8]), and they find interesting a priori estimates involving f and ϕ.

Our paper is organized as follows. In the next section we isolate some useful preliminary results concerning the map ϕ. Then, in Section 3 we consider our problem in the particular case when f is independent of u and u'. The study of this simplified case is the first step in the direction of applying the Leray–Schauder degree, as done in Section 4. That section is, in particular, devoted to the main theorem of this work, that is, an existence result for system (1.1). In the last section we present an application of the main theorem to a particular system.

We refer to e.g. [6] or [7] for the definition and the main properties of the Leray–Schauder degree.

Standing notation. In what follows I will denote the closed interval $[0,T]$, with T fixed. In addition, we will put $\mathcal{C} = C(I,\mathbb{R}^N)$, $\mathcal{C}^1 = C^1(I,\mathbb{R}^N)$, $\mathcal{C}_{T,0} = \{u \in \mathcal{C} : u(0) = u(T) = 0\}$,

$\mathcal{C}^1_T = \{u \in \mathcal{C}^1 : u(0) = u(T),\ u'(0) = u'(T)\}$, $L^1 = L^1(I, \mathbb{R}^N)$, and, finally, $L^1_m = \{h \in L^1 : \int_0^T h(t)\,dt = 0\}$. The norm in $\mathcal{C}$ and $\mathcal{C}_{T,0}$ is defined by

$$\|u\|_0 = \max_{t\in I} \|u(t)\|_{\mathbb{R}^N},$$

the norm in $\mathcal{C}^1$ and $\mathcal{C}^1_T$ by

$$\|u\|_1 = \|u\|_0 + \|u'\|_0,$$

and the norm in L^1 and L^1_m by

$$\|h\|_{L^1} = \left[\sum_{i=1}^N \int_0^T \|h_i(t)\|^2\,dt\right]^{1/2} = \left(\sum_{i=1}^N \|h_i\|^2_{L_1}\right)^{1/2}.$$

Finally, by $\|\cdot\|$ we simply denote the Euclidean norm in $\mathbb{R}^N$.

REMARK 1.1. By a solution of (1.1) we mean a C^1 map u on $[0,T]$, with values in $\mathbb{R}^N$, satisfying the boundary conditions, such that $\phi(u')$ is absolutely continuous and verifies $(\phi(u'))' = f(t,u,u')$ a.e. on $[0,T]$.

2. Preliminary results. In this section we show some important consequences following from assumption (H1).

LEMMA 2.1. *Let $\phi : \mathbb{R}^N \to B(0,1)$ be a homeomorphism between $\mathbb{R}^N$ and the open ball in $\mathbb{R}^N$ with center zero and radius* 1, *verifying condition* (H1). *Then*

i) $\|Ax\| = \|Ay\|$ *if* $\|x\| = \|y\|$;

ii) $\langle Ax, Ay\rangle = 0$ *if* $\langle x,y\rangle = 0$.

Proof. i) Let x and y be such that $\|x\| = \|y\|$. Given $\lambda > 0$, we have

$$\|\phi(\lambda x)\| = \lambda w(\lambda\,\|x\|)\|Ax\| \quad \text{and} \quad \|\phi(\lambda y)\| = \lambda w(\lambda\,\|y\|)\|Ay\|.$$

Since $\|\phi(\lambda x)\|$ and $\|\phi(\lambda y)\|$ converge to 1 when $\lambda \to +\infty$, and since $w(\lambda\,\|x\|) = w(\lambda\,\|y\|)$ for any $\lambda > 0$, one has

$$\frac{1}{\|Ax\|} = \lim_{\lambda\to+\infty} \lambda w(\lambda\,\|x\|) = \lim_{\lambda\to+\infty} \lambda w(\lambda\,\|y\|) = \frac{1}{\|Ay\|},$$

and the claim follows.

ii) Observe first that, by i), we can assume without loss of generality that $\|Ax\| = \|x\|$ for any x. Consider now x and y such that $\langle x,y\rangle = 0$ and $\|x\| = \|y\|$. If $\langle Ax, Ay\rangle \neq 0$ then $\|x+y\| \neq \|Ax+Ay\|$, but this is a contradiction. ∎

REMARK 2.2. By the above argument, from now on and without loss of generality we will suppose that A is an orthonormal linear isomorphism.

The following lemma concerns the particular case when A is the identity.

LEMMA 2.3. *Let $\psi : \mathbb{R}^N \to B(0,1)$ be a homeomorphism between $\mathbb{R}^N$ and the open ball in $\mathbb{R}^N$ with center zero and radius* 1 *of the form $\psi(x) = w(\|x\|)x$, where $w : [0,+\infty) \to [0,+\infty)$ is continuous. Then, for any $x, y \in \mathbb{R}^N$ with $x \neq y$, one has*

$$\langle \psi(x) - \psi(y), x-y\rangle > 0.$$

Proof. Consider first the particular case $y = \lambda x$, with $\lambda \geq 0$, $\lambda \neq 1$ and $x \neq 0$. One has

$$\begin{aligned}&\langle \psi(x) - \psi(\lambda x), x - \lambda x\rangle \\ &\quad = \big\langle w(\|x\|)x - w(\|\lambda x\|)\lambda x, (1-\lambda)x\big\rangle = \big(w(\|x\|)\|x\| - w(\|\lambda x\|)\|\lambda x\|\big)(1-\lambda)\|x\|.\end{aligned}$$

