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MODULUS OF DENTABILITY IN $L^1 + L^\infty$

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Abstract. We introduce the notion of the modulus of dentability defined for any point of the unit sphere $S(X)$ of a Banach space X. We calculate effectively this modulus for denting points of the unit ball of the classical interpolation space $L^1 + L^\infty$. Moreover, a criterion for denting points of the unit ball in this space is given. We also show that no denting point of the unit ball of $L^1 + L^\infty$ is a LUR-point. Consequently, the set of LUR-points of the unit ball of $L^1 + L^\infty$ is empty.

1. Introduction. Let $S(X)$ (resp. $B(X)$, $B(X)^0$) be the unit sphere (resp. the closed unit ball, the open unit ball) of a real Banach space $(X, \|\cdot\|_X)$.

A point $x \in S(X)$ is called

a) an *extreme point of* $B(X)$ (written $x \in \delta_e B(X)$) if for any $y, z \in B(X)$ the equality $2x = y + z$ implies $y = z$.

b) a *strongly extreme point of* $B(X)$ (written $x \in \delta_{se} B(X)$) if for any sequences (y_n), (z_n) in $B(X)$ we have $\|y_n - x\|_X \to 0$ whenever $\|y_n + z_n - 2x\|_X \to 0$.

c) a *denting point of* $B(X)$ (written $x \in \delta_d B(X)$) if $x \notin \overline{co}\{B(X)\backslash[x + \varepsilon B(X)^0]\}$ for each $\varepsilon > 0$.

d) a *point of local uniform rotundity of* $B(X)$ (LUR-*point* for short, written $x \in \delta_{LUR} B(X)$) if for each $\varepsilon > 0$ there is $\delta(x, \varepsilon) > 0$ such that for all $y \in B(X)$ the inequality $\|x - y\|_X \geq \varepsilon$ implies $\|(x + y)/2\|_X \leq 1 - \delta(x, \varepsilon)$.

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e) an *H-point of* $S(X)$ (written $x \in \delta_H S(X)$) if every sequence (x_n) in $S(X)$ we have $\|x_n - x\|_X \to 0$ whenever x_n is weakly convergent to x (write $x_n \xrightarrow{w} x$).

f) a *point of continuity of* $S(X)$ (written $x \in \delta_{pc} S(X)$) if the identity map from $(S(X), \tau_w)$ into $(S(X), \tau_{\|\cdot\|_X})$ (where τ_w is the weak topology and $\tau_{\|\cdot\|_X}$ the norm topology) is continuous at x.

It is known (see [12]) that

$$\delta_d B(X) = \delta_{se} B(X) \cap \delta_{pc} S(X).$$

Moreover,

$$\delta_{LUR} B(X) \subset \delta_d B(X) \subset \delta_{se} B(X) \subset \delta_e B(X), \; \delta_d B(X) \subset \delta_{pc} S(X) \subset \delta_H S(X). \tag{1.1}$$

A Banach space X is said to have *property* **R** (resp. **MLUR**, **G**, **LUR**, **H**, **PC**) if $S(X) = \delta_e B(X)$ (resp. $S(X) = \delta_{se} B(X)$, $S(X) = \delta_d B(X)$, $S(X) = \delta_{LUR} B(X)$, $S(X) = \delta_H S(X)$, $S(X) = \delta_{pc} S(X)$).

Given any $x \in S(X)$ a function $\delta_x : [0,1) \to [0,1)$ defined for any $\varepsilon \in [0,1)$ by

$$\delta_x(\varepsilon) = \inf\{\|x - y\|_X : y \in \overline{co}\{B(X) \backslash [x + \varepsilon B(X)^0]\}\}$$

is said to be the *modulus of dentability of* x. Clearly, $x \in \delta_d B(X)$ if and only if $\delta_x(\varepsilon) > 0$ for any $\varepsilon \in [0,1)$.

It is worth mentioning that the Radon-Nikodým property (RNP), one of the most important properties of Banach spaces, can be defined in terms of denting points. Namely, H. B. Maynard [10] proved that a Banach space X has the RNP if and only if every non-empty bounded closed set K in X has at least one denting point. It is also known that a Banach space X has the RNP if and only if for any equivalent norm in X the respective unit ball $B(X)$ has a denting point (see, e.g., [2, p. 30]).

2. Some results in Köthe spaces. Let (T, Σ, μ) be a σ-finite and complete measure space and $L^0 = L^0(T)$ be the space of all μ-equivalence classes of real valued measurable functions defined on T. The notation $x \le y$ for $x, y \in L^0$ will mean that $x(t) \le y(t)$ μ-a.e. in T.

A Banach space $E = (E, \|\cdot\|_E)$ is said to be a *Köthe space* if E is a linear subspace of L^0 and

(i) if $x \in E, y \in L^0$ and $|y| \le |x|$ μ-a.e., then $y \in E$ and $\|y\|_E \le \|x\|_E$;
(ii) there exists a function x in E that is positive on the whole T.

Every Köthe space is a Banach lattice under the obvious partial order ($x \ge 0$ if $x(t) \ge 0$ for μ-a.e. $t \in T$). The set $E_+ = \{x \in E : x \ge 0\}$ is called the *positive cone of* E. For any subset $A \subset E$ define $A_+ = A \cap E_+$. For any $x, y \in E$,

$$(x \vee y)(t) = \max\{x(t), y(t)\}, \qquad (x \wedge y)(t) = \min\{x(t), y(t)\}$$

for μ-a.e. $t \in T$.

The next lemma is a generalization of Lemma 1 from [4].

