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THE COMMON FIXED POINT SET OF COMMUTING NONEXPANSIVE MAPPINGS IN CARTESIAN PRODUCTS OF SEPARABLE SPACES

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Abstract. In this note we derive a theorem about the common fixed point set of commuting nonexpansive mappings defined in Cartesian products of separable spaces. The proof is based on a method due to R. E. Bruck.

1. Introduction. In [1] R. E. Bruck noticed, for the first time, the connection between fixed point sets of nonexpansive mappings and nonexpansive retracts and applied this fact in his proof of a deep result about the common fixed point set of any commuting family of nonexpansive mappings ([3], see also [2] and [4]). In our paper we extend Bruck's latter result about the common fixed point set to the case of a Cartesian product of Banach spaces with the maximum norm.

2. Basic notation and facts. We begin with a few definitions about some classes of mappings and their fixed points.

Let C be a nonempty closed convex subset of a Banach space $(X, \|\cdot\|)$. A mapping $T : C \to C$ is *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$.

If there exists $x \in C$ such that $Tx = x$, then we say that x is a *fixed point* of T. The set of all fixed points of T is denoted by $\mathrm{Fix}(T)$.

Let $T_1, T_2 : C \to C$ be nonexpansive. We say that a nonempty closed convex subset D of C is T_1*-invariant* if $T_1(D) \subset D$. If every nonempty, closed, convex and T_1-invariant subset D of C is also T_2-invariant, then we say that the mapping T_2 is T_1*-invariant*.

Let $\mathcal{S}$ be a family of nonexpansive self-mappings of C. If D is a nonempty closed convex subset of C and D is S-invariant for each $S \in \mathcal{S}$, then we say that D is $\mathcal{S}$*-invariant*. If $T : C \to C$ is nonexpansive and each nonempty, closed, convex and $\mathcal{S}$-invariant subset D of C is also T-invariant, then we say that the mapping T is $\mathcal{S}$*-invariant*.

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Let $\mathcal{S}$ and $\mathcal{S}'$ be two families of nonexpansive self-mappings of C. If each mapping $S' \in \mathcal{S}'$ is $\mathcal{S}$-invariant, then we say that the family $\mathcal{S}'$ is *$\mathcal{S}$-invariant*.

The set C has the *fixed point property for nonexpansive mappings* (*FPP*) if each nonexpansive mapping $T : C \to C$ has a fixed point.

In [2] R. E. Bruck introduced a definition of a conditional fixed point property for nonexpansive mappings.

Let C be a nonempty closed convex subset of a Banach space $(X, \|\cdot\|)$. C has the *conditional fixed point property for nonexpansive mappings* (*CFPP*) if each nonexpansive $T : C \to C$ either has no fixed points in C or has a fixed point in every nonempty bounded closed convex T-invariant subset of C.

We also have the following definition of a generic fixed point property [10].

Let C be a nonempty bounded closed convex subset of a Banach space $(X, \|\cdot\|)$. C has the *generic fixed point property for nonexpansive mappings* (*GFPP*) if C has FPP and CFPP.

Now, we recall a basic definition of a nonexpansive retract.

Let C be a nonempty bounded closed convex subset of a Banach space $(X, \|\cdot\|)$ and $\emptyset \neq F \subset C$. If there exists a retraction of C onto F which is nonexpansive, then we call F a *nonexpansive retract*.

If a mapping $T : C \to C$ is nonexpansive and if there exists a T-invariant nonexpansive retraction of C onto F, then we call F a *T-invariant nonexpansive retract*.

In [2] (see also [3] and [4]), using the above notions, Bruck obtained a characterization of the fixed point set of a nonexpansive mapping. Here we give a slightly weaker version of his result in a form which is suitable for our considerations.

THEOREM 2.1.

i) *If C is a nonempty, bounded, closed, convex and separable subset of a Banach space $(X, \|\cdot\|)$ and if C has GFPP, then the fixed point set* $\mathrm{Fix}(T)$ *of any nonexpansive $T : C \to C$ is a nonexpansive retract of C.*

ii) *If C is a weakly compact convex subset of a Banach space $(X, \|\cdot\|)$ and C has GFPP, then the fixed point set* $\mathrm{Fix}(T)$ *of any nonexpansive $T : C \to C$ is a nonexpansive and T-invariant retract of C.*

As we mentioned earlier, in this paper we will generalize the above result to the product spaces in the separable case only. An analogous result for the product of weakly compact sets will be established elsewhere.