Using the fact that $t \mapsto w(t)t$ is strictly increasing, one can easily show that

$$\big[w(\|x\|)\|x\| - w(\|\lambda x\|)\|\lambda x\|\big](1-\lambda) > 0 \quad \forall \lambda \geq 0,\ \lambda \neq 1.$$

Consider now any $x, y \in \mathbb{R}^N$ with $\|x\| \neq \|y\|$. We have

$$\begin{aligned}\langle \psi(x) - \psi(y), x - y\rangle &= w(\|x\|)\|x\|^2 + w(\|y\|)\|y\|^2 - \big(w(\|x\|) + w(\|y\|)\big)\langle x, y\rangle \\ &\geq w(\|x\|)\|x\|^2 + w(\|y\|)\|y\|^2 - \big(w(\|x\|) + w(\|y\|)\big)\|x\|\,\|y\|.\end{aligned}$$

Take $y_1 = \lambda x$ such that $\|y_1\| = \|y\|$, with $\lambda \geq 0$. It follows that

$$\begin{aligned}&w(\|x\|)\|x\|^2 + w(\|y\|)\|y\|^2 - \big(w(\|x\|) + w(\|y\|)\big)\|x\|\,\|y\| \\ &\qquad = w(\|x\|)\|x\|^2 + w(\|y_1\|)\|y_1\|^2 - \big(w(\|x\|) + w(\|y_1\|)\big)\|x\|\,\|y_1\| \\ &\qquad\qquad = \langle \psi(x) - \psi(y_1), x - y_1\rangle > 0\end{aligned}$$

(the last inequality holds by the first case).

Finally, if $x \neq y$, but $\|x\| = \|y\|$, we have

$$\langle \psi(x) - \psi(y), x - y\rangle = \langle w(\|x\|)x - w(\|y\|)y, x - y\rangle = w(\|x\|)\langle x - y, x - y\rangle > 0$$

and the lemma is proved. ■

Let us point out that the above result is *always false* for any homeomorphism $\phi(x) = w(\|x\|)Ax$ *if A is not the identity.*

3. An auxiliary problem. Consider the following periodic boundary value problem

$$\begin{cases} (\phi(u'))' = h(t) \\ \quad u(0) = u(T) \\ \quad u'(0) = u'(T), \end{cases} \tag{3.1}$$

where h is in L^1_m and ϕ is a homeomorphism between $\mathbb{R}^N$ and the open ball of $\mathbb{R}^N$, with center zero and radius 1, verifying condition (H1) (see also Remark 2.2). If a C^1 function $u : I \to \mathbb{R}^N$ solves the equation $(\phi(u'))' = h(t)$, then there exists $a \in \mathbb{R}^N$ such that

$$\phi(u'(t)) = a + H(h)(t), \tag{3.2}$$

where H is the integral operator

$$H(h)(t) = \int_0^t h(s)\,ds.$$

REMARK 3.1. Notice that the condition $u'(0) = u'(T)$ implies that $\int_0^T h(t)\,dt = 0$ and this justifies the assumption that $h \in L^1_m$.

By the inversion of ϕ in (3.2) we have

$$u'(t) = \phi^{-1}(a + H(h)(t)),$$

and thus the image of $H(h)$, which contains the origin of $\mathbb{R}^N$, is included in an open ball of radius 1. In addition, any a verifying the above equality is such that $\|a\| < 1$. Call $\widetilde{D}$ the set of functions h in L^1_m such that there exists $a \in \mathbb{R}^N$ with

$$\|a + H(h)(t)\| < 1 \quad \forall t \in I.$$

The set $\widetilde{D}$ is unbounded in L^1_m. Indeed, take for simplicity $T = 1$ and consider the sequence of real functions $\{h_n\}_{n\in\mathbb{N}}$, where

$$h_n : [0,1] \to \mathbb{R}, \quad h_n(t) = \begin{cases} n & t \in [k/n, (2k+1)/(2n)) \\ -n & t \in [(2k+1)/(2n), (k+1)/n) \cup \{1\}, \end{cases} \tag{3.3}$$

$k = 0, \dots, n-1$. Consider the sequence $\{k_n\} \subseteq L^1_m$, where $k_n = (h_n, 0, \dots, 0)$. A straightforward computation shows that, for each n,

$$\|k_n\|_{L^1} = n \quad \text{and} \quad \|H(k_n)\|_0 = 1/2.$$

Thus, $\{k_n\}$ is an unbounded sequence contained in $\widetilde{D}$. On the other hand, even if $\widetilde{D}$ is unbounded, it is easy to see that it does not contain any one-dimensional subspace of L^1_m.

Moreover, $\widetilde{D}$ is open in L^1_m. To see this, let $h \in \widetilde{D}$ be given. One has

$$\max_{t_1,t_2\in I} \|H(h(t_2)) - H(h(t_1))\| = \max_{t_1,t_2\in I} \left| \int_{t_1}^{t_2} h(t)\, dt \right| = \delta < 2.$$

Then, given ε in L^1_m, it follows that

$$\max_{t_1,t_2\in I} \big\|H(h(t_2) + \varepsilon(t_2)) - H(h(t_1) + \varepsilon(t_1))\big\| \le \delta + \|\varepsilon\|_{L^1}.$$

Therefore the open ball in L^1_m of center h and radius $2 - \delta$ is contained in $\widetilde{D}$ and the claim follows.