LEMMA 1. *Let* $x \in \delta_{se} B(E)$, $x \in E_+$ *and* (p_n) *be a sequence such that* $p_n \in E_+$ *for every* $n \in \mathbb{N}$ *and* $p_n \to x$ *in* E. *If* $y_n \ge p_n$ *and* $y_n \in B(E)$ *for any* $n \in \mathbb{N}$, *then* $y_n \to x$ *in* E.

Proof. Since $1 = \|x\|_E \leftarrow \|p_n\|_E \le \|y_n\|_E \le 1$, $\|y_n\|_E \to 1$. Define $z_n = 2p_n - y_n$ and $A_n = \{t \in T : z_n(t) \ge 0\}$ for any $n \in \mathbb{N}$. Then, by the fact that $p_n \le y_n$, we have

$$|z_n| = (2p_n - y_n)\chi_{A_n} + (y_n - 2p_n)\chi_{T\setminus A_n} \le p_n\chi_{A_n} + y_n\chi_{T\setminus A_n} \le y_n.$$

Consequently, $\|z_n\|_E \le \|y_n\|_E \le 1$ and $\|z_n\|_E \ge |2\|p_n\|_E - \|y_n\|_E|$ for any $n \in \mathbb{N}$, whence

$$1 = \lim_{n\to\infty} |2\|p_n\|_E - \|y_n\|_E| \le \liminf_{n\to\infty}\|z_n\|_E \le \limsup_{n\to\infty}\|z_n\|_E \le 1,$$

i.e. $\lim_{n\to\infty} \|z_n\|_E = 1$. Moreover $z_n + y_n = 2p_n \to 2x$ in E. Therefore, by the fact that $x \in \delta_{se}B(E)$, we conclude that $y_n \to x$ in E. ■

We use Lemma 1 to prove the following.

LEMMA 2. *Let $x \in S(E)$ be such that $|x| \in \delta_d B(E)$. Then for any $\varepsilon > 0$ there is $\delta > 0$ such that*

$$\overline{co}\{y \in E : |y| \in B(E)\backslash[|x| + \varepsilon B(E)^0]\} \cap \{z \in E : |z| \in [|x| + \delta B(E)^0]\} = \varnothing.$$

Proof. If the lemma is not true, then there exist $\varepsilon > 0$ and sequences (a_i^m) and (x_i^m) in $\mathbb{R}_+$ and X, respectively, such that

$$\sum_{i=1}^{N(m)} a_i^m = 1, \qquad |x_i^m| \in B(E)\backslash[|x| + \varepsilon B(E)^0] \tag{2.1}$$

for any $m \in \mathbb{N}$, $i = 1, 2, \ldots, N(m)$ and

$$\lim_{m\to\infty} \Big|\sum_{i=1}^{N(m)} a_i^m x_i^m\Big| = |x|. \tag{2.2}$$

Since every denting point of $B(E)$ is a strongly extreme point of $B(E)$, by our assumptions $|x| \in \delta_{se}B(E)$. Define $y_m = \sum_{i=1}^{N(m)} a_i^m |x_i^m|$. Clearly, by the triangle inequality for the norm, $\|y_m\|_E \le 1$. Moreover

$$y_m = \sum_{i=1}^{N(m)} a_i^m |x_i^m| \ge \Big|\sum_{i=1}^{N(m)} a_i^m x_i^m\Big|$$

for every $m \in \mathbb{N}$. Hence, in virtue of Lemma 1, we have

$$\lim_{m\to 0} y_m = |x|,$$

which is impossible, because $|x|$ is a denting point and y_m is a convex combination of points from the set $B(E)\backslash[|x| + \varepsilon B(E)^0]$. This contradiction finishes the proof. ■

REMARK 1. Lemma 2 is a special case of Lemma 4.2.2 from [11], but our method of proof is much simpler. Moreover, Lemma 4.2.2 from [11] can be proved in the same way also.

PROPOSITION 1. *Let E be a Köthe space. Then $\delta_x(\varepsilon) = \delta_{|x|}(\varepsilon)$ for any $x \in S(E)$ and $\varepsilon \in [0, 1)$. In particular, $x \in \delta_d B(E)$ if and only if $|x| \in \delta_d B(E)$.*

Proof. Take $x \in S(E)$ and $\varepsilon \in [0, 1)$. Then there are sequences (a_i^m) and (y_i^m) in $\mathbb{R}_+$ and X, respectively, such that

$$\sum_{i=1}^{N(m)} a_i^m = 1, \qquad y_i^m \in B(E)\backslash[x + \varepsilon B(E)^0]$$

for any $m \in \mathbb{N}$, $i = 1, 2, \ldots, N(m)$ and

$$\delta_x(\varepsilon) = \lim_{m\to\infty} \Big\| \sum_{i=1}^{N(m)} a_i^m y_i^m - x \Big\|_E.$$

Define $x_i^m = y_i^m \operatorname{sign} x$. Then

$$\|x_i^m - |x|\|_E = \|\operatorname{sign} x (y_i^m - x)\|_E = \|y_i^m - x\|_E \geq \varepsilon$$

and consequently $x_i^m \in B(E) \backslash [|x| + \varepsilon B(E)^0]$. Moreover,

$$\delta_x(\varepsilon) = \lim_{m\to\infty} \Big\| \sum_{i=1}^{N(m)} a_i^m x_i^m \operatorname{sign} x - x \Big\|_E = |\operatorname{sign} x| \lim_{m\to\infty} \Big\| \sum_{i=1}^{N(m)} a_i^m x_i^m - |x| \Big\|_E \geq \delta_{|x|}(\varepsilon).$$