Let $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ be Banach spaces. Then $(X_1 \times X_2, \|\cdot\|_\infty)$ denotes the Banach space obtained by setting for $(x_1, x_2) \in X_1 \times X_2$:

$$\|(x_1, x_2)\|_\infty = \max\{\|x_1\|_1, \|x_2\|_2\}.$$

Throughout this paper we always use the maximum-norm $\|\cdot\|_\infty$ in product spaces. We also denote by P_1 and P_2 the coordinate projections of $X_1 \times X_2$ onto X_1 and X_2, respectively.

Now, we recall the fixed point theorem for nonexpansive mappings in product spaces. W. A. Kirk [9] used a retraction approach based on a method due to Bruck [4] to prove the following theorems (the analogous results for the product of convex weakly compact sets was obtained by the first author [10], see also [5], [6], [7], [8], [11], [12], [13] and [14]).

Here, we only give his results in the weaker form suitable for our considerations in the next section. The first result is the following.

THEOREM 2.2. *Let $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ be Banach spaces. Let $C_1 \times C_2$ be a nonempty, bounded, closed, convex and separable subset of $(X_1 \times X_2, \|\cdot\|_\infty)$. Assume that C_1 has the GFPP and C_2 has the FPP. Suppose $T : C_1 \times C_2 \to C_1$ is nonexpansive. Then there exists a nonexpansive mapping $R : C_1 \times C_2 \to C_1$ such that for all $(x_1, x_2) \in C_1 \times C_2$:*

(a) $T(R(x_1, x_2), x_2) = R(x_1, x_2)$,
and
(b) *if $T(x_1, x_2) = x_1$, then $R(x_1, x_2) = x_1$.*

Directly from this result we get

THEOREM 2.3. *Let $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ be Banach spaces. If $C_1 \times C_2$ is a nonempty, bounded, closed, convex and separable subset of the Banach space $(X_1 \times X_2, \|\cdot\|_\infty)$, and C_1 and C_2 both have the GFPP, then for any nonexpansive $T : C_1 \times C_2 \to C_1 \times C_2$ the fixed point set* $\mathrm{Fix}(T)$ *is nonempty.*

In the product case we need a small modification of definitions of T-invariant sets and T-invariant maps.

Let $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ be Banach spaces and C_1 and C_2 nonempty closed convex subsets of X_1 and X_2 respectively. If $T : C_1 \times C_2 \to C_1 \times C_2$, $D_1 \subset C_1$ and $D_2 \subset C_2$ are nonempty, closed and convex and $T(D_1 \times D_2) \subset D_1 \times D_2$, then we say that the product set $D_1 \times D_2$ is *T-invariant*.

Let $T_1, T_2 : C_1 \times C_2 \to C_1 \times C_2$ be nonexpansive. We say that a nonempty closed convex subset $D_1 \times D_2$ of $C_1 \times C_1$ is *T_1-invariant* if $T_1(D_1 \times D_2) \subset D_1 \times D_2$.

If every nonempty, closed, convex and T_1-invariant subset $D_1 \times D_2$ of $C_1 \times C_2$ is also T_2-invariant, then we say that the mapping T_2 is *T_1-invariant*.

In the same way we can modify definitions of a $\mathcal{S}$-invariant nonexpansive mapping and a T-invariant ($\mathcal{S}$-invariant) nonexpansive retract in product spaces.

3. The fixed point set of one mapping. Theorem 2.2 allows us to prove a much stronger result than that one given in Theorem 2.3. See [12], Theorem 5 for the weakly compact case.

