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A SPECTRAL APPROACH TO THE KAPLANSKY PROBLEM

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Let ϕ be a unital invertibility preserving linear surjective map onto a semisimple Banach algebra. The famous Kaplansky conjecture [5] claims that ϕ must be a Jordan homomorphism, that is,

$$\phi(x^2) = \phi(x)^2$$

for all x.

It has been recently observed [6, Theorem 2.2] that the question is equivalent to showing that

$$\phi(x^{-1}) = \phi(x)^{-1}$$

for all invertible x, i.e., that the assumed inverse of $\phi(x)$ is precisely $\phi(x^{-1})$.

Let us ask a weaker question: is it true that the difference

$$\phi(x^{-1}) - \phi(x)^{-1}$$

is quasi-nilpotent? If so, then the next step would be to show that these differences are in the radical (hence equal to zero) by using various spectral characterizations of the radical [8], [2, Theorem 5.3.1]. By the way, this was a significant tool in proving that ϕ is automatically continuous [2, Theorem 5.5.2].

Some related examples and partial results can be found in [1, p. 28], [3] , [4], [7].

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