The open ball of L^1_m of center zero and radius 2 is contained in $\widetilde{D}$. To see this, consider first any map $g \in L^1(I, \mathbb{R})$ such that $\int_0^T g(t)\, dt = 0$. Then, define

$$g_+(t) = \begin{cases} g(t) & \text{if } g(t) \ge 0 \\ 0 & \text{if } g(t) < 0 \end{cases} \qquad \text{and} \qquad g_-(t) = \begin{cases} 0 & \text{if } g(t) \ge 0 \\ -g(t) & \text{if } g(t) < 0. \end{cases}$$

As $\int_0^T g(t)\, dt = 0$, one has $\|g_+\|_{L^1} = \|g_-\|_{L^1} = \frac{1}{2}\|g\|_{L^1}$. In addition,

$$\left| \int_0^t g(s)\, ds \right| \le \|g_+\|_{L^1} \quad \forall t \in I.$$

Hence

$$\|H(g)\|_0 \le \frac{1}{2}\|g\|_{L^1}.$$

Now consider any $h = (h_1, \dots, h_N) \in L^1_m$, with $\|h\|_{L^1} < 2$. It is immediate to check that

$$\|H(h)\|_0 \le \frac{1}{2}\|h\|_{L^1} < 1,$$

and this proves the assert.

The closure of the open ball of L^1_m of center zero and radius 2 is not contained in $\widetilde{D}$. Indeed, it is obvious that the constant map $h(t) = 2/T$ is not in $\widetilde{D}$.

Coming back to problem (3.1), we have seen that it admits a solution only if h belongs to $\widetilde{D}$. Then, any C^1 solution u can be written as

$$u(t) = u(0) + \int_0^t \phi^{-1}(a + H(h)(s))\,ds.$$

The boundary condition $u(0) = u(T)$ implies that

$$\int_0^T \phi^{-1}(a + H(h)(t))\,dt = 0. \tag{3.4}$$

Thus (3.1) has a solution in $\mathcal{C}^1_T$ if and only if h belongs to the subset D of $\widetilde{D}$ defined as the set of functions $h \in \widetilde{D}$ such that there exists $a \in \mathbb{R}^N$ verifying (3.4). The next result lists some properties of D.

Proposition 3.2. *The following conditions hold.*

(1) *For any $h \in D$ the point $a \in \mathbb{R}^N$ such that*

$$\int_0^T \phi^{-1}\big(a + H(h)(t)\big)\,dt = 0$$

is unique and then defines a map $\alpha : D \to \mathbb{R}^N$ which is bounded and continuous.

(2) *The set D is open, unbounded in L^1_m, and contains the open ball in L^1_m with center zero and radius $2/3$.*

Proof. (1) Recalling Remark 2.2, the map $\psi(x) = w(\|x\|)x$ is a homeomorphism between $\mathbb{R}^N$ and the open ball in $\mathbb{R}^N$ with center zero and radius 1, and we have

$$\phi(x) = A\psi(x) = \psi(Ax).$$

Let $h \in D$ be given and consider the function

$$G_{H(h)}(a) = \int_0^T \psi^{-1}\big(a + H(h)(t)\big)\,dt \tag{3.5}$$

which is well defined and continuous on the set

$$\big\{a \in \mathbb{R}^N : \|a + H(h)(t)\| < 1 \ \forall t \in I\big\}.$$

We have

$$\big\langle G_{H(h)}(a_1) - G_{H(h)}(a_2), a_1 - a_2\big\rangle > 0, \quad \text{if } a_1 \neq a_2. \tag{3.6}$$

Indeed,

$$\begin{aligned}&\big\langle G_{H(h)}(a_1) - G_{H(h)}(a_2), a_1 - a_2\big\rangle\\ &= \int_0^T \langle \psi^{-1}(a_1 + H(h)(t)) - \psi^{-1}(a_2 + H(h)(t)), a_1 + H(h)(t) - (a_2 + H(h)(t))\rangle\,dt > 0.\end{aligned}$$

The last inequality is a consequence of Lemma 2.3 and thus, by (3.6), $G_{H(h)}(a) = 0$ has a unique solution. Since

$$\int_0^T \phi^{-1}(a + H(h)(t))\,dt = A^{-1}G_{H(h)}(a) \tag{3.7}$$

and A is an isomorphism, it follows that the unique a such that $G_{H(h)}(a) = 0$ coincides with the unique a such that

$$\int_0^T \phi^{-1}(a + H(h)(t))\, dt = 0.$$

Thus it turns out we defined well the map $\alpha : D \to \mathbb{R}^N$ which associates to any $h \in D$ the unique $a \in \mathbb{R}^N$ such that the above equality holds. Clearly α is a bounded map, whose image is contained in the open ball in $\mathbb{R}^N$ with center zero and radius 1.