THEOREM 3.1. *Let $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ be Banach spaces. If $C_1 \times C_2$ is a nonempty, bounded, closed, convex and separable subset of the Banach space $(X_1 \times X_2, \|\cdot\|_\infty)$, and C_1 and C_2 have both the GFPP, then for any nonexpansive $T : C_1 \times C_2 \to C_1 \times C_2$ the fixed point set* $\mathrm{Fix}(T)$ *is a nonempty nonexpansive retract.*

Proof. We recall the proof from [12] (see also [13]). For $T = (T_1, T_2) : C_1 \times C_2 \to C_1 \times C_2$, by Theorem 2.2 there exists a nonexpansive mapping $R_1 : C_1 \times C_2 \to C_1$ such that for all $(x_1, x_2) \in C_1 \times C_2$

(a) $T_1\big(R_1(x_1, x_2), x_2\big) = R_1(x_1, x_2)$,
and
(b) if $T_1(x_1, x_2) = x_1$, then $R_1(x_1, x_2) = x_1$.

Applying once more Theorem 2.2 to the nonexpansive mapping

$$C_1 \times C_2 \ni (x_1, x_2) \mapsto T_2\big(R_1(x_1, x_2), x_2\big) \in C_2$$

we get the second mapping R_2 which has properties similar to those mentioned above, i.e.,

$$T_2\big(R_1(x_1, R_2(x_1, x_2)), R_2(x_1, x_2)\big) = R_2(x_1, x_2) \quad \text{for } (x_1, x_2) \in C_1 \times C_2$$

and $R_2(x_1, \cdot)$ is a retraction of C_2 onto $\mathrm{Fix}\big(T_2\big(R_1(x_1, R_2(x_1, \cdot)), R_2(x_1, \cdot)\big)\big)$. Now it is obvious that

$$C_1 \times C_2 \ni (x_1, x_2) \mapsto \big(R_1(x_1, R_2(x_1, x_2)), R_2(x_1, x_2)\big) \in C_1 \times C_2$$

is the claimed retraction. ∎

However, the above result is still too weak for our aims. Namely, we need to show that in $C_1 \times C_2$ the fixed point set $\mathrm{Fix}(T)$ is a nonempty nonexpansive T-invariant retract. So, first we have to prove a stronger version of Theorem 2.1(i).

Theorem 3.2. *If C is a nonempty, bounded, closed, convex and separable subset of a Banach space $(X, \|\cdot\|)$ and C has GFPP, then the fixed point set $\mathrm{Fix}(T)$ of any nonexpansive $T : C \to C$ is a nonexpansive T-invariant retract of C.*

Proof. To prove this theorem we follow Bruck's arguments presented in the proof of Theorem 2 of [3], changing his construction of the semigroup $\mathcal{S}$ (p. 65_{14} in [3]) in the following way:

$$\mathcal{S} = \{s : C \to C : s \text{ is nonexpansive, } T\text{-invariant and } \mathrm{Fix}(T) \subset \mathrm{Fix}(s)\}. \qquad ∎$$

As a corollary we get

Corollary 3.3. *If C is a nonempty, bounded, closed, convex and separable subset of a Banach space $(X, \|\cdot\|)$, C has GFPP and if $\mathcal{S}$ is a nonempty family of nonexpansive self-mappings of C, then the fixed point set $\mathrm{Fix}(T)$ of any nonexpansive $\mathcal{S}$-invariant mapping $T : C \to C$ is a nonexpansive $\mathcal{S}$-invariant retract of C.*

Theorem 3.4. *Let $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ be Banach spaces. Let $C_1 \times C_2$ be a nonempty, bounded, closed, convex and separable subset of $(X_1 \times X_2, \|\cdot\|_\infty)$, and C_1 and C_2 both have the GFPP. If $T : C_1 \times C_2 \to C_1$ is nonexpansive, then there exists a nonexpansive mapping $R : C_1 \times C_2 \to C_1$ such that for all $(x_1, x_2) \in C_1 \times C_2$,*

(a) *$T(R(x_1, x_2), x_2) = R(x_1, x_2)$,*
(b) *if $T(x_1, x_2) = x_1$, then $R(x_1, x_2) = x_1$,*
(c) *$R(D_1 \times D_2) \subset D_1$ for each nonempty, convex and closed subset $D_1 \times D_2$ of $C_1 \times C_2$ such that $T(D_1 \times D_2) \subset D_1$.*

Proof. The proof is a simple modification of the proof of Theorem 3 in [9]—the only difference is that we have to use Theorem 3.2 instead of Theorem 2.2. ∎

