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REMARKS ON A BOUNDARY VALUE PROBLEM IN BANACH SPACES ON THE HALF-LINE

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Abstract. In this paper we investigate the linear initial value problem in Banach spaces. In order to obtain existence results the Fredholm operator technique is used.

In the paper [3] the authors consider the problem of the existence of solutions in Sobolev space $H^1(\overline{\mathbb{R}}_+, \mathbb{R}^N)$ for the ODE system

$$\begin{cases} \dot{u} + F(t,u) = f(t), & t \in \overline{\mathbb{R}}_+ = [0,\infty), \\ u_1(0) = \zeta, \end{cases}$$

where $F : \overline{\mathbb{R}}_+ \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a C^1 mapping such that $F(t,0) = 0$ for all $t \in \overline{\mathbb{R}}_+$, and u_1 is the component of u along the first factor of a given splitting $\mathbb{R}^N = X_1 \oplus X_2$. Both $f \in L^2(\mathbb{R}_+, \mathbb{R}^N)$ and $\zeta \in X_1$ are given.

In the case $X_1 = \mathbb{R}^N$ and $X_2 = \{0\}$ the above problem is the classical initial value problem. Our aim is to generalize this problem to the infinite-dimensional case. In this work we investigate only the linear equation.

Let us fix the notation.

Let $(E, \|\cdot\|)$ be a real separable Banach space. Denote by $\mathcal{L}(E)$ the space of linear bounded operators on E and by $\mathcal{L}_c(E)$ the subspace of $\mathcal{L}(E)$ consisting of compact operators.

For a measurable function $u : (a,b) \longrightarrow E$, $-\infty \le a < b \le \infty$, the integral $\int_a^b u(t)\,dt$ means the integral in Bochner sense. We recall that a measurable function $u(\cdot)$ is (*Bochner*) *integrable* on (a,b) if and only if the real function $\|u(\cdot)\|$ is (Lebesgue) inte-

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grable on (a,b). The vector space $L^2 = L^2(\overline{\mathbb{R}}_+, E)$ consists of all functions $u : \overline{\mathbb{R}}_+ \longrightarrow E$ such that $\|u\| \in L^2(\overline{\mathbb{R}}_+, \mathbb{R})$ with the norm $\|u\|_2 = \left(\int_0^\infty \|u(t)\|^2\,dt\right)^{1/2}$. We denote by $H^1 = H^1(\overline{\mathbb{R}}_+, E)$ the Sobolev space of such functions $u : \overline{\mathbb{R}}_+ \longrightarrow E$ that $u, \dot{u} \in L^2$. The space H^1 is equipped with the norm $\|u\|_{H^1} = (\|u\|_2^2 + \|\dot{u}\|_2^2)^{1/2}$. We recall that the derivative $\dot{u}$ means the derivative in the distributional sense.

It can be shown that if $u \in H^1$ and $\widetilde{u}$ is a continuous function such that $u = \widetilde{u}$ a.e. on $\overline{\mathbb{R}}_+$, then $\lim_{t\to\infty} \|\widetilde{u}(t)\| = 0$.

The main problem. Let $A \in \mathcal{L}(E)$ be such that $\sigma(A) \cap \mathbb{R}i = \emptyset$, where $\sigma(A)$ denotes the spectrum of the operator A. Then A gives the standard unique decomposition $E = X_+ \oplus X_-$, where the subspaces X_+ and X_- are invariant with respect to A. The operators $A_+ := A|_{X_+}$, $A_- := A|_{X_-}$ have their spectra contained in $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) > 0\}$ and $\mathbb{C}_-$, respectively. Every function $u : \overline{\mathbb{R}}_+ \longrightarrow E$ has a unique decomposition of the form $u = u_+ + u_-$, where $u_\pm : \overline{\mathbb{R}}_+ \longrightarrow X_\pm$.

PROPOSITION 1. *Under the above conditions the linear mapping*

$$H^1 \ni u \longmapsto (\dot{u} + Au, u_+(0)) \in L^2 \times X_+$$

is an isomorphism.