To see the continuity of α we proceed as follows. Define the set

$$C = \Big\{ l \in \mathcal{C}_{T,0} : \exists a \in \mathbb{R}^N \text{ with } \|a + l(t)\| < 1 \ \forall t \in I \text{ and } \int_0^T \psi^{-1}(a + l(t))\, dt = 0 \Big\}, \tag{3.8}$$

where ψ is as above. Recalling the equality (3.7), we have

$$C = \Big\{ l \in \mathcal{C}_{T,0} : \exists a \in \mathbb{R}^N \text{ with } \|a + l(t)\| < 1 \ \forall t \in I \text{ and } \int_0^T \phi^{-1}(a + l(t))\, dt = 0 \Big\}.$$

Consider the function $\widetilde{\alpha} : C \to \mathbb{R}^N$ such that, for each $l \in C$,

$$\int_0^T \psi^{-1}(\widetilde{\alpha}(l) + l(t))\, dt = 0.$$

Let us prove the continuity of $\widetilde{\alpha}$. Let $\{l_n\}$ be a sequence in C, converging to $l \in C$. Since $\widetilde{\alpha}$ is bounded, any subsequence of $\widetilde{\alpha}(l_n)$ admits a convergent subsequence, say $\widetilde{\alpha}(l_{n_j}) \to \widehat{a}$ as $j \to \infty$. Let us show that $\psi^{-1}(\widehat{a} + l(t))$ is well defined. To this purpose, define $\overline{a} = \widetilde{\alpha}(l)$ and call B an open ball centered at $\overline{a}$ such that G_l is well defined on $\overline{B}$, where G_l is given in (3.5). As seen for (3.6), Lemma 2.3 implies that $\langle G_l(a), a - \overline{a} \rangle > 0$ for each $a \in \overline{B}$, $a \neq \overline{a}$. In particular

$$\langle G_l(a), a - \overline{a} \rangle > 0 \quad \forall a \in \partial B. \tag{3.9}$$

Observe that there exists a neighborhood U of l in $\mathcal{C}_{T,0}$ such that, for each $x \in U$, G_x is well defined on $\overline{B}$. In addition, the map

$$x \mapsto \inf_{a \in \partial B} \langle G_x(a), a - \overline{a} \rangle,$$

is easily seen to be continuous on U. Then

$$\langle G_m(a), a - \overline{a} \rangle > 0 \quad \forall a \in \partial B,$$

for each function m in a suitable neighborhood $V \subseteq U$ of l. This implies, by a simple application of the homotopy invariance property of the Brouwer degree, that the equation $G_m(a) = 0$ has its (unique) solution in $\overline{B}$, given m in V. Hence $\widetilde{\alpha}(l_{n_j}) \in \overline{B}$, for j sufficiently large, and thus $\widehat{a}$ belongs to $\overline{B}$. Therefore $\psi^{-1}(\widehat{a} + l(t))$ is well defined. Now, by letting $j \to \infty$ in

$$\int_0^T \psi^{-1}(\widetilde{\alpha}(l_{n_j}) + l_{n_j}(t))\, dt = 0,$$

we have

$$\int_0^T \psi^{-1}(\widehat{\alpha} + l(t))\, dt = 0,$$

and this proves the continuity of $\widetilde{\alpha}$. Finally, the definition of C implies that $\alpha = \widetilde{\alpha} \circ H$ and this shows the continuity of α, being H continuous.

(2) To prove that D is open in L^1_m, we first observe that the set C, defined by (3.8), is open. Indeed, this can be proved by the same argument following inequality (3.9). Now, as $D = H^{-1}(C)$, we see that D is open in L^1_m.

The unboundedness of D can be proved in the same way as done for $\widetilde{D}$. Precisely, for simplicity let $T = 1$, and take the sequence of real functions $\{h_n\}$, defined by formula (3.3). Then, let $\{k_n\} \subseteq L^1_m$ be given by $k_n = (h_n, 0, \dots, 0)$, $n \in \mathbb{N}$. For any n the function

$$G_n(a) = \int_0^1 \psi^{-1}(a + H(k_n)(t))\, dt$$

is well defined, in particular, for any a of the form $a = (a_1, 0, \dots, 0)$, with $a_1 \in (-1, 1/2)$. Denote by $G_{n,j}$, $j = 1, \dots, N$, the j-th component of G_n. If a is selected as above, we have

$$G_{n,j}(a) = 0$$

for any a and any $j \geq 2$. In addition, $G_{n,1}(a) > 0$ if $a_1 \geq 0$ and $G_{n,1}(a) < 0$ if $a_1 \leq -1/2$. As $G_{n,1}$ is continuous, it admits a zero for a suitable a. Therefore $\{k_n\} \subseteq D$, which turns out to be not bounded.

In order to show that D contains the open ball in L^1_m centered at zero with radius $2/3$ we first prove that the set C, defined by (3.8), contains the open ball in $\mathcal{C}_{T,0}$ of center zero and radius $1/3$. Let $l \in \mathcal{C}_{T,0}$, with $\|l\|_0 < 1/3$, be given. If l is identically zero, then it clearly belongs to C. Thus, suppose that l is not zero for some t. Denote $\delta = \|l\|_0$. Then consider $2\delta < \delta' < 2/3$ and let A be the closed ball in $\mathbb{R}^N$ with center zero and radius δ'. Observe that $\|a + l(t)\| < 1$ for any $t \in I$ and any $a \in A$. We show now that

$$\langle G_l(a), a \rangle > 0, \quad \text{if } \|a\| = \delta'. \tag{3.10}$$

To this purpose denote by $v : [0, 1) \to \mathbb{R}$ the function such that $\psi^{-1}(x) = v(\|x\|)x$. We have

$$\begin{aligned}
\langle G_l(a), a \rangle &= \int_0^T \langle \psi^{-1}(a + l(t)), a + l(t) \rangle\, dt - \int_0^T \langle \psi^{-1}(a + l(t)), l(t) \rangle\, dt \\
&\geq \int_0^T v(\|a + l(t)\|)\|a + l(t)\|^2\, dt - \int_0^T v(\|a + l(t)\|)\|a + l(t)\|\, \|l(t)\|\, dt \\
&= \int_0^T v(\|a + l(t)\|)\|a + l(t)\|\, (\|a + l(t)\| - \|l(t)\|)\, dt.
\end{aligned}$$