Corollary 3.5. *Let $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ be Banach spaces. Let $C_1 \times C_2$ be a nonempty, bounded, closed, convex and separable subset of $(X_1 \times X_2, \|\cdot\|_\infty)$, and C_1 and C_2 both have the GFPP. Let $\mathcal{S}$ be a family of nonexpansive self-mappings of $C_1 \times C_2$. If $T : C_1 \times C_2 \to C_1$ is nonexpansive and $\mathcal{S}$-invariant, then there exists a nonexpansive mapping $R : C_1 \times C_2 \to C_1$ such that for all $(x_1, x_2) \in C_1 \times C_2$*

(a) $T(R(x_1, x_2), x_2) = R(x_1, x_2)$,

(b) *if* $T(x_1, x_2) = x_1$, *then* $R(x_1, x_2) = x_1$,

(c) $R(D_1 \times D_2) \subset D_1$ *for each nonempty, convex, closed and* $\mathcal{S}$*-invariant subset* $D_1 \times D_2$ *of* $C_1 \times C_2$.

Now we are ready to prove the main result of this section.

THEOREM 3.6. *Let* $(X_1, \|\cdot\|_1)$ *and* $(X_2, \|\cdot\|_2)$ *be Banach spaces. If* $C_1 \times C_2$ *is a nonempty, bounded, closed, convex and separable subset of the Banach space* $(X_1 \times X_2, \|\cdot\|_\infty)$ *and* C_1 *and* C_2 *both have the GFPP, then for any nonexpansive and* $\mathcal{S}$*-invariant mapping* $T : C_1 \times C_2 \to C_1 \times C_2$ *the fixed point set* $\mathrm{Fix}(T)$ *is a nonempty nonexpansive* T*-invariant retract.*

Proof. The proof is a simple modification of the proof of Theorem 3.1—here we apply Theorem 3.4 in place of Theorem 2.2. ■

COROLLARY 3.7. *Let* $(X_1, \|\cdot\|_1)$ *and* $(X_2, \|\cdot\|_2)$ *be Banach spaces. Let* $C_1 \times C_2$ *be a nonempty, bounded, closed, convex and separable subset of* $(X_1 \times X_2, \|\cdot\|_\infty)$, C_1 *and* C_2 *both have the GFPP and let* $\mathcal{S}$ *be a family of nonexpansive self-mappings of* $C_1 \times C_2$. *Then for any nonexpansive and* $\mathcal{S}$*-invariant mapping* $T : C_1 \times C_2 \to C_1 \times C_2$ *the fixed point set* $\mathrm{Fix}(T)$ *is a nonempty nonexpansive* $\mathcal{S}$*-invariant retract.*

4. The fixed point set of a family of commuting nonexpansive mappings. In this section we will consider a family of commuting nonexpansive mappings. We begin with an auxiliary lemma. The proof of this lemma runs as the proof of Lemma 6 in [2] and therefore we omit it.

LEMMA 4.1. *Let* $(X_1, \|\cdot\|_1)$ *and* $(X_2, \|\cdot\|_2)$ *be Banach spaces,* $C_1 \times C_2$ *a nonempty, bounded, closed, convex and separable subset of the Banach space* $(X_1 \times X_2, \|\cdot\|_\infty)$ *and let* C_1 *and* C_2 *both have GFPP. Let* $\mathcal{S}$ *be a family of nonexpansive self-mappings of* $C_1 \times C_2$. *Suppose* $T : C_1 \times C_2 \to C_1 \times C_2$ *is an* $\mathcal{S}$*-invariant and nonexpansive mapping,* A *is a nonempty subset of* $C_1 \times C_2$ *such that there exists* $\mathcal{S}$*-invariant and nonexpansive retraction of* $C_1 \times C_2$ *onto* A *and* $T(A) \subset A$. *Then the set* $\mathrm{Fix}(T) \cap A$ *is an* $\mathcal{S}$*-invariant and nonexpansive retract of* $C_1 \times C_2$.

Similarly we can modify Theorem 7 in [2] to the following one. Once more the proof of this fact is similar to that given in [2].

