FUNCTION SPACES VIII
BANACH CENTER PUBLICATIONS, VOLUME 79
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2008

ON THE POINTWISE STRONG APPROXIMATION BY $(C, 1)(E, 1)$ MEANS

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Abstract. We present an estimate of the $(C, 1)(E, 1)$-strong means with mixed powers of the Fourier series of a function $f \in L_{2\pi}$ as a generalization of the result obtained by M. Yildrim and F. Karakus. Some corollaries on the norm approximation are also given.

1. Introduction. Let $L^p_{2\pi}$ $(1 \le p < \infty)$ [resp. $C_{2\pi}$] be the class of all 2π-periodic real-valued functions p-integrable [resp. continuous] over $Q = [-\pi, \pi]$ and let $X^p = L^p_{2\pi}$ when $1 \le p < \infty$ or $X^p = C_{2\pi}$ when $p = \infty$. Let us define the norm of $f \in X^p$ as

$$\|f\|_p = \begin{cases} (\int_Q |f(x)|^p dx)^{1/p} & \text{when } 1 \le p < \infty, \\ \sup_{x\in Q} |f(x)| & \text{when } p = \infty. \end{cases}$$

Consider the trigonometric Fourier series

$$S[f](x) = \frac{a_0(f)}{2} + \sum_{k=0}^{\infty}(a_k(f)\cos kx + b_k(f)\sin kx) = \sum_{k=0}^{\infty} C_k[f](x)$$

and denote by $S_k[f](x)$ the k-th partial sum of $S[f](x)$. Denote

$$S_{r,p}[f](x) := \frac{1}{r+1}\sum_{k=p}^{r+p} S_k[f](x) \qquad \text{for } r, p = 0, 1, 2, \dots$$

and let

$$|C_1^{q_1} E_1^{q_2}|_{n,p}[f](x) := \left\{ \frac{1}{n+1}\sum_{k=0}^{n}\left[\frac{1}{2^k}\sum_{r=0}^{k}\binom{k}{r}|S_{r,p}[f](x) - f(x)|^{q_1}\right]^{q_2/q_1}\right\}^{1/q_2}$$

2000 *Mathematics Subject Classification*: Primary 42A24.

Key words and phrases: strong approximation, rate of pointwise strong summability.
The paper is in final form and no version of it will be published elsewhere.

for $n, p = 0, 1, 2, \ldots$ and $q_1, q_2 > 0$ be the $(C, 1)$ transform of the $(E, 1)$ transform of $S_{r,p}[f](x)$ in the strong sense with the mixed q_1, q_2-powers (cf. [2], [3]).

As a measure of approximation by the above quantities we use the pointwise characteristic

$$w_x[f](\delta)_p := \left\{ \frac{1}{\delta} \int_0^\delta |\varphi_x(t)|^p dt \right\}^{1/p},$$

where

$$\varphi_x(t) := f(x+t) + f(x-t) - 2f(x),$$

constructed on the base of definition of Lebesgue points (L^p-points) (cf. [3]).

We can observe that with $\widetilde{p} \geq p$ for $f \in X^{\widetilde{p}}$, by the Minkowski inequality

$$\|w_{\cdot}[f](\delta)_p\|_{\widetilde{p}} \leq \omega_{\widetilde{p}}[f](\delta),$$

where $\omega_{\widetilde{p}}[f]$ is the modulus of continuity of f in the space $X^{\widetilde{p}}$ defined by the formula

$$\omega_{\widetilde{p}}[f](\delta) := \sup_{0<|h|\leq\delta} \|\varphi_{\cdot}(h)\|_{\widetilde{p}} \, .$$

By K we shall designate either an absolute constant or a constant depending on some parameters, not necessarily the same at each occurrence.

2. Statement of results.

We can now formulate our main result:

THEOREM 1. *If $f \in L^1_{2\pi}$ then there exists a constant $K > 0$ such that*

$$|C_1^{q_1} E_1^{q_2}|_{n,p}[f](x) \leq K \Bigg(w_x[f]\left(\frac{\pi}{n+2p+1}\right)_1 + w_x[f]\left(\frac{\pi}{(n+2p+1)^\delta}\right)_1 + \int_{(n+2p+1)^\delta}^{(n+2p+1)} \frac{w_x[f](\frac{\pi}{t})_1}{t} dt + \begin{cases} \frac{1}{n+1} \int_1^{(n+2p+1)^\delta} w_x[f](\frac{\pi}{t})\, dt & \text{when } 0 < q_2 < 1, \\ \frac{\log(n+1)}{n+1} \int_1^{(n+2p+1)^\delta} w_x[f](\frac{\pi}{t})\, dt & \text{when } q_2 = 1, \\ \frac{1}{(n+1)^{1/q_2}} \int_1^{(n+2p+1)^\delta} w_x[f](\frac{\pi}{t})\, dt & \text{when } q_2 > 1 \end{cases} \Bigg)$$

with a $\delta \in (0,\ 1)$ and $q_1, q_2 > 0$ for any $x \in \mathbf{R}$ and $n, p = 0, 1, 2, \ldots$