Conversely, by the definition of $\delta_{|x|}(\varepsilon)$, there are sequences (a_i^m) and (x_i^m) in $\mathbb{R}_+$ and X, respectively, such that

$$\sum_{i=1}^{N(m)} a_i^m = 1, \ x_i^m \in B(E) \backslash [|x| + \varepsilon B(E)^0]$$

for any $m \in \mathbb{N}$, $i = 1, 2, \ldots, N(m)$ and

$$\delta_{|x|}(\varepsilon) = \lim_{m\to\infty} \Big\| \sum_{i=1}^{N(m)} a_i^m x_i^m - |x| \Big\|_E.$$

Defining $y_i^m = x_i^m \operatorname{sign} x$ and repeating the above argument, we prove that $\delta_{|x|}(\varepsilon) \geq \delta_x(\varepsilon)$. Hence $\delta_x(\varepsilon) = \delta_{|x|}(\varepsilon)$ and consequently $x \in \delta_d B(E)$ if and only if $|x| \in \delta_d B(E)$. ∎

3. Denting points in $L^1 + L^\infty$. Consider the classical interpolation space $L^1 + L^\infty$ equipped with the norm

$$\|x\|_{L^1+L^\infty} = \inf\{\|y\|_1 + \|z\|_\infty : y + z = x, \ y \in L^1, \ z \in L^\infty\}$$

(see [1], [9]). It is well known that if $\mu(T) \geq 1$, then

$$\|x\|_{L^1+L^\infty} = \int_0^1 x^*(t)dt,$$

where x^* is the nonincreasing rearrangement of x (see [9]). Recall that d_x denotes the distribution function of $|x|$, that is,

$$d_x(\lambda) = \mu(\{t \in [0, \mu(T)] : |x(t)| > \lambda\})$$

for all $\lambda \geq 0$. The decreasing rearrangement of x is denoted by x^* and is defined by $x^*(t) = \inf\{\lambda > 0 : d_x(\lambda) \leq t\}$. We also know (see [7]) that $L^1 + L^\infty$ is an Orlicz space generated by the Orlicz function $\Phi_{\infty,1}$ defined by the formula $\Phi_{\infty,1}(u) = \max\{0, |u| - 1\}$. Moreover, $\|\cdot\|_{L^1+L^\infty} = \|\cdot\|^0_{\Phi_{\infty,1}}$. The local geometry of $L^1 + L^\infty$ was considered e.g. in [5], [6], [8]. We are going to characterize the denting points of the unit ball of this space. To do it, we will start with the following

LEMMA 3. *If $1 < \mu(T) < \infty$, then the norms $\|\cdot\|_1$ and $\|\cdot\|_{L^1+L^\infty}$ are equivalent and*

$$\|x\|_{L^1+L^\infty} \leq \|x\|_1 \leq \mu(T)\|x\|_{L^1+L^\infty}$$

for any $x \in L^1 + L^\infty$. *The equality* $\|x\|_1 = \mu(T)\|x\|_{L^1+L^\infty}$ *holds only in the case when* $|x(t)| = c\chi_T$ *μ-a.e. in* T *for some real number* $c \geq 0$.

Proof. The left hand inequality is obvious in view of the definition of the norm $\|\cdot\|_{L^1+L^\infty}$. Take an arbitrary $x \in L^1 + L^\infty$ such that $\|x\|_1 = c$. Denote

$$A = \{t \in T : |x(t)| > c/\mu(T)\} \text{ and } B = \{t \in T : |x(t)| < c/\mu(T)\}.$$

Then $x^*(t) > c/\mu(T)$ for $t \in \langle 0, \mu(A)\rangle$, $x^*(t) = c/\mu(T)$ for $t \in (\mu(A), \mu(T\setminus B))$ and $x^*(t) < c/\mu(T)$ for $t \in \langle \mu(T\setminus B), \mu(T)\rangle$. By our assumptions, we have

$$\begin{aligned}
&\int_0^{\mu(A)} \left(x^*(t) - \frac{c}{\mu(T)}\right) dt - \int_{\mu(T\setminus B)}^{\mu(T)} \left(\frac{c}{\mu(T)} - x^*(t)\right) dt \\
&= \int_0^{\mu(T)} \left(x^*(t) - \frac{c}{\mu(T)}\right) dt = \int_0^{\mu(T)} x^*(t)dt - c = \|x\|_1 - c = 0.
\end{aligned}$$

Hence

$$\int_0^{\mu(A)} \left(x^*(t) - \frac{c}{\mu(T)}\right) dt = \int_{\mu(T\setminus B)}^{\mu(T)} \left(\frac{c}{\mu(T)} - x^*(t)\right) dt. \tag{3.1}$$

To prove the assertion, consider the following cases.

i) $\mu(A) \geq 1$. Then

$$\|x\|_{L^1+L^\infty} = \int_0^1 x^*(t)dt = \int_0^1 \frac{cdt}{\mu(T)} + \int_0^1 \left(x^*(t) - \frac{c}{\mu(T)}\right) dt > \frac{c}{\mu(T)}.$$

ii) $0 < \mu(A) < 1$ and $\mu(B) < \mu(T) - 1$. Then

$$\begin{aligned}
\|x\|_{L^1+L^\infty} &= \int_0^{\mu(A)} x^*(t)\,dt + \int_{\mu(A)}^1 x^*(t)\,dt \\
&= \int_0^{\mu(A)} \left(\frac{c}{\mu(T)} + \left(x^*(t) - \frac{c}{\mu(T)}\right)\right) dt + \int_{\mu(A)}^1 \frac{c}{\mu(T)}\,dt \\
&= \int_0^1 \frac{c}{\mu(T)}\,dt + \int_0^{\mu(A)} \left(x^*(t) - \frac{c}{\mu(T)}\right) dt > \frac{c}{\mu(T)}.
\end{aligned}$$

iii) $0 < \mu(A) < 1$ and $\mu(B) > \mu(T) - 1$. Then, by the inequality (3.1), we get