THEOREM 4.2. *Let* $(X_1, \|\cdot\|_1)$ *and* $(X_2, \|\cdot\|_2)$ *be Banach spaces,* $C_1 \times C_2$ *a nonempty, bounded, closed, convex and separable subset of the Banach space* $(X_1 \times X_2, \|\cdot\|_\infty)$ *and let* C_1 *and* C_2 *both have GFPP. Let* $\mathcal{S}$ *be a family of nonexpansive self-mappings of* $C_1 \times C_2$. *Suppose* $\{T_j : 1 \le j \le n\}$ *is a finite family of commuting* $\mathcal{S}$*-invariant and nonexpansive mappings* $T_j : C_1 \times C_2 \to C_1 \times C_2$. *Then* $\bigcap_{j=1}^n \mathrm{Fix}(T_j)$ *is a nonempty* $\mathcal{S}$*-invariant and nonexpansive retract of* $C_1 \times C_2$.

Next we get (see [2])

THEOREM 4.3. *Let* $(X_1, \|\cdot\|_1)$ *and* $(X_2, \|\cdot\|_2)$ *be Banach spaces,* $C_1 \times C_2$ *a nonempty, bounded, closed, convex and separable subset of the Banach space* $(X_1 \times X_2, \|\cdot\|_\infty)$ *and let* C_1 *and* C_2 *both have GFPP. Let* $\mathcal{S}$ *be a family of nonexpansive self-mappings of* $C_1 \times C_2$.

Suppose $\{F_n\}$ *is a descending sequence of nonempty,* $\mathcal{S}$*-invariant and nonexpansive retracts of* $C_1 \times C_2$*. Then* $\bigcap_{j=1}^{\infty} F_n$ *is a nonempty,* $\mathcal{S}$*-invariant and nonexpansive retract of* $C_1 \times C_2$*.*

THEOREM 4.4. *Let* $(X_1, \|\cdot\|_1)$ *and* $(X_2, \|\cdot\|_2)$ *be Banach spaces,* $C_1 \times C_2$ *a nonempty, bounded, closed, convex and separable subset of the Banach space* $(X_1 \times X_2, \|\cdot\|_\infty)$ *and let* C_1 *and* C_2 *both have GFPP. Let* $\mathcal{S}$ *be a family of nonexpansive self-mappings of* $C_1 \times C_2$*. Suppose* $\mathcal{F}$ *is a family of nonempty,* $\mathcal{S}$*-invariant and nonexpansive retracts of* $C_1 \times C_2$ *and suppose* $\mathcal{F}$ *is directed by* $\supset$*. Then* $\bigcap_{F \in \mathcal{F}} F$ *is a nonempty,* $\mathcal{S}$*-invariant and nonexpansive retract of* $C_1 \times C_2$*.*

Applying the above facts we get the main theorem of this paper, whose proof is analogous to the proof of Theorem 1 in [3].

THEOREM 4.5. *Let* $(X_1, \|\cdot\|_1)$ *and* $(X_2, \|\cdot\|_2)$ *be Banach spaces,* $C_1 \times C_2$ *a nonempty, bounded, closed, convex and separable subset of the Banach space* $(X_1 \times X_2, \|\cdot\|_\infty)$ *and let* C_1 *and* C_2 *both have GFPP. Let* $\mathcal{S}$ *be a family of commuting nonexpansive self-mappings of* $C_1 \times C_2$*. Then* $\bigcap_{T \in \mathcal{S}} \mathrm{Fix}(T)$ *is a nonempty* $\mathcal{S}$*-invariant nonexpansive retract of* $C_1 \times C_2$*.*

Finally, we have a very simple remark. It seems that the following modifications of the generic fixed point property should give a result stronger than the last theorem.

DEFINITION 4.1. Let $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ be Banach spaces and let $C_1 \times C_2$ be a nonempty, bounded, closed, convex subset of the Banach space $(X_1 \times X_2, \|\cdot\|_\infty)$. If $C_1 \times C_2$ has the FPP and if each nonexpansive $T : C_1 \times C_2 \to C_1 \times C_2$ has a fixed point in every nonempty closed convex T-invariant subset $D_1 \times D_2$ of $C_1 \times C_2$, then the set $C_1 \times C_2$ has a *product generic fixed point property* (*PGFPP*).

However, it is easy to notice (see Theorem 3.6) that under assumptions given in the above definition the set $C_1 \times C_2$ has PGFPP if and only if the sets C_1 and C_2 both have GFPP and therefore using the PGFPP instead of the GFPP in Theorem 4.5 we do not get anything new.

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