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ON THE APPROXIMATION OF FUNCTIONS BY MATRIX MEANS IN THE GENERALIZED HÖLDER METRIC

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Abstract. Under some assumptions on the matrix of a summability method, whose rows are sequences of bounded variation, we obtain a generalization and an improvement of some results of Xie-Hua Sun and L. Leindler.

1. Introduction. Let f be a continuous and 2π-periodic function ($f \in C_{2\pi}$) and let

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1.1}$$

be its Fourier series. Denote by $S_n(x) = S_n(f, x)$ the n-th partial sum of (1.1) and by $\omega(f, \delta)$ the modulus of continuity of $f \in C_{2\pi}$.

The usual supremum norm will be denoted by $\|\cdot\|_C$.

Let ω be a nondecreasing continuous function on the interval $[0, 2\pi]$ having the properties

$$\omega(0) = 0, \qquad \omega(\delta_1 + \delta_2) \le \omega(\delta_1) + \omega(\delta_2).$$

Such a function will be called a modulus of continuity.

Denote by H^ω the class of functions

$$H^\omega := \{f \in C_{2\pi};\ |f(x) - f(y)| \le C\omega(|x - y|)\},$$

where C is a positive constant. For $f \in H^\omega$, we define the norm $\|\cdot\|_\omega$ by the formula

$$\|f\|_\omega := \|f\|_C + \sup_{x,\ y} |\Delta^\omega f(x, y)|,$$

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where

$$\Delta^{\omega} f(x, y) = \frac{|f(x) - f(y)|}{\omega(|x - y|)}, \quad x \neq y.$$

If $\omega(t) = C_1 |t|^{\alpha}$ $(0 < \alpha \leq 1)$, where $C_1$is a positive constant, then

$$H^{\alpha} = \{f \in C_{2\pi};\ |f(x) - f(y)| \leq C_1 |x - y|^{\alpha},\ 0 < \alpha \leq 1\}$$

is a Banach space and the metric induced by the norm $\|\cdot\|_{\alpha}$ on H^{α} is said to be the Hölder metric.

Let $A := (a_{nk})$ $(k, n = 0, 1, \dots)$ be a lower triangular infinite matrix of real numbers satisfying the following conditions:

$$\text{i) } a_{nk} \geq 0\ \ (k = 0, 1, \dots), a_{nk} = 0, (k > n) \text{ and } \sum_{k=0}^{n} a_{nk} = 1, \tag{1.2}$$

where $n = 0, 1, 2, \dots,$

$$\text{ii) for every } k,\ a_{nk} \to 0\,(n \to \infty). \tag{1.3}$$

Let the A-transformation of $(S_n(f; x))$ be given by

$$T_n(f) := T_n(f; x) := \sum_{k=0}^{n} a_{nk} S_k(f; x) \quad (n = 0, 1, \dots).$$

and let

$$A_{nk} = \sum_{r=n-k+1}^{n} a_{nr}, \ \ (k = 1, 2, \dots, n+1).$$

Let $\omega(t)$ be a given modulus of continuity satisfying the following condition:

$$\frac{(\omega(f; t))^{\frac{p}{q}}}{\omega(t)} = O(1) \ \ (t \to 0_+), \tag{1.4}$$

for $0 \leq p < q \leq 1$ and $f \in C$.

In [6] Xie-Hua Sun proved the following theorems:

Theorem 1. *Let $A = (a_{nk})$ satisfy the conditions (1.2), (1.3) and $a_{nk} \leq a_{nk+1}$ for $k = 0, 1, \dots, n-1$ and $n = 0, 1, \dots$. If $f \in C$ and $\omega(f; t)$, $\omega(t)$ satisfy (1.4) then*

$$\|T_n(f) - f\|_{\omega} = O\left(\left\{\sum_{k=1}^{n} \frac{A_{nk}}{k}\right\}^{\frac{p}{q}} \left\{\sum_{k=1}^{n} A_{nk} \frac{\omega\left(f, \frac{1}{k}\right)}{k}\right\}^{1-\frac{p}{q}}\right). \tag{1.5}$$

Theorem 2. *Let $A = (a_{nk})$ satisfy the conditions (1.2), (1.3) and $a_{nk} \geq a_{nk+1}$ for $k = 0, 1, \dots, n-1$ and $n = 0, 1, \dots$. If $f \in C$ and $\omega(f; t)$, $\omega(t)$ satisfy (1.4) then*

$$\|T_n(f) - f\|_{\omega} = O\left(\left\{\sum_{k=0}^{n} a_{nk} \omega\left(f, \frac{1}{k+1}\right)\right\}^{1-\frac{p}{q}}\right). \tag{1.6}$$

Now we define two classes of sequences ([3]).