Proof. Fix $\zeta_+ \in X_+$, $f \in L^2$. We will show that the problem

$$\begin{cases} \dot{u} + Au = f \\ u_+(0) = \zeta_+ \end{cases} \tag{1}$$

has a unique solution $u \in H^1$. The above problem is equivalent to the conjunction of two problems:

$$\begin{cases} \dot{u}_+ + A_+u_+ = f_+ \\ u_+(0) = \zeta_+ \end{cases} \qquad \text{and} \qquad \dot{u}_- + A_-u_- = f_- \,. \tag{2}$$

The first is a Cauchy problem in the Banach space X_+ and has a unique solution given by the formula

$$u_+(t) = e^{-tA_+}\zeta_+ + \int_0^t e^{-(t-s)A_+} f_+(s)\,ds. \tag{3}$$

We will show that $u_+ \in H^1$. First we consider the function

$$\overline{\mathbb{R}}_+ \ni t \longmapsto e^{-tA_+}\zeta_+ \in X_+. \tag{4}$$

Because $\sigma(-tA|_{X_+}) = \sigma(-tA_+) \subset \mathbb{C}_-$, there are $\alpha > 0$ and $M > 0$ such that $\|e^{-tA_+}\zeta_+\| \le \|e^{-tA_+}\| \cdot \|\zeta_+\| \le Me^{-\alpha t} \cdot \|\zeta_+\|$. Therefore

$$\int_0^\infty \|e^{-tA_+}\zeta_+\|^2\,dt \le \int_0^\infty (Me^{-\alpha t} \cdot \|\zeta_+\|)^2\,dt < \infty$$

and it follows that the function (4) belongs to L^2. In order to estimate the second component of the function defined in (3) we will use the Young inequality:

$$\|g * f\|_r \le \|g\|_p \cdot \|f\|_q$$

for $1 \le p, q, r \le \infty$ such that $1/p + 1/q = 1 + 1/r$ ($\|\cdot\|_p$ is the norm in L^p).

Let us denote by g the function

$$\mathbb{R}_+ \ni s \longmapsto g(s) = e^{-sA_+} \in \mathcal{L}(E).$$

Then

$$\int_0^t e^{-(t-s)A_+} f_+(s)\, ds = (g * f_+)(t).$$

Taking in the Young inequality $r = 2$, $q = 2$, $p = 1$, we have the estimate

$$\begin{aligned}\|g * f_+\|_2 \le \|g\|_1 \cdot \|f_+\|_2 &= \int_0^\infty \|e^{-sA_+}\|\, ds \cdot \Big(\int_0^\infty \|f_+\|^2\, ds\Big)^{1/2} \\ &\le \int_0^\infty Me^{-s\alpha}\, ds \cdot \Big(\int_0^\infty \|f_+(s)\|^2\, ds\Big)^{1/2} < \infty.\end{aligned}$$

It follows that the function

$$\overline{\mathbb{R}}_+ \ni t \longmapsto \int_0^t e^{-(t-s)A_+} f_+(s)\, ds \in X_+$$

belongs to L^2, and therefore the function (3) belongs to L^2. Because $\dot{u}_+ = f_+ - A_+u_+ \in L^2$ we deduce that $u_+ \in H^1$.

The second problem of (2) is not an initial value problem but we will show that it has a unique solution in the class H^1. Any solution of

$$\dot{u}_- + A_-u_- = f_- \tag{5}$$

is given by the formula

$$u_-(t) = e^{-tA_-}u_-(0) + \int_0^t e^{-(t-s)A_-} f_-(s)\, ds = e^{-tA_-}\Big(u_-(0) + \int_0^t e^{sA_-} f_-(s)\, ds\Big).$$

We observe first that the limit

$$\lim_{t\to\infty} \int_0^t e^{sA_-} f_-(s)\, ds = \int_0^\infty e^{sA_-} f_-(s)\, ds \in X_-$$

exists (this follows from the fact that $\sigma(A_-) \subset \mathbb{C}_-$).

Next, let us notice that

$$u_-(0) + \int_0^\infty e^{sA_-} f_-(s)\, ds = 0.$$

Indeed, if the above does not hold then the norm of the expression

$$e^{-tA_-}\Big(u_-(0) + \int_0^t e^{sA_-} f_-(s)\, ds\Big)$$

tends to the infinity and the function u_- is not in L^2. Consequently u_- is not in H^1. Therefore, in order to have $u_- \in H^1$ the condition

$$u_-(0) = -\int_0^\infty e^{sA_-} f_-(s)\, ds \tag{6}$$

has to be satisfied. We can write the solution of (5) in the form

$$u_-(t) = e^{-tA_-}\Big(u_-(0) + \int_0^\infty e^{sA_-} f_-(s)\, ds - \int_t^\infty e^{sA_-} f_-(s)\, ds\Big)$$

and if we allow (6) it will take the form

$$u_-(t) = -\int_t^\infty e^{(s-t)A} f_-(s)\, ds. \tag{7}$$

Using again the Young inequality and the fact that $\sigma(A_-) \subset \mathbb{C}_-$ we prove that $u_- \in L^2$ and in consequence $u_- \in H^1$. Then the condition (6) is also sufficient to have $u_- \in H^1$. Therefore the solutions of both problems in (2) are uniquely determined in H^1. ■