$$\begin{aligned}
\|x\|_{L^1+L^\infty} &= \int_0^{\mu(A)} x^*(t)dt + \int_{\mu(A)}^{\mu(T\setminus B)} x^*(t)dt + \int_{\mu(T\setminus B)}^1 x^*(t)\,dt \\
&= \int_0^{\mu(A)} \left(\frac{c}{\mu(T)} + \left(x^*(t) - \frac{c}{\mu(T)}\right)\right) dt + \int_{\mu(A)}^{\mu(T\setminus B)} \frac{c}{\mu(T)}dt \\
&\quad + \int_{\mu(T\setminus B)}^1 (\frac{c}{\mu(T)} - (\frac{c}{\mu(T)} - x^*(t)))\,dt \\
&= \int_0^1 \frac{c}{\mu(T)}\,dt + \int_0^{\mu(A)} \left(x^*(t) - \frac{c}{\mu(T)}\right) dt - \int_{\mu(T\setminus B)}^1 \left(\frac{c}{\mu(T)} - x^*(t)\right) dt \\
&= \frac{c}{\mu(T)} + \int_1^{\mu(T)} \left(\frac{c}{\mu(T)} - x^*(t)\right) dt > \frac{c}{\mu(T)}
\end{aligned}$$

iv) If $\mu(A)=0$, then $\mu(B)=0$ and

$$\|x\|_{L^1+L^\infty}=\int_0^1\frac{c}{\mu(T)}dt=\frac{c}{\mu(T)}$$

and consequently $\|x\|_1\le\mu(T)\|x\|_{L^1+L^\infty}$ for any $x\in L^1+L^\infty$, which finishes the proof of the lemma. ■

LEMMA 4. *If* $1<\mu(T)<\infty$, *then*

$$\mu(T)\int_T((|x(t)|-1)\vee 0)d\mu\le\int_T((1-|x(t)|)\vee 0)d\mu$$

for any $x\in B(L^1+L^\infty)$.

Proof. Fix $x\in B(L^1+L^\infty)$ and define

$$A=\{t\in T:|x(t)|-1>0\}\text{ and }B=\{t\in T:|x(t)|-1<0\}.$$

Then $x^*(t)>1$ for $t\in\langle 0,\mu(A)\rangle$, $x^*(t)=1$ for $t\in(\mu(A),\mu(T\setminus B))$ and $x^*(t)<1$ for $t\in\langle\mu(T\setminus B),\mu(T)\rangle$. We claim that $\mu(A)<1$ and $\mu(B)>\mu(T)-1$. Indeed, supposing that $\mu(A)\ge 1$, we get, by $x^*(t)>1$ for $t\in[0,\mu(A)]$,

$$\|x\|_{L^1+L^\infty}=\int_0^1(1+(x^*(t)-1))dt>1,$$

i.e. $x\notin B(L^1+L^\infty)$. This contradiction proves that $\mu(A)<1$. Further, if $\mu(B)\le\mu(T)-1$, then

$$\|x\|_{L^1+L^\infty}=\int_0^{\mu(A)}(1+(x^*(t)-1))dt+\int_{\mu(A)}^1 dt=1+\int_0^{\mu(A)}(x^*(t)-1)dt>1$$

and consequently $x\notin B(L^1+L^\infty)$. This contradiction finishes the proof of the claim.

Notice that

$$\int_T((|x(t)|-1)\vee 0)d\mu=\int_0^{\mu(A)}(x^*(t)-1)\,dt$$

and

$$\int_T((1-|x(t)|)\vee 0)d\mu=\int_{\mu(T\setminus B)}^{\mu(T)}(1-x^*(t))dt.$$

Since

$$\int_0^{\mu(A)}(x^*(t)-1)dt-\int_{\mu(T\setminus B)}^1(1-x^*(t))dt=\int_0^1x^*(t)dt-\int_0^1dt=\|x\|_{L^1+L^\infty}-1\le 0,$$

we have

$$\int_{\mu(T\setminus B)}^1(1-x^*(t))dt\ge\int_0^{\mu(A)}(x^*(t)-1)dt. \tag{3.2}$$

Moreover, it is easy to see that

$$x^*(t)\le x^*(1)\le 1-\int_{\mu(T\setminus B)}^1(1-x^*(t))dt \tag{3.3}$$

for all $t > 1$. Finally, applying inequalities (3.2) and (3.3), we have

$$\begin{aligned}\int_T ((1-|x(t)|)\vee 0)d\mu &= \int_{\mu(T\setminus B)}^1 (1-x^*(t))dt + \int_1^{\mu(T)} (1-x^*(t))dt\\ &\geq \int_0^{\mu(A)} (x^*(t)-1)dt + \int_1^{\mu(T)} (1-x^*(1))dt\\ &= \int_0^{\mu(A)} (x^*(t)-1)dt + (\mu(T)-1)(1-x^*(1))\\ &\geq \int_0^{\mu(A)} (x^*(t)-1)dt + (\mu(T)-1)\int_{\mu(T\setminus B)}^1 (1-x^*(t))dt\\ &\geq \int_0^{\mu(A)} (x^*(t)-1)dt + (\mu(T)-1)\int_0^{\mu(A)} (x^*(t)-1)dt\\ &= \mu(T)\int_0^{\mu(A)} (x^*(t)-1)dt = \mu(T)\int_T ((|x(t)|-1)\vee 0)d\mu,\end{aligned}$$

so the conclusion of the lemma follows. ∎

THEOREM 1. *a) The set $\delta_d B(L^1+L^\infty)$ is non-empty if and only if $1 < \mu(T) < \infty$.*

b) Let $1 < \mu(T) < \infty$. A point $x \in S(L^1+L^\infty)$ is a denting point if and only if $|x| = \chi_T$. Moreover, for any $\varepsilon \in (0,1)$ and any $x \in \delta_d B(L^1+L^\infty)$ we have

$$\delta_x(\varepsilon) = \min\left\{\frac{(\mu(T)-1)\varepsilon}{2\mu(T)}, \frac{\varepsilon}{\mu(T)}\right\}.$$