The last integral turns out to be positive if we show that, given a with $\|a\| = \delta'$,

$$\|a + l(t)\| > \|l(t)\| \quad \forall t \in I. \tag{3.11}$$

We have

$$\|a + l(t)\|^2 \geq \|a\|^2 + \|l(t)\|^2 - 2\|a\|\, \|l(t)\| \geq \|l(t)\|^2$$

because $\|a\| > 2\|l(t)\|$ for each t. Hence (3.11) holds and this proves (3.10). Therefore, by an elementary topological degree argument, the equation $G_l(a) = 0$ has a solution in A and hence $l \in C$. Thus, C contains the open ball in $\mathcal{C}_{T,0}$ of center zero and radius $1/3$.

Now, let $h = (h_1, \dots, h_N) \in L^1_m$ with $\|h\|_{L^1} < 2/3$. Define, for $i = 1, \dots, N$,

$$h_i^+(t) = \begin{cases} h_i(t) & \text{if } h_i(t) \geq 0 \\ 0 & \text{if } h_i(t) < 0 \end{cases} \quad \text{and} \quad h_i^-(t) = \begin{cases} 0 & \text{if } h_i(t) \geq 0 \\ -h_i(t) & \text{if } h_i(t) < 0. \end{cases}$$

As $\int_0^T h(t)\,dt = 0$, one has, for any i, $\|h_i^+\|_{L^1} = \|h_i^-\|_{L^1} = \frac{1}{2}\|h_i\|_{L^1}$ and thus

$$\|h\|_{L^1} = 2\left(\sum_{i=1}^N \|h_i\|_{L_1}^2\right)^{1/2}.$$

In addition,

$$\left|\int_0^t h_i(t)\,dt\right| \leq \|h_i^+\|_{L^1} \quad \forall t \in I,\ i = 1, \dots N.$$

Then

$$2\,\|H(h)(t)\|_{\mathbb{R}^N} \leq \|h\|_{L^1} \quad \forall t \in I,$$

and, finally,

$$2\,\|H(h)\|_0 \leq \|h\|_{L^1}.$$

This proves that D contains the open ball in L^1_m with center zero and radius $2/3$. ∎

For any $h \in D$, we have infinite solutions of (3.1) which differ by a constant and can be written as

$$u(t) = u(0) + H\big(\phi^{-1}[\alpha(h) + H(h)]\big)(t),$$

where, by an abuse of notation, $\phi^{-1}[\alpha(h) + H(h)]$ is the continuous map $t \mapsto \phi^{-1}[\alpha(h) + H(h)(t)]$.

Define $P : \mathcal{C}^1_T \to \mathcal{C}^1_T$ as $Pu = u(0)$. Observe that $\mathcal{C}^1_T$ admits the splitting

$$\mathcal{C}^1_T = E_1 \oplus E_2, \tag{3.12}$$

where E_1 contains the maps $\widetilde{u}$ such that $\widetilde{u}(0) = 0$ and E_2 is the N-dimensional subspace of constant maps. It is immediate to see that P is the continuous projection onto E_2 by the above decomposition.

Consider $Q : L^1 \to L^1$, defined as $Qh = \frac{1}{T}\int_0^T h(t)\,dt$. One can split L^1 as

$$L^1 = L^1_m \oplus F_2,$$

where F_2 is the N-dimensional subspace of constant maps([1]). The operator Q turns easily out to be the continuous projection on F_2 in the above splitting of L^1. Then, consider the subset $\widehat{D}$ of L^1, given by

$$\widehat{D} = D + F_2, \tag{3.13}$$

and the nonlinear operator $K : \widehat{D} \to \mathcal{C}^1_T$, defined as

$$K(\widehat{h})(t) = H\big(\phi^{-1}\big[\alpha((I-Q)\widehat{h}) + H((I-Q)\widehat{h})\big]\big)(t).$$

([1]) The reader could notice that E_2 and F_2 are actually different, being contained in different Banach spaces.

If a C^1 function u is a solution of (3.1), for a given $h \in D$, of course u solves the equation

$$u = Pu + Qh + K(h). \tag{3.14}$$

Conversely, if $u \in \mathcal{C}^1_T$ is a solution of (3.14), for a given $h \in \widehat{D}$, it follows that h belongs to D and u solves (3.1). The idea of studying equation (3.14), in order to find a solution of (3.1), is particularly important if we consider an abstract periodic problem

$$\begin{cases} (\phi(u'))' = G(u)(t) \\ u(0) = u(T) \\ u'(0) = u'(T), \end{cases} \tag{3.15}$$

where $G : \mathcal{C}^1 \to \widehat{D}$ can be supposed continuous. In fact, if we define $\mathcal{G} : \mathcal{C}^1_T \to \mathcal{C}^1_T$ by

$$\mathcal{G}(u) = Pu + QG(u) + K(G(u)),$$

we observe that problem (3.15) is equivalent to the fixed point problem $u = \mathcal{G}(u)$, which can be studied, under suitable conditions, by topological methods. Following this idea, in the next section we will apply the Leray–Schauder degree to obtain our main result, that is, as said in the Introduction, an existence theorem for (1.1).

We conclude this section showing some important properties of K.