We have actually proved

Corollary 1. *The semi-Cauchy problem*

$$\begin{cases} \dot{u} + Au = f \\ u_+(0) = \zeta_+; \quad \zeta_+ \in X_+,\ f \in L^2 \end{cases}$$

has a unique solution $u \in H^1$.

We are going to investigate the above problem if the pair of spaces (X_+, X_-) is perturbed via maps which are compact perturbations of the identity. For this purpose we introduce the following

Definition 1. Let (X_1, X_2), (Y_1, Y_2) be two pairs of closed subspaces of the space E such that $E = X_1 \oplus X_2 = Y_1 \oplus Y_2$. We say that the pairs (X_1, X_2) and (Y_1, Y_2) are *equivalent* if there exist a compact mapping $B \in \mathcal{L}_c(E)$ and a finite-dimensional subspace V of E such that the mapping $I + B \in \mathcal{L}(E)$ is an isomorphism and one of the following conditions (a) or (b) is satisfied

$$\begin{aligned} &\text{(a)} \quad \widehat{X}_1 = Y_1 \oplus V \text{ and } \widehat{X}_2 \oplus V = Y_2 \\ &\text{(b)} \quad \widehat{X}_1 \oplus V = Y_1 \text{ and } \widehat{X}_2 = Y_2 \oplus V, \end{aligned}$$

where $\widehat{X}_i = (I + B)(X_i)$, $i = 1, 2$.

Proposition 2. *The above relation is an equivalence relation.*

The proof is elementary, but it needs some calculations. We omit it.

Remark. If $E = \mathbb{R}^N$ then any two splittings (X_1, X_2) and (Y_1, Y_2) are equivalent in the above sense.

In what follows we will consider two decompositions of E, $X_+ \oplus X_- = X_1 \oplus X_2$. The first one is associated with the operator $A \in \mathcal{L}(E)$ satisfying $\sigma(A) \cap \mathbb{R}i = \emptyset$ and the second decomposition $E = X_1 \oplus X_2$ is such that the pairs (X_1, X_2), (X_+, X_-) are equivalent in the above sense. Choose a compact operator $B \in \mathcal{L}_c(E)$ and a finite-dimensional subspace $V \subset E$ such that $I + B$ is an isomorphism and one of the following conditions holds

$$\begin{aligned} &\text{(a)} \quad \widehat{X}_+ = X_1 \oplus V \;\text{ and } \widehat{X}_- \oplus V = X_2 \\ &\text{(b)} \quad \widehat{X}_+ \oplus V = X_1 \;\text{ and } \widehat{X}_- = X_2 \oplus V, \end{aligned} \tag{8}$$

where as earlier $\widehat{X}_\pm = (I + B)(X_\pm)$.

Let X, Y be closed subspaces of E such that $E = X \oplus Y$. By $P_X : E \longrightarrow E$ we will denote the projection onto X along Y.

Lemma 1. *The operators* $P_{X_+} - P_{\widehat{X}_+}$, $P_{X_-} - P_{\widehat{X}_-}$, $P_{\widehat{X}_+} - P_{X_1}$, $P_{\widehat{X}_-} - P_{X_2}$ *are compact.*

Proof. We have of course $E = \widehat{X}_+ \oplus \widehat{X}_-$. Then for each $w = z + Bz \in E$ we have $w = P_{\widehat{X}_+} w + P_{\widehat{X}_-} w = P_{\widehat{X}_+}(z + Bz) + P_{\widehat{X}_-}(z + Bz) = (P_{\widehat{X}_+} z + P_{\widehat{X}_+} Bz) + (P_{\widehat{X}_-} z + P_{\widehat{X}_-} Bz) = w_+ + w_- \in \widehat{X}_+ \oplus \widehat{X}_-$.