Hence by the Minkowski inequality and our observation we can state

THEOREM 2. *If $f \in L^q_{2\pi}$ with $1 \leq q \leq \infty$, then there exists a constant $K > 0$ such that*

$$\| \, |C_1^{q_1} E_1^{q_2}|_{n,p}[f](\cdot)\|_q \leq K \Bigg(\omega_q[f]\left(\frac{\pi}{(n+2p+1)^\delta}\right) + \int_{(n+2p+1)^\delta}^{(n+2p+1)} \frac{\omega_q[f](\frac{\pi}{t})}{t} dt. + \begin{cases} \frac{1}{n+1} \int_1^{(n+2p+1)^\delta} \omega_q[f](\frac{\pi}{t})\, dt & \text{when } 0 < q_2 < 1, \\ \frac{\log(n+1)}{n+1} \int_1^{(n+2p+1)^\delta} \omega_q[f](\frac{\pi}{t})\, dt & \text{when } q_2 = 1, \\ \frac{1}{(n+1)^{1/q_2}} \int_1^{(n+2p+1)^\delta} \omega_q[f](\frac{\pi}{t})\, dt & \text{when } q_2 > 1 \end{cases} \Bigg)$$

with a $\delta \in (0,\ 1)$ and $q_1, q_2 > 0$ for any $n, p = 0, 1, 2, \ldots$

From Theorem 1 we can derive the following corollary:

COROLLARY 3. *If we additionally assume that $f \in L^1_{2\pi}$ and $x \in \mathbf{R}$ are such that $w_x[f](t) = o_x\left(\frac{1}{\log \frac{\pi}{t}}\right)$ in case $0 < q_2 \leq \frac{1}{1-\delta}$ and $w_x[f](t) = o_x\left(t^{1-\frac{1}{(1-\delta)q_2}}\right)$ in case $q_2 > \frac{1}{1-\delta}$*

as $t \to 0$ *and if there exists an* $s \geq 1$ *such that* $p = O(n^s)$ *as* $n \to \infty$, *then*

$$|C_1^{q_1} E_1^{q_2}|_{n,p}[f](x) = o_x(1) \quad as \quad n \to \infty.$$

REMARK 4. We note that in the case $q_1 = q_2 = 1$ the above corollary is a generalization of the result obtained by M. Yildrim and F. Karakus in [1].

3. Proofs of the results.

We only prove Theorem 1 and Corollary 3.

3.1. *Proof of Theorem 1.* By simple calculations we obtain

$$\begin{aligned}
&|C_1^{q_1} E_1^{q_2}|_{n,p}[f](x) \\
&\quad \leq \left\{ \frac{1}{n+1} \sum_{k=0}^{n} \left[\frac{1}{2^k} \sum_{r=0}^{k} \binom{k}{r} \left| \frac{1}{2\pi(r+1)} \left[\int_0^{\pi/(n+2p+1)} + \int_{\pi/(n+2p+1)}^{\pi/(n+2p+1)^\delta} + \int_{\pi/(n+2p+1)^\delta}^{\pi} \right] \right.\right.\right. \\
&\qquad\qquad \left.\left.\left. \varphi_x(t) \frac{\sin(r+2p+1)t/2 \, \sin(r+1)t/2}{\sin^2 t/2} \, dt \right|^{q_1} \right]^{q_2/q_1} \right\}^{1/q_2} \\
&\quad \leq |I_1| + |I_2| + |I_3|, \qquad \text{with } 0 < \delta < 1.
\end{aligned}$$

First we have

$$\begin{aligned}
|I_1| &\leq \left\{ \frac{1}{n+1} \sum_{k=0}^{n} \left[\frac{1}{2^k} \sum_{r=0}^{k} \binom{k}{r} \left| \frac{1}{2\pi(r+1)} \right.\right.\right. \\
&\quad \times \left.\left.\left. \int_0^{\pi/(n+2p+1)} |\varphi_x(t)| \frac{(r+2p+1)t/2 \; (r+1)t/2}{t^2/\pi^2} \, dt \right|^{q_1} \right]^{q_2/q_1} \right\}^{1/q_2} \\
&= \frac{\pi^2}{8} \left\{ \frac{1}{n+1} \sum_{k=0}^{n} \left[\frac{1}{2^k} \sum_{r=0}^{k} \binom{k}{r} \left(\frac{(r+2p+1)}{2\pi} \int_0^{\pi/(n+2p+1)} |\varphi_x(t)| \, dt \right)^{q_1} \right]^{q_2/q_1} \right\}^{1/q_2} \\
&\leq \frac{\pi^2}{8} w_x[f] \left(\frac{\pi}{n+2p+1} \right)_1 \left\{ \frac{1}{n+1} \sum_{k=0}^{n} \left[\frac{1}{2^k} \sum_{r=0}^{k} \binom{k}{r} \right]^{q_2/q_1} \right\}^{1/q_2} \\
&= \frac{\pi^2}{8} w_x[f] \left(\frac{\pi}{n+2p+1} \right)_1.
\end{aligned}$$