Proposition 3.3. *The map K is continuous and sends equi-integrable sets of $\widehat{D}$ into relatively compact sets in $\mathcal{C}^1_T$.*

Proof. The continuity of K as valued in $\mathcal{C}$ is a straightforward consequence of the fact that this map is a composition of continuous maps. In addition

$$(K(\widehat{h}))'(t) = \phi^{-1}\big[\alpha((I-Q)\widehat{h}) + H((I-Q)\widehat{h})\big](t).$$

That is, K' is a composition of continuous operators and thus K is continuous. Consider an equi-integrable set S of L^1, contained in $\widehat{D}$, and let $g \in L^1(I, \mathbb{R})$ be such that

$$\forall h \in S \quad \|h(t)\| \le g(t) \quad \text{a.e. in } I.$$

Let us show that $\overline{K(S)}$ is compact. To see this consider first a sequence $\{k_n\}$ of $K(S)$ and let $\{h_n\}$ be such that $K(h_n) = k_n$. For any $t_1, t_2 \in I$ we have

$$\begin{aligned} \|H(I-Q)(h_n)(t_1) - H(I-Q)(h_n)(t_2)\| &\le \left\| \int_{t_2}^{t_1} h_n(s)\,ds \right\| + \|Qh_n\|\,|t_1 - t_2| \\ &\le \left| \int_{t_2}^{t_1} g(s)\,ds \right| + \frac{|t_1 - t_2|}{T} \int_0^T g(s)\,ds. \end{aligned}$$

Therefore the sequence $\{H(I-Q)(h_n)\}$ is bounded and equicontinuous and then, by Ascoli–Arzelà Theorem, it admits a convergent subsequence in $\mathcal{C}$, say $\{H(I-Q)(h_{n_j})\}$. Up to a subsequence, $\{\alpha((I-Q)(h_{n_j})) + H((I-Q)(h_{n_j})\}$ converges in $\mathcal{C}$. In addition

$$(K(h_{n_j}))'(t) = \phi^{-1}\big[\{\alpha((I-Q)(h_{n_j})) + H((I-Q)(h_{n_j})\}\big](t)$$

and, by the continuity of ϕ^{-1}, $(K(h_{n_j}))'$ is convergent in $\mathcal{C}$. Therefore $\{k_{n_j}\} = \{K(h_{n_j})\}$ converges in $\mathcal{C}^1_T$. Now consider a sequence $\{k_n\}$ belonging to $\overline{K(S)}$ (that is, not necessarily to $K(S)$). Let $\{l_n\} \subseteq K(S)$ be such that $\|l_n - k_n\|_1 \to 0$ as $n \to \infty$. Let in addition $\{l_{n_j}\}$

be a subsequence of $\{l_n\}$ that converges to l. Therefore, $l \in \overline{K(S)}$ and $\{k_{n_j}\} \to l$, and this completes the proof. ■

4. Main result. In this section we present the main result of this paper, that is, an existence theorem for the periodic boundary value problem

$$\begin{cases} (\phi(u'))' = f(t,u,u') \\ u(0) = u(T) \\ u'(0) = u'(T), \end{cases} \tag{4.1}$$

where ϕ is as in the above section and $f : I \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function, that is,

i) for almost every $t \in I$, $f(t,\cdot,\cdot)$ is continuous;

ii) for any $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$, $f(\cdot,x,y)$ is measurable;

iii) for any $\rho > 0$ there exists $g \in L^1(I,\mathbb{R})$ such that, for almost every $t \in I$ and every$(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$, with $\|x\| \le \rho$ and $\|y\| \le \rho$, we have

$$\|f(t,x,y)\| \le g(t).$$

THEOREM 4.1. *Let Ω be an open subset of $\mathcal{C}^1_T$ such that the following conditions hold:*

(1) *for any $u \in \Omega$ the map $t \mapsto f(t,u(t),u'(t))$ belongs to $\widehat{D}$, where $\widehat{D}$ is given by* (3.13);

(2) *denoting by S the set of pairs (u,λ) such that $\lambda \in (0,1]$, $u \in \Omega$ and solves the problem*

$$\begin{cases} (\phi(u'))' = \lambda f(t,u,u') \\ u(0) = u(T) \\ u'(0) = u'(T), \end{cases} \tag{4.2}$$

suppose that the closure $\overline{S}$ in $\mathcal{C}^1_T \times [0,1]$ is bounded and contained in $\Omega \times [0,1]$;

(3) *the set of the solutions of the equation*

$$F(a) := \int_0^T f(t,a,0)\,dt = 0 \tag{4.3}$$

is compact in Ω_2, where $\Omega_2 := \Omega \cap E_2$ and E_2 is the subspace of $\mathcal{C}^1_T$ in the splitting (3.12);

(4) *the Brouwer degree $\deg_B(F,\Omega_2,0)$ is well defined and non-zero.*

Then problem (4.1) *has a solution in Ω.*

Proof. Let N_f denote the Nemytski operator associated to f, that is,

$$N_f : \mathcal{C}^1_T \to L^1, \quad N_f(u)(t) = f(t,u(t),u'(t)).$$

Consider the problem

$$\begin{cases} (\phi(u'))' = \lambda N_f(u) + (1-\lambda)QN_f(u) \\ u(0) = u(T) \\ u'(0) = u'(T). \end{cases} \tag{4.4}$$