On the other hand, $w = z + Bz = P_{X_+}z + P_{X_-}z + B(P_{X_+}z + P_{X_-}z) = P_{X_+}z + BP_{X_+}z + P_{X_-}z + BP_{X_-}z = (I + B)P_{X_+}z + (I + B)P_{X_-}z = \widehat{w}_+ + \widehat{w}_- \in \widehat{X}_+ \oplus \widehat{X}_-$. From the uniqueness of the decomposition we get $w_+ = \widehat{w}_+$ and $w_- = \widehat{w}_-$. Therefore $P_{\widehat{X}_+} z + P_{\widehat{X}_+} Bz = P_{X_+}z + BP_{X_+}z$ and $P_{X_+} - P_{\widehat{X}_+} = P_{\widehat{X}_+} B - BP_{X_+} \in \mathcal{L}_c(E)$. Similarly $P_{\widehat{X}_-} z + P_{\widehat{X}_-} Bz = P_{X_-}z + BP_{X_-}z$ and $P_{X_-} - P_{\widehat{X}_-} = P_{\widehat{X}_-} B - BP_{X_-} \in \mathcal{L}_c(E)$.

Now let condition (a) from (8) be fulfilled. Then $E = X_1 \oplus \widehat{X}_- \oplus V$. For any $w \in E$ we have $w = P_{\widehat{X}_+} w + P_{\widehat{X}_-} w \in \widehat{X}_+ \oplus \widehat{X}_-$ and $w = P_{X_1}w + P_{X_2}w = P_{X_1}w + P_{\widehat{X}_-} P_{X_2}w + P_V P_{X_2}w = (P_{X_1}w + P_V P_{X_2}w) + P_{\widehat{X}_-} P_{X_2}w \in \widehat{X}_+ \oplus \widehat{X}_-$. It follows that $P_{\widehat{X}_+} w = P_{X_1}w + P_V P_{X_2}w$. Therefore $P_{\widehat{X}_+} - P_{X_1} = P_V P_{X_2} \in \mathcal{L}_c(E)$ (since P_V is finite-dimensional). Similarly $w = P_{\widehat{X}_+} w + P_{\widehat{X}_-} w = P_{X_1} P_{\widehat{X}_+} w + (P_V P_{\widehat{X}_+} w + P_{X_2}w) \in X_1 \oplus X_2$ and $w = P_{X_1}w + P_{X_2}w \in X_1 \oplus X_2$. Hence $P_V P_{\widehat{X}_+} w + P_{\widehat{X}_-} w = P_{X_2}w$ and $P_{\widehat{X}_-} - P_{X_2} = -P_V P_{\widehat{X}_+} \in \mathcal{L}_c(E)$.

If condition (b) from (8) is fulfilled the proof is similar. ∎

THEOREM 1. *Let $A \in \mathcal{L}(E)$, $\sigma(A) \cap \mathbb{R}i = \emptyset$ and (X_+, X_-), (X_1, X_2) be equivalent pairs. Then the continuous linear mapping $T : H^1 \longrightarrow L^2 \times X_1$ defined by*

$$Tu := (\dot{u} + Au, u_1(0))$$

is a Fredholm map of index

$$\operatorname{ind}(T) = \begin{cases} \dim V & \text{if } \widehat{X}_+ = X_1 \oplus V \text{ and } \widehat{X}_- \oplus V = X_2 \\ -\dim V & \text{if } \widehat{X}_+ \oplus V = X_1 \text{ and } \widehat{X}_- = X_2 \oplus V. \end{cases}$$

Proof. Assume condition (a) holds, i.e. $\widehat{X}_+ = X_1 \oplus V$ and $\widehat{X}_- \oplus V = X_2$.