Proof. By Corollary 5 from [5], the set of strongly extreme points of $B(L^1+L^\infty)$ is empty whenever $\mu(T) \leq 1$. Since every denting point of the unit ball $B(X)$ is a strongly extreme point of $B(X)$ for any Banach X, the unit ball $B(L^1+L^\infty)$ has no denting points if $\mu(T) \leq 1$. By Theorem 2.5 from [8], none of extreme points is an H-point provided $\mu(T) = \infty$. Hence if $\mu(T) = \infty$, the set of denting points of $B(L^1+L^\infty)$ is also empty because any denting point is always an H-point. Therefore denting points of $B(L^1+L^\infty)$ can exist only in the case when $1 < \mu(T) < \infty$. By Corollary 5 from [5], x is a strongly extreme point of $B(L^1+L^\infty)$ if and only if $|x(t)| = 1$ for μ-a.e. $t \in T$. Hence, to finish the proof of both assertions, it is enough to show that a point x such that $|x| = \chi_T$ is a denting point of $B(L^1+L^\infty)$. But, by Proposition 1, we can reduce our considerations to the case $x = \chi_T$. Fix $\varepsilon > 0$, take $z \in B(L^1+L^\infty)$ such that $\|\chi_T - z\|_{L^1+L^\infty} \geq \varepsilon$ and denote $\alpha = \int_T((|z(t)|-1)\vee 0)d\mu$. Defining $A = \{t \in T : |z(t)|-1 > 0\}$ and taking into account that $z \in B(L^1+L^\infty)$, similarly to the beginning of the proof of Lemma 4, we conclude that $\mu(A) < 1$. Then

$$\alpha = \int_0^{\mu(A)} (z^*(t)-1)dt < \int_0^{\mu(A)} z^*(t)dt \leq \int_0^1 z^*(t)dt = \|z\|_{L^1+L^\infty} \leq 1,$$

whence $\alpha \in [0,1)$. Define the function

$$F_z(\alpha) = \int_T ((1-|z(t)|)\vee 0)d\mu - \int_T ((|z(t)|-1)\vee 0)d\mu = \int_T ((1-|z(t)|)\vee 0)d\mu - \alpha.$$

Then, by Lemma 4, we have

$$F_z(\alpha) \geq \mu(T) \int_T ((|z(t)| - 1) \vee 0) d\mu - \alpha = \alpha(\mu(T) - 1). \tag{3.4}$$

Since $z \in B(L^1 + L^\infty)$, we have

$$\begin{aligned} 0 \leq 1 - \|z\|_{L^1+L^\infty} &= \int_0^1 dt - \int_0^1 z^*(t) dt \\ &= \int_0^1 ((1 - z^*(t)) \vee 0) dt - \int_0^1 ((z^*(t) - 1) \vee 0) dt = \int_0^1 ((1 - z^*(t)) \vee 0) dt - \alpha. \end{aligned}$$

This implies that

$$\int_0^1 ((1 - z^*(t)) \vee 0) dt \geq \alpha. \tag{3.5}$$

Hence

$$z^*(1) = 1 - \int_0^1 (1 - z^*(1)) dt \leq 1 - \int_0^1 ((1 - z^*(t)) \vee 0) dt \leq 1 - \alpha. \tag{3.6}$$

Assume additionally that $\alpha < \varepsilon$ and consider the following two cases.

1. Suppose that $\mu(T) \geq 2$. Then we have

$$\begin{aligned} \varepsilon \leq \|\chi_T - z\|_{L^1+L^\infty} &= \int_0^1 (\chi_T - z)^*(t) dt \\ &\leq \int_0^{\mu(A)} (z^*(t) - 1) dt + \int_{\mu(T)-1}^{\mu(T)} (1 - z^*(t)) dt = \alpha + \int_{\mu(T)-1}^{\mu(T)} (1 - z^*(t)) dt, \end{aligned}$$

whence

$$\int_{\mu(T)-1}^{\mu(T)} (1 - z^*(t)) dt \geq \varepsilon - \alpha.$$

Moreover, by (3.5) and (3.6), we obtain

$$\begin{aligned} \int_0^{\mu(T)-1} ((1 - z^*(t)) \vee 0) dt &= \int_0^1 ((1 - z^*(t)) \vee 0) dt + \int_1^{\mu(T)-1} (1 - z^*(t)) dt \\ &\geq \alpha + \int_1^{\mu(T)-1} (1 - z^*(1)) dt \\ &\geq \alpha + \alpha(\mu(T) - 2) = \alpha(\mu(T) - 1). \end{aligned}$$

Therefore

$$\begin{aligned} F_z(\alpha) &= \int_T ((1 - |z(t)|) \vee 0) d\mu - \alpha = \int_0^{\mu(T)} ((1 - z^*(t)) \vee 0) dt - \alpha \\ &= \int_0^{\mu(T)-1} ((1 - z^*(t)) \vee 0) dt + \int_{\mu(T)-1}^{\mu(T)} (1 - z^*(t)) dt - \alpha \\ &\geq \alpha(\mu(T) - 1) + \varepsilon - 2\alpha = \varepsilon + \alpha(\mu(T) - 3). \end{aligned}$$