For $\lambda \in (0,1]$, if u is a solution of (4.2), then, as seen in the previous section, condition $u'(0) = u'(T)$ implies $QN_f(u) = 0$ and hence u solves problem (4.4) as well. Conversely, if u is a solution of (4.4), then $QN_f(u) = 0$ since it is easy to see that

$$Q\big[\lambda N_f(u) + (1-\lambda)QN_f(u)\big] = QN_f(u),$$

and thus u solves (4.2) (λ still belongs to $(0,1]$). Let us now consider problem (4.4). It can be written in the equivalent form

$$u = \mathcal{K}(u,\lambda), \tag{4.5}$$

where

$$\begin{aligned}\mathcal{K}(u,\lambda) = Pu + QN_f(u) + (K\circ[\lambda N_f + (1-\lambda)QN_f])(u) \\ = Pu + QN_f(u) + \big(K\circ[\lambda(I-Q)N_f]\big)(u)\end{aligned}$$

is well defined in $\Omega\times[0,1]$. Observe that the last equality in the above formula is a consequence of the fact that $K(g) = K(g+Qg)$ for any $g\in D$.

Since f is Carathéodory, the nonlinear map $\mathcal{N} : \mathcal{C}^1_T\times[0,1]\to L^1$, defined by

$$\mathcal{N}(u,\lambda) = \lambda N_f(u) + (1-\lambda)QN_f(u), \tag{4.6}$$

is continuous and takes bounded sets into equi-integrable sets. This implies that, recalling Proposition 3.3, $\mathcal{K}$ is completely continuous and, consequently, the map $\mathcal{F} : \Omega\times[0,1]\to \mathcal{C}^1_T$, defined as

$$\mathcal{F}(u,\lambda) = u - \mathcal{K}(u,\lambda)$$

is proper on closed bounded subsets of its domain (we mean closed in $\mathcal{C}^1_T\times[0,1]$).

Take $\lambda = 0$. We have

$$\mathcal{F}(u,0) = u - u(0) - \frac{1}{T}\int_0^T f(t,u(t),u'(t))\,dt$$

(recall that $K(c) = 0$ if c is a constant map). Therefore, u is a solution of the equation $\mathcal{F}(u,0) = 0$ if and only if u is constant. It follows that $\int_0^T f(t,u(t),u'(t))\,dt = 0$, that is,

$$\int_0^T f(t,c,0)\,dt = 0,$$

where $u(t) = c$. Thus, by assumption (3), we can say that the set of solutions of $\mathcal{F}(u,0) = 0$ is a compact subset of Ω. Then, by assumption (2) and the above argument, we deduce that $\mathcal{F}^{-1}(0)$ is bounded and closed in the topology of $\mathcal{C}^1_T\times[0,1)$. It is not difficult to prove, by the properness of $\mathcal{F}$, that $\mathcal{F}^{-1}(0)$ is also compact and contained in $\Omega\times[0,1]$. Thus, we can apply the homotopy invariance property of the Leray–Schauder degree to $\mathcal{F}$ obtaining

$$\deg_{LS}(I-\mathcal{K}(\cdot,0),\Omega,0) = \deg_{LS}(I-\mathcal{K}(\cdot,1),\Omega,0). \tag{4.7}$$

Therefore, (4.1) has a solution in Ω if we prove that $\deg_{LS}(I-\mathcal{K}(\cdot,1),\Omega,0)\neq 0$. To this purpose we show that $\deg_{LS}(I-\mathcal{K}(\cdot,0),\Omega,0)\neq 0$. To see this we apply a finite-dimensional reduction property of the Leray–Schauder degree, associated with assumption (3). The operator $I-\mathcal{K}(\cdot,0)$ can be represented in block-matrix form as

$$I-\mathcal{K}(\cdot,0) = \begin{pmatrix} I_{E_1} & -\mathcal{K}_{12} \\ 0 & -F \end{pmatrix}.$$

By the properties of the Leray–Schauder degree we have

$$\deg_{LS}(I - \mathcal{K}(\cdot, 0), \Omega, 0) = (-1)^N \deg_B(F, \Omega_2, 0)$$

and this completes the proof. ■

5. An application. In this section we show an application of Theorem 4.1 to the two-dimensional problem

$$\begin{cases} \left(\dfrac{u_1'}{\sqrt{1+|u'|^2}}\right)' = g_1(t)(\arctan u_1 - h_1(u_1')), \\ \left(\dfrac{u_2'}{\sqrt{1+|u'|^2}}\right)' = g_2(t)(\arctan u_2 - h_2(u_2')), \\ u_1(0) = u_1(1), \quad u_1'(0) = u_1'(1), \\ u_2(0) = u_2(1), \quad u_2'(0) = u_2'(1), \end{cases} \tag{5.1}$$

where g_1, g_2 are positive, continuous, real functions on $[0, 1]$, h_1, h_2 are bounded, continuous, real functions defined on $\mathbb{R}$. Suppose, in addition, that $\sup|h_1(t)|$ and $\sup|h_2(t)|$ are less than $\pi/2$, and assume that $g_1(t)$ and $g_2(t)$ are less than $2/(3\pi)$ for any $t \in [0, 1]$.

