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A REDUCIBILITY PROBLEM FOR THE CLASSICAL RESIDUE FORMULA

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Let $z_1, \ldots, z_n$ be n distinct points in $\mathbb{C}$ and let

$$G(z) = \prod_{k=1}^{n} (z - z_k).$$

Denote by Γ a simple contour surrounding $\{z_k\}_{k=1}^n$. The residue formula

$$\frac{1}{2\pi i} \int_\Gamma f(z) \frac{G'(z)}{G(z)}\, dz = \sum_{k=1}^{n} f(z_k) \tag{1}$$

is valid in a class of analytic functions, in particular, it is true for all polynomials of degree $\le 2n-1$. In this sense (1) is a Gauss type quadratic formula of order n.

Definition 1. Let m be an integer, $2 \le m \le n$. A configuration $\{z_k\}_{k=1}^n$ is called *m-reducible* if there exists another configuration $\{w_j\}_{j=1}^m$ such that

$$\frac{1}{n} \sum_{k=1}^{n} f(z_k) = \sum_{j=1}^{m} \alpha_j f(w_j), \qquad f \in \mathcal{P}ol(\mathbb{C}), \quad \deg f \le 2m-1, \tag{2}$$

with some complex coefficients $\alpha_1, \ldots, \alpha_m$. Obviously, (2) implies that $\sum_{j=1}^m \alpha_j = 1$.

Remark. It does not make sense to extend Definition 1 to $m = 1$ since in this case the barycenter w_1 of the system $\{z_k\}$ satisfies (2) with $\alpha_1 = 1$. Thus one can say that every configuration is 1-reducible.

Definition 2. A configuration $\{z_k\}_{k=1}^n$ is called *irreducible* if for each $m \in [2, n)$ it is not m-reducible.

Note that these properties are affine invariant, i.e. they are invariant with respect to transformations $z \mapsto az + b$.

It is shown in [1] that a triangle $\{z_k\}_{k=1}^3$ is irreducible if and only if it is either equilateral or isosceles with the angle between the equal sides which is equal to

$$\alpha = \frac{\pi}{2} + \arctan \frac{\eta}{\sqrt{4 - \eta^2}}$$

where η is the unique real root of the cubic equation

$$4\eta^3 - 12\eta^2 + 9\eta + 2 = 0$$

(so that $\eta \approx 0,5283\pi$).

Also it turns out that for every $n \in \mathbb{N}, n \geq 3$ the regular n-gon is irreducible. It would be interesting to find other examples for $n \geq 4$ and, maybe, to describe explicitly all of them for small n. In general there is a characterization of irreducibility by a union of systems of algebraic equations. (This can be easily extracted from [1, Theorem 6].)

CONJECTURE. *For every n the set of irreducible configurations is finite up to affine equivalence.*

To support formally this conjecture let me indicate that each system mentioned above consists of $n-2$ equations. On the other hand, the affine class of n-configuration depends exactly on $n-2$ complex parameters.

References

[1] Yu. I. Lyubich, *Gauss type quadrature formula, power moment problem and elliptic curves*, Mat. Fizika, Analiz, Geometriya 9 (2002), 128–145.