Combining the last inequality with inequality (3.4) we get the system of inequalities

$$\begin{cases} F_z(\alpha) \geq \varepsilon + \alpha(\mu(T) - 3), \\ F_z(\alpha) \geq \alpha(\mu(T) - 1). \end{cases}$$

If $\mu(T) \geq 3$, then the smallest number $F_z(\alpha)$ corresponds to $\alpha = 0$. Hence

$$F_z(\alpha) \geq \varepsilon \tag{3.7}$$

for any $\alpha \in [0, \varepsilon)$. If $2 \leq \mu(T) \leq 3$, then the slopes $\mu(T) - 3$ and $\mu(T) - 1$ have opposite signs, and the smallest number $F_z(\alpha)$ is attained for $\alpha = \frac{\varepsilon}{2}$. We can find it by solving the equation

$$\varepsilon + \alpha(\mu(T) - 3) = \alpha(\mu(T) - 1).$$

Hence

$$F_z(\alpha) \geq \frac{\varepsilon}{2}(\mu(T) - 1) \tag{3.8}$$

for any $\alpha \in [0, \varepsilon)$, whenever $2 \leq \mu(T) \leq 3$.

2. Now suppose that $1 < \mu(T) < 2$. As in the previous case, it can be proved that

$$\int_T ((1 - |z(t)|) \vee 0) d\mu \geq \int_{\mu(T)-1}^{\mu(T)} ((1 - z^*(t)) \vee 0) dt \geq \varepsilon - \alpha. \tag{3.9}$$

If $\alpha \leq \varepsilon \frac{2-\mu(T)}{3-\mu(T)}$, then

$$F_z(\alpha) \geq (\varepsilon - \alpha) - \alpha \geq \varepsilon - 2\varepsilon \frac{2 - \mu(T)}{3 - \mu(T)} = \varepsilon \frac{\mu(T) - 1}{3 - \mu(T)} \geq \frac{\varepsilon}{2}(\mu(T) - 1).$$

Now suppose that $\alpha > \varepsilon \frac{2-\mu(T)}{3-\mu(T)}$. This inequality is equivalent to

$$(\varepsilon - \alpha)(2 - \mu(T)) < \alpha.$$

Since the function $(1 - z^*(t)) \vee 0$ is non-decreasing, by (3.9), we obtain

$$\begin{aligned} \int_T ((1 - |z(t)|) \vee 0) d\mu &\geq \int_0^1 ((1 - z^*(t)) \vee 0) dt + \int_1^{\mu(T)} (1 - z^*(t)) dt \\ &\geq \alpha + (\mu(T) - 1)(\varepsilon - \alpha). \end{aligned}$$

Hence

$$F_z(\alpha) \geq (\mu(T) - 1)(\varepsilon - \alpha).$$

Combining the last inequality with inequality (3.4), we get again the system of inequalities

$$\begin{cases} F_z(\alpha) \geq \varepsilon(\mu(T) - 1) - \alpha(\mu(T) - 1), \\ F_z(\alpha) \geq \alpha(\mu(T) - 1). \end{cases}$$

Solving the equation

$$\varepsilon(\mu(T) - 1) - \alpha(\mu(T) - 1) = \alpha(\mu(T) - 1),$$

we conclude that the smallest number $F_z(\alpha)$ is attained at $\alpha = \frac{\varepsilon}{2}$. It leads again to inequality (3.8) for any $\alpha \in [0, \varepsilon)$, whenever $1 < \mu(T) < 2$.

Summing up, we have

$$F_z(\alpha) \geq \begin{cases} \frac{\varepsilon}{2}(\mu(T) - 1) & \text{if } 1 < \mu(T) < 3, \\ \varepsilon & \text{if } \mu(T) \geq 3, \end{cases} \tag{3.10}$$

for any $\alpha \in [0, \varepsilon)$, i.e. $F_z(\alpha) \geq \min\{\frac{\varepsilon}{2}(\mu(T) - 1), \varepsilon\}$.

If $\alpha \geq \varepsilon$, then $\|\chi_T - z\|_{L^1+L^\infty} \geq 2\alpha \geq 2\varepsilon > \varepsilon$. Moreover, by (3.4), the smallest number $F_z(\alpha)$ is taken for $\alpha = \varepsilon$. Hence

$$F_z(\alpha) \geq \varepsilon(\mu(T) - 1)$$

for any $\alpha \in [\varepsilon, 1)$. But

$$\varepsilon(\mu(T) - 1) \geq \min\left\{\frac{\varepsilon}{2}(\mu(T) - 1), \varepsilon\right\},$$

so (3.10) is satisfied for any $\alpha \in [0, 1)$.

Now, we will show that $x = \chi_T$ is a denting point. Take a finite sequence $(z_i)_{i=1}^m$ in the unit ball $B(L^1 + L^\infty)$ such that $\|\chi_T - z_i\|_{L^1+L^\infty} \geq \varepsilon$ for $i = 1, 2, \ldots, m$, and a finite sequence of positive numbers $(a_i)_{i=1}^m$ such that $\sum_{i=1}^m a_i = 1$. Set $\alpha_i = \int_T((|z_i(t)|-1)\vee 0)d\mu$ for $i = 1, 2, \ldots, m$. Then, by Lemma 3 and the inequality (3.10), we have