To estimate the term I_2, we note that

$$\begin{aligned}
|I_2| &\leq \left\{ \frac{1}{n+1} \sum_{k=0}^{n} \left[\frac{1}{2^k} \sum_{r=0}^{k} \binom{k}{r} \left| \frac{1}{2\pi(r+1)} \right.\right.\right. \\
&\quad \times \left.\left.\left. \int_{\pi/(n+2p+1)}^{\pi/(n+2p+1)^\delta} |\varphi_x(t)| \frac{(r+1)t/2}{t^2/\pi^2} \, dt \right|^{q_1} \right]^{q_2/q_1} \right\}^{1/q_2} \\
&\leq \frac{1}{2\pi} \int_{\pi/(n+2p+1)}^{\pi/(n+2p+1)^\delta} \frac{|\varphi_x(t)|}{t} \, dt \left\{ \frac{1}{n+1} \sum_{k=0}^{n} \left[\frac{1}{2^k} \sum_{r=0}^{k} \binom{k}{r} \right]^{q_2/q_1} \right\}^{1/q_2}
\end{aligned}$$

$$\leq \frac{\pi}{4}\int_{\pi/(n+2p+1)}^{\pi/(n+2p+1)^{\delta}} \frac{|\varphi_x(t)|}{t}dt = \frac{\pi}{4}\int_{\pi/(n+2p+1)}^{\pi/(n+2p+1)^{\delta}} \frac{1}{t}\frac{d}{dt}\Big(\int_0^t |\varphi_x(u)|du\Big)dt$$
$$= \frac{\pi}{4}\Bigg[w_x[f]\Big(\frac{\pi}{(n+2p+1)^{\delta}}\Big)$$
$$-w_x[f]\Big(\frac{\pi}{n+2p+1}\Big) + \int_{\pi/(n+2p+1)}^{\pi/(n+2p+1)^{\delta}} \frac{w_x[f](t)}{t}\,dt\Bigg]$$
$$= \frac{\pi}{4}\Bigg[w_x[f]\Big(\frac{\pi}{(n+2p+1)^{\delta}}\Big) + \int_{(n+2p+1)^{\delta}}^{(n+2p+1)} \frac{w_x[f](\frac{\pi}{t})}{t}\,dt\Bigg].$$

The integral I_3 can be estimated as follows:

$$|I_3| \leq \Bigg\{\frac{1}{n+1}\sum_{k=0}^{n}\Bigg[\frac{1}{2^k}\sum_{r=0}^{k}\binom{k}{r}\Bigg|\frac{1}{2\pi(r+1)}$$
$$\times\int_{\pi/(n+2p+1)^{\delta}}^{\pi} |\varphi_x(t)|\frac{1}{t^2/\pi^2}\,dt\Bigg|^{q_1}\Bigg]^{q_2/q_1}\Bigg\}^{1/q_2}$$
$$= \frac{\pi}{2}\int_{\pi/(n+2p+1)^{\delta}}^{\pi} \frac{|\varphi_x(t)|}{t^2}\,dt\Bigg\{\frac{1}{n+1}\sum_{k=0}^{n}\Bigg[\frac{1}{2^k}\sum_{r=0}^{k}\Big(\frac{1}{r+1}\Big)^{q_1}\binom{k}{r}\Bigg]^{q_2/q_1}\Bigg\}^{1/q_2}$$
$$= \frac{\pi}{2}\int_{\frac{\pi}{(n+2p+1)}^{\delta}}^{\pi} \frac{1}{t^2}\frac{d}{dt}\Big(\int_0^t |\varphi_x(u)|du\Big)\,dt$$
$$\times\Bigg\{\frac{1}{n+1}\sum_{k=0}^{n}\Bigg[\frac{1}{2^k}\sum_{r=0}^{k}\Big(\frac{1}{r+1}\Big)^{q_1}\binom{k}{r}\Bigg]^{q_2/q_1}\Bigg\}^{1/q_2}$$
$$\leq \Bigg[\frac{1}{\pi}w_x[f](\pi) - \frac{(n+2p+1)^{\delta}}{\pi}w_x[f]\Big(\frac{\pi}{(n+2p+1)^{\delta}}\Big)$$
$$+2\int_{\pi/(n+2p+1)^{\delta}}^{\pi} \frac{w_x[f](t)}{t^2}\,dt\Bigg]\Bigg\{\frac{1}{n+1}\sum_{k=0}^{n}\Bigg[\frac{1}{2^k}\sum_{r=0}^{k}\Big(\frac{1}{r+1}\Big)^{q_1}\binom{k}{r}\Bigg]^{q_2/q_1}\Bigg\}^{1/q_2}$$
$$\leq \frac{\pi}{2}\Bigg[\frac{1}{\pi}w_x[f](\pi) + \frac{2}{\pi}\int_1^{(n+2p+1)^{\delta}} w_x[f]\Big(\frac{\pi}{t}\Big)\,dt\Bigg]$$
$$\times\Bigg\{\frac{1}{n+1}\sum_{k=0}^{n}\Bigg[\frac{1}{2^k}\sum_{r=0}^{k}\Big(\frac{1}{r+1}\Big)^{q_1}\binom{k}{r}\Bigg]^{q_2/q_1}\Bigg\}^{1/q_2}.$$