REMARK 5.1. Recalling Remark 1.1, if $u \in \mathcal{C}^1_T$ solves system (5.1), $\phi(u')$ is absolutely continuous (ϕ being defined as $\phi(t) = t/\sqrt{1+t^2}$). It is immediate to verify that u' is absolutely continuous as well. Now, observe that u'' coincides a.e. with a continuous function and thus it can be continuously extended to $[0, 1]$. This implies that u' is actually C^1 and then any solution of the problem is actually a C^2 function. Therefore, system (5.1) can be written in the following equivalent way:

$$\begin{cases} u_1''\left(1 + (u_2')^2\right) - u_1'u_2'u_2'' = \left(1 + |u'|^2\right)^{3/2} [g_1(t)(\arctan u_1 - h_1(u_1'))] \\ u_2''\left(1 + (u_1')^2\right) - u_1'u_2'u_1'' = \left(1 + |u'|^2\right)^{3/2} [g_2(t)(\arctan u_2 - h_2(u_2'))], \\ u_1(0) = u_1(1), \quad u_1'(0) = u_1'(1), \\ u_2(0) = u_2(1), \quad u_2'(0) = u_2'(1). \end{cases} \tag{5.2}$$

In the attempt of applying Theorem 4.1 to problem (5.1), fix $\alpha > 0$ and let Ω be the open subset of $\mathcal{C}^1_T$ of maps u such that $\|u'\|_0 < \alpha$. Our purpose is to prove the existence in Ω of a solution of (5.1).

Suppose that a given $u \in \Omega$ solves (5.1). If u is not identically zero, without loss of generality let $t_0 \in [0, 1]$ be such that

$$|u_1(t_0)| = \max\{|u_1(t)|, |u_2(t)|, \ t \in [0, 1]\}.$$

Observe that $u_1'(t_0) = 0$ (this holds even in the case when t_0 coincides with 0 or 1, because of the condition $u_1'(0) = u_1'(1)$). Now, if $\|u\|_1$ is sufficiently large, that is, if $|u_1(t_0)|$ is sufficiently large, then

$$\begin{aligned} &\arctan u_1(t_0) - h_1(u_1'(t_0)) > 0 \quad \text{if } u_1(t_0) > 0, \\ &\arctan u_1(t_0) - h_1(u_1'(t_0)) < 0 \quad \text{if } u_1(t_0) < 0. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} u_1''(t_0) &\le 0 \quad \text{if } u_1(t_0) > 0, \\ u_1''(t_0) &\ge 0 \quad \text{if } u_1(t_0) < 0, \end{aligned}$$

and this contradicts the fact that u is a solution of (5.1). Therefore the set of solutions of (5.1) is bounded and closed in Ω (of course Ω is unbounded in $\mathcal{C}^1_T$). For the same reason the set S of pairs (u, λ) such that $0 < \lambda \le 1$, $u \in \Omega$ and solves the problem

$$\begin{cases} \left(\dfrac{u_1'}{\sqrt{1 + |u'|^2}} \right)' = \lambda g_1(t)(\arctan u_1 - h_1(u_1')), \\ \left(\dfrac{u_2'}{\sqrt{1 + |u'|^2}} \right)' = \lambda g_2(t)(\arctan u_2 - h_2(u_2')), \\ u_1(0) = u_1(1), \quad u_1'(0) = u_1'(1), \\ u_2(0) = u_2(1), \quad u_2'(0) = u_2'(1) \end{cases} \tag{5.3}$$

is such that $\overline{S}$ (the closure is in $\mathcal{C}^1_T \times [0, 1]$) is bounded and contained in $\Omega \times [0, 1]$.

Moreover, recalling points (3) and (4) in the statement of Theorem 4.1, the set of solutions of the equation

$$F(a, b) = \Big(\int_0^1 g_1(t)\, dt \arctan a + h_1(0), \int_0^1 g_2(t)\, dt \arctan b + h_2(0) \Big) = (0, 0)$$

is bounded and compact in $\Omega_2 = \Omega \cap E_2$, where E_2 is the subspace of $\mathcal{C}^1_T$ in the splitting (3.12). It is immediate to see that $\deg_B(F, \Omega_2, 0) = 1$.

Thus we can apply Theorem 4.1 to conclude that (5.1) admits a solution in Ω. It is also immediate to observe that any solution is non-trivial if $h_1(0)$ and $h_2(0)$ are not both zero.

References

[1] P. Benevieri, J. M. do Ó, E. S. Medeiros, *Periodic solutions for nonlinear equations with mean curvature-like operators*, preprint.

[2] P. Benevieri, J. M. do Ó, E. S. Medeiros, *Periodic solutions for nonlinear systems with mean curvature-like operators*, Nonlinear Anal., to appear.

[3] C. Bereanu, J. Mawhin, *Nonlinear Neumann boundary value problems with ϕ-Laplacian operators*, An. Ştiinţ. Univ. Ovidius Constanţa Ser. Mat. 12 (2004), 73–82.

[4] C. Bereanu, J. Mawhin, *Boundary value problems with non-surjective ϕ-Laplacian and one-sided bounded nonlinearity*, Adv. Differential Equations 11 (2006), 35–60.

[5] C. Bereanu, J. Mawhin, *Dirichlet or periodic problems for some nonlinear perturbations of diffeomorphic one-dimensional mean curvature-like operators.*

[6] K. Deimling, *Nonlinear Functional Analysis*, Springer, Berlin 1985.

[7] N. Lloyd, *Degree Theory*, Cambridge Univ. Press, Cambridge 1978.

[8] R. Manásevich, J. Mawhin, *Periodic solutions for nonlinear systems with p-Laplacian-like operators*, J. Differential Equations 145 (1998), 367–393.

[9] J. Mawhin, *Periodic solutions of systems with p-Laplacian-like operators*, in: Nonlinear Analysis and its Application to Differential Equations (Lisbon 1998), Progr. Nonlinear Differential Equations Appl. 43, Birkhäuser, Boston 2001, 37–63.