$$\begin{aligned}
\Big\|\chi_T - \sum_{i=1}^m a_i z_i\Big\|_{L^1+L^\infty} &\geq \frac{1}{\mu(T)}\Big\|\chi_T - \sum_{i=1}^m a_i z_i\Big\|_1 = \frac{1}{\mu(T)}\int_T \Big|\chi_T(t) - \sum_{i=1}^m a_i z_i(t)\Big| d\mu \\
&\geq \frac{1}{\mu(T)}\int_T \sum_{i=1}^m a_i(\chi_T(t) - z_i(t))d\mu \\
&= \frac{1}{\mu(T)}\sum_{i=1}^m a_i \int_T (\chi_T(t) - z_i(t))d\mu \\
&\geq \frac{1}{\mu(T)}\sum_{i=1}^m a_i\Big(\int_T((1 - |z_i(t)|) \vee 0)d\mu - \int_T((|z_i(t)| - 1) \vee 0)d\mu\Big) \\
&= \frac{1}{\mu(T)}\sum_{i=1}^m a_i F_{z_i}(\alpha_i) \geq \frac{1}{\mu(T)}\sum_{i=1}^m a_i \min\left\{\frac{\varepsilon}{2}(\mu(T) - 1), \varepsilon\right\} \\
&= \min\left\{\frac{(\mu(T) - 1)\varepsilon}{2\mu(T)}, \frac{\varepsilon}{\mu(T)}\right\}.
\end{aligned}$$

Consequently, by Proposition 1, every $x \in L^1 + L^\infty$ such that $|x| = \chi_T$ is a denting point of $B(L^1 + L^\infty)$ and

$$\delta_x(\varepsilon) \geq \min\left\{\frac{(\mu(T) - 1)\varepsilon}{2\mu(T)}, \frac{\varepsilon}{\mu(T)}\right\}. \tag{3.11}$$

Now, we will show the converse inequality by considering two cases.

1. Let $1 < \mu(T) < 3$. Take $\varepsilon \in (0, 1)$. Define $A_i^n = [\frac{i-1}{n}\mu(T), \frac{i}{n}\mu(T))$ and

$$x_i^n = \left(\frac{n\varepsilon}{2\mu(T)} + 1\right)\chi_{A_i^n} + \left(1 - \frac{n\varepsilon}{2(n - \mu(T))}\right)\chi_{T\setminus A_i^n}$$

for $n = 6, 7, \ldots$ and $i = 1, 2, \ldots, n$. Then

$$\begin{aligned}
\|x_i^n\|_{L^1+L^\infty} &= \frac{1}{n}\mu(T)\left(\frac{n\varepsilon}{2\mu(T)} + 1\right) + \left(1 - \frac{1}{n}\mu(T)\right)\left(1 - \frac{n\varepsilon}{2(n - \mu(T))}\right) \\
&= \frac{\varepsilon}{2} + \frac{1}{n}\mu(T) + \frac{n - \mu(T)}{n} - \frac{\varepsilon}{2} = 1
\end{aligned}$$

for any integer $n \geq 6$ and $i = 1, 2, \ldots, n$. Moreover,

$$\begin{aligned}\|x_i^n - \chi_T\|_{L^1+L^\infty} &= \left\| \frac{n\varepsilon}{2\mu(T)}\chi_{A_i^n} - \frac{n\varepsilon}{2(n-\mu(T))}\chi_{T\setminus A_i^n} \right\|_{L^1+L^\infty} \\ &= \frac{\mu(T)}{n} \cdot \frac{n\varepsilon}{2\mu(T)}\chi_{A_i^n} + \left(1 - \frac{1}{n}\mu(T)\right)\frac{n\varepsilon}{2(n-\mu(T))} = \varepsilon\end{aligned}$$

for any integer $n \geq 6$ and $i = 1, 2, \ldots, n$. Now, for any $n \geq 6$ we consider the convex linear combination with coefficients $a_i^n = \frac{1}{n}$ for $i = 1, 2, \ldots, n$, namely

$$\begin{aligned}\sum_{i=1}^n a_i^n x_i^n(t) &= \frac{1}{n}\left(\frac{n\varepsilon}{2\mu(T)} + 1\right) + \frac{n-1}{n}\left(1 - \frac{n\varepsilon}{2(n-\mu(T))}\right) \\ &= \frac{\varepsilon}{2\mu(T)} + \frac{1}{n} + \frac{n-1}{n} - \frac{(n-1)\varepsilon}{2(n-\mu(T))} \\ &= 1 + \frac{\varepsilon}{2\mu(T)} - \frac{(n-1)\varepsilon}{2(n-\mu(T))}\end{aligned}$$

for any $n \geq 6$ and $t \in [0, \mu(T))$. Hence

$$\begin{aligned}\left\| \sum_{i=1}^n a_i^n x_i^n - \chi_T \right\|_{L^1+L^\infty} &= \left\| \left(1 + \frac{\varepsilon}{2\mu(T)} - \frac{(n-1)\varepsilon}{2(n-\mu(T))}\right)\chi_T - \chi_T \right\|_{L^1+L^\infty} \\ &= \left| \frac{\varepsilon}{2\mu(T)} - \frac{(n-1)\varepsilon}{2(n-\mu(T))} \right| \|\chi_T\|_{L^1+L^\infty} = \frac{(n-1)\varepsilon}{2(n-\mu(T))} - \frac{\varepsilon}{2\mu(T)}\end{aligned}$$

for any $n \geq 6$. Consequently,

$$\delta_x(\varepsilon) \leq \lim_{n\to\infty} \frac{(n-1)\varepsilon}{2(n-\mu(T))} - \frac{\varepsilon}{2\mu(T)} = \frac{\varepsilon}{2} - \frac{\varepsilon}{2\mu(T)} = \frac{(\mu(T)-1)\varepsilon}{2\mu(T)}. \tag{3.12}$$