Now, simple calculations lead us to the estimation

$$\Bigg\{\frac{1}{n+1}\sum_{k=0}^{n}\Bigg[\frac{1}{2^k}\sum_{r=0}^{k}\Big(\frac{1}{r+1}\Big)^{q_1}\binom{k}{r}\Bigg]^{q_2/q_1}\Bigg\}^{1/q_2}$$
$$\leq 2^{1/q_1}(2+2q_1)\Bigg\{\frac{1}{n+1}\sum_{k=0}^{n}\frac{1}{(k+1)^{q_2}}\Bigg\}^{1/q_2}.$$

Hence,

$$|I_3| \le K \int_1^{(n+2p+1)^\delta} w_x[f]\left(\frac{\pi}{t}\right)\, dt \left\{\frac{1}{n+1}\sum_{k=0}^{n}\left(\frac{1}{r+1}\right)^{q_2}\right\}^{1/q_2}$$

$$\le \begin{cases} K\frac{1}{n+1}\int_1^{(n+2p+1)^\delta} w_x[f](\frac{\pi}{t})\, dt & \text{when } 0<q_2<1,\\ K\frac{\log(n+1)}{n+1}\int_1^{(n+2p+1)^\delta} w_x[f](\frac{\pi}{t})\, dt & \text{when } q_2=1,\\ K\frac{1}{(n+1)^{1/q_2}}\int_1^{(n+2p+1)^\delta} w_x[f](\frac{\pi}{t})\, dt & \text{when } q_2>1.\end{cases}$$

This completes the proof. ∎

3.2. *Proof of Corollary 3.* Since $w_x[f](t) = o_x\big(\frac{1}{\log\frac{\pi}{t}}\big)$ as $t \to 0$ we estimate the terms in Theorem 1 as follows:

$$w_x[f](\frac{\pi}{n+2p+1})_1 \le \frac{o_x(1)}{\log(n+2p+1)} \le \frac{o_x(1)}{\log(n+1)},$$

$$w_x[f](\frac{\pi}{(n+2p+1)^\delta})_1 \le \frac{o_x(1)}{\delta\ \log(n+2p+1)},$$

$$\int_{(n+2p+1)^\delta}^{(n+2p+1)} \frac{w_x[f](\frac{\pi}{t})_1}{t}\, dt \le o_x(1)\int_{(n+2p+1)^\delta}^{(n+2p+1)} \frac{dt}{t\log t}$$

$$= o_x(1)\int_{\delta\log(n+2p+1)}^{\log(n+2p+1)} \frac{dt}{t} = o_x(1)\ \log\frac{1}{\delta}.$$

For the last term we can observe that

$$\int_1^{(n+2p+1)^\delta} w_x[f]\left(\frac{\pi}{t}\right)_1 dt \le o_x(1) w_x[f](\pi)(n+2p+1)^\delta$$

and if we put $\delta = \frac{1}{s+1}$, then $(n+2p+1)^\delta \le (n+2Kn^s+1)^{1/(s+1)} \le K(n+1)^{s/(s+1)}$ and

$$\int_1^{(n+2p+1)^\delta} w_x[f]\left(\frac{\pi}{t}\right)_1 \le o_x(1)K(n+1)^{s/(s+1)}.$$

Hence, for $0 < q_2 \le \frac{1}{1-\delta}$ our corollary follows but otherwise, since $w_x[f](t) = o_x\big(t^{1-\frac{1}{(1-\delta)q_2}}\big)$ as $t \to 0$ we obtain

$$\frac{1}{(n+1)^{1/q_2}}\int_1^{(n+2p+1)^\delta} w_x[f]\left(\frac{\pi}{t}\right)_1 \le o_x(1)K\frac{1}{(n+1)^{1/q_2}}(n+1)^{(1-\delta)\frac{1}{(1-\delta)q_2}} = o_x(1),$$

and our proof is complete. ∎

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