A sequence $c := (c_n)$ of nonnegative numbers tending to zero is called the Rest Bounded Variation Sequence, or briefly $c \in RBVS$, if it has the property

$$\sum_{k=m}^{\infty} |c_n - c_{n+1}| \leq K(c) c_m \tag{1.7}$$

for all natural numbers m, where $K(c)$ is a constant depending only on c.

A sequence $c := (c_n)$ of nonnegative numbers will be called the Head Bounded Variation Sequence, or briefly $c \in HBVS$, if it has the property

$$\sum_{k=0}^{m-1} |c_n - c_{n+1}| \leq K(c)c_m \tag{1.8}$$

for all natural numbers m, or only for all $m \leq N$ if the sequence c has only finite nonzero terms and the last nonzero term is c_N.

Therefore we assume that the sequence $(K(\alpha_n))_{n=0}^{\infty}$ is bounded, that is, that there exists a constant K such that

$$0 \leq K(\alpha_n) \leq K$$

holds for all n, where $K(\alpha_n)$ denote the sequence of constants appearing in the inequalities (1.7) or (1.8) for the sequence $\alpha_n := (a_{nk})_{k=0}^{\infty}$. Now we can give the conditions to be used later on. We assume that for all n and $0 \leq m \leq n$

$$\sum_{k=m}^{\infty} |a_{nk} - a_{nk+1}| \leq K a_{nm} \tag{1.9}$$

and

$$\sum_{k=0}^{m-1} |a_{nk} - a_{nk+1}| \leq K a_{nm} \tag{1.10}$$

hold if $\alpha_n := (a_{nk})_{k=0}^{\infty}$ belongs to $RBVS$ or $HBVS$, respectively.

In [3] L. Leindler proved the following theorem:

Theorem 3. *Let us assume that (1.2) and (1.9) hold. Then for $f \in C_{2\pi}$*

$$\|T_n(f) - f\|_C = O\left(\omega\left(f; \frac{\pi}{n}\right) + \sum_{k=1}^{n} \frac{\omega\left(f, \frac{\pi}{k}\right)}{k} \sum_{r=0}^{k+1} a_{nr}\right). \tag{1.11}$$

In the paper we present estimates of the deviation $T_n(f) - f$ in the norm $\|\cdot\|_\omega$ under more general assumptions on the elements of the matrix A.

Throughout the paper we shall use the following notations:

$$\phi_x(t) = f(x+t) + f(x-t) - 2f(x), \qquad A_n(u) = \sum_{k=0}^{n} a_{nk} \frac{\sin\left(k + \frac{1}{2}\right)u}{\sin \frac{1}{2}u}.$$

By $K_1, K_2, \ldots$ we shall designate either an absolute constant or a constant depending on the indicated parameters, not necessarily the same at each occurrence.

2. Main results.

Our main results are the following.

Theorem 4. *Let (1.2), (1.10) hold. If $f \in C$ and $\omega(f;t)$, $\omega(t)$ satisfy (1.4) then*

$$\|T_n(f) - f\|_\omega = O\left(\left\{\sum_{k=1}^{n+1} \frac{A_{nk}}{k}\right\}^{\frac{p}{q}} \left\{\sum_{k=1}^{n+1} A_{nk} \frac{\omega\left(f, \frac{1}{k}\right)}{k}\right\}^{1-\frac{p}{q}}\right). \tag{2.1}$$

THEOREM 5. *Let (1.2), (1.9) hold. If $f \in C$ and $\omega(f;t)$, $\omega(t)$ satisfy (1.4) then*

$$\|T_n(f) - f\|_\omega = O\left(\left\{\sum_{k=0}^{n} a_{nk}\omega\left(f, \frac{1}{k+1}\right)\right\}^{1-\frac{p}{q}}\right). \tag{2.2}$$

REMARK 1. If the elements of the matrix A satisfy the condition $a_{nk} \le a_{nk+1}$