2. Let $\mu(T) > 3$. Take $\varepsilon \in (0,1)$ and denote by $[\mu(T)]$ the greatest integer not bigger than $\mu(T)$. For any $n \geq 2$ define $k(n) \in \{0, 1, \ldots, n-1\}$ such that

$$\frac{k(n)}{n} \leq \mu(T) - [\mu(T)] \leq \frac{k(n)+1}{n}. \tag{3.13}$$

For any integer $n \geq 2$ and $i = 1, 2, \ldots, n[\mu(T)] + k(n)$, let

$$B_i^n = \begin{cases} [\frac{i-1}{n}, \frac{n+i-1}{n}) & \text{if } \frac{n+i-1}{n} \leq \mu(T), \\ [\frac{i-1}{n}, \frac{n[\mu(T)]+k(n)}{n}] \cup [0, \frac{n+i-1-n[\mu(T)]-k(n)}{n}] & \text{if } \frac{n+i-1}{n} > \mu(T), \end{cases}$$

and define

$$x_i^n = \chi_T - \varepsilon\chi_{B_i^n}.$$

Notice that

$$\|x_i^n\|_{L^1+L^\infty} = \int_0^1 (x_i^n)^*(t)dt = \int_0^1 dt = 1$$

and

$$\|x_i^n - \chi_T\|_{L^1+L^\infty} = \varepsilon\|\chi_{B_i^n}\|_{L^1+L^\infty} = \varepsilon$$

for any integer $n \geq 2$ and $i = 1, 2, \ldots, n[\mu(T)] + k(n)$. Taking $a_i^n = \frac{1}{n[\mu(T)]+k(n)}$, we get

$$\begin{aligned}\sum_{i=1}^{n[\mu(T)]+k(n)} a_i^n x_i^n(t) &= \frac{n}{n[\mu(T)]+k(n)}(1-\varepsilon) + \frac{n[\mu(T)]+k(n)-n}{n[\mu(T)]+k(n)} \\ &= 1 - \frac{n\varepsilon}{n[\mu(T)]+k(n)}\end{aligned}$$

for any integer $n \geq 2$ and $t \in [0, \mu(T)]$. Hence

$$\begin{aligned}\Big\| \sum_{i=1}^{n[\mu(T)]+k(n)} a_i^n x_i^n - \chi_T \Big\|_{L^1+L^\infty} &= \Big\| \Big(1 - \frac{n\varepsilon}{n[\mu(T)]+k(n)}\Big)\chi_T - \chi_T \Big\|_{L^1+L^\infty} \\ &= \frac{n\varepsilon}{n[\mu(T)]+k(n)} \|\chi_T\|_{L^1+L^\infty} = \frac{n\varepsilon}{n[\mu(T)]+k(n)}\end{aligned}$$

for any integer $n \geq 2$. Notice that

$$\frac{n\varepsilon}{n[\mu(T)]+k(n)} = \frac{\varepsilon}{[\mu(T)] + \frac{k(n)}{n}}$$

whence, by (3.13),

$$\begin{aligned}\frac{\varepsilon}{\mu(T)} &= \frac{\varepsilon}{[\mu(T)] + (\mu(T) - [\mu(T)])} \\ &\leq \frac{\varepsilon}{[\mu(T)] + \frac{k(n)}{n}} = \frac{\varepsilon}{[\mu(T)] + \frac{k(n)+1}{n} - \frac{1}{n}} \\ &\leq \frac{\varepsilon}{[\mu(T)] + (\mu(T) - [\mu(T)]) - \frac{1}{n}} = \frac{\varepsilon}{\mu(T) - \frac{1}{n}}\end{aligned}$$

for any $n \geq 2$. Hence, by the Sandwich Theorem for infinite sequences, we get

$$\delta_x(\varepsilon) \leq \lim_{n\to\infty} \frac{n\varepsilon}{n[\mu(T)]+k(n)} = \frac{\varepsilon}{\mu(T)}. \tag{3.14}$$

Combining inequalities (3.12) and (3.14), we have

$$\delta_x(\varepsilon) \leq \min\Big\{ \frac{(\mu(T)-1)\varepsilon}{2\mu(T)}, \frac{\varepsilon}{\mu(T)} \Big\},$$

whence, by (3.12), we obtain the desired conclusion. ∎

Corollary 1. *The set $\delta_{LUR}B(L^1 + L^\infty)$ is empty.*

Proof. Since for any Banach X every LUR-point of $B(X)$ is a denting point of $B(X)$, the unit ball $B(L^1 + L^\infty)$ has no LUR-points whenever $\mu(T) \leq 1$ or $\mu(T) = \infty$. If $1 < \mu(T) < \infty$, by Theorem 1, every $x \in S(L^1 + L^\infty)$ such that $|x| = \chi_T$ is a denting point of $B(L^1 + L^\infty)$. Therefore, if $1 < \mu(T) < \infty$, the only LUR-points of $B(L^1 + L^\infty)$ are those $x \in L^0$ with $|x| = \chi_T$. Take $x \in L^1 + L^\infty$ such that $|x(t)| = 1$ for μ-a.e. $t \in T$. Let $A \subset T$ be a set such that $\mu(A) = 1$ and define $y = x\chi_A$. Obviously, $\|x\|_{L^1+L^\infty} = \|y\|_{L^1+L^\infty} = 1$. Moreover, for any positive $\varepsilon < \min\{1, \mu(T\backslash A)\}$ we have

$$\|x - y\|_{L^1+L^\infty} = \|x\chi_{T\backslash A}\|_{L^1+L^\infty} = \|\chi_{T\backslash A}\|_{L^1+L^\infty} = \min\{1, \mu(T\backslash A)\} > \varepsilon.$$

On the other hand

$$\|x + y\|_{L^1+L^\infty} = \|2\chi_A + \chi_{T\backslash A}\|_{L^1+L^\infty} = \int_0^1 2d\mu = 2.$$

Hence $x \notin \delta_{LUR}B(L^1 + L^\infty)$ and consequently $\delta_{LUR}B(L^1 + L^\infty) = \varnothing$. ∎

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