Then $E = X_1 \oplus V \oplus \widehat{X}_-$. The map

$$H^1 \ni u \xrightarrow{\widehat{T}} (\dot{u} + Au, (I + B)u_+(0)) \in L^2 \times \widehat{X}_+$$

is a superposition of isomorphisms

$$H^1 \ni u \xrightarrow{T_+} (\dot{u} + Au, u_+(0)) \xrightarrow{(I, I+B)} (\dot{u} + Au, (I + B)u_+(0)) \in L^2 \times \widehat{X}_+,$$

where $T_+(u) := (\dot{u} + Au, u_+(0))$ is the isomorphism given by Proposition 1. Let $\pi : L^2 \times (X_1 \oplus V) \longrightarrow L^2 \times X_1$ be the epimorphism given by the formula $\pi(f, \widehat{\zeta}_+) = (f, P_{X_1}(\widehat{\zeta}_+))$, where $P_{X_1} : X_1 \oplus V \longrightarrow X_1$ is the projection operator and let $T_1 := \pi \circ \widehat{T}$. Then T_1 is the epimorphism which can be written by the formula

$$H^1 \ni u \longmapsto T_1 u = (\dot{u} + Au, P_{X_1}(I + B)u_+(0)) \in L^2 \times X_1.$$

We illustrate all above mappings in the following diagram

$$\begin{tikzcd}
H^1 \arrow[r, "T_+"] \arrow[rr, bend left=50, "\widehat{T}", "\text{isom}"'] \arrow[drr, "T_1"'] & L^2 \times X_+ \arrow[r, "{(I, I+B)}"] & L^2 \times \widehat{X}_+ = L^2 \times (X_1 \oplus V) \arrow[d, "{\pi = (I, P_{X_1})}"] \\
& & L^2 \times X_1
\end{tikzcd}$$

Directly from the definition of the Fredholm index one easily obtains

$$\operatorname{ind}(T_1) = \dim \operatorname{Ker}(T_1) - \operatorname{codim} \operatorname{Im}(T_1) = \dim V.$$

We will show that $\operatorname{ind}(T_1) = \operatorname{ind}(T)$. It is enough to show that $T_1 - T \in \mathcal{L}_c(E)$. Indeed,

$$\begin{aligned}(T_1 - T)(u) &= (0, (I + B)u_+(0) - u_1(0)) \\ &= (0, (I + B)P_{X_+}u(0) - P_{X_1}u(0)) = (0, (P_{X_+} - P_{X_1} + BP_{X_+})u(0)).\end{aligned}$$

Since $P_{X_+} - P_{X_1} = P_{X_+} - P_{\widehat{X}_+} + P_{\widehat{X}_+} - P_{X_1}$ and $P_{X_+} - P_{\widehat{X}_+}$, $P_{\widehat{X}_+} - P_{X_1}$, BP_{X_+} are compact, the map $T_1 - T \in \mathcal{L}_c(E)$.

Now let condition (b): $\widehat{X}_+ \oplus V = X_1$ and $\widehat{X}_- = X_2 \oplus V$ be satisfied. Then $E = \widehat{X}_+ \oplus V \oplus X_2$. Let ι: $L^2 \times \widehat{X}_+ \hookrightarrow L^2 \times (\widehat{X}_+ \oplus V) = L^2 \times X_1$ be the embedding map and let $T_2 := \iota \circ \widehat{T}$,

$$H^1 \ni u \longmapsto T_2 u = (\dot{u} + Au, (I + B)u_+(0)) \in L^2 \times X_1.$$

Consider the following diagram

$$\begin{tikzcd}
H^1 \arrow[r, "T_+"] \arrow[rr, bend left=50, "\widehat{T}", "\text{izom}"'] \arrow[drr, "T_2"'] & L^2 \times X_+ \arrow[r, "{(I, I+B)}"] & L^2 \times \widehat{X}_+ \arrow[d, "\iota"] \\
& & L^2 \times X_1 = L^2 \times (\widehat{X}_+ \oplus V)
\end{tikzcd}$$

The map T_2 is a Fredholm operator of $\operatorname{ind}(T_2) = -\dim V$ ($\dim \operatorname{Ker}(T_2) = 0$ and $\operatorname{codim} \operatorname{Im}(T_2) = \dim V$). As earlier it can be shown that $T_2 - T \in \mathcal{L}_c(E)$ and it follows that $\operatorname{ind}(T_2) = \operatorname{ind}(T)$. ∎

We can reformulate the above theorem to obtain the following result.

Corollary 2. *If T from Theorem 1 is an isomorphism then the space of solutions of the problem*

$$\begin{cases} \dot{u} + Au = f \\ u_1(0) = \zeta_1; \quad \zeta_1 \in X_1,\ f \in L^2 \end{cases}$$

is of dimension $\operatorname{ind}(T)$. *In particular, if* $\operatorname{ind}(T) = 0$ *then this problem has a unique solution in H^1.*

The nonlinear case of (1) will be discussed in the next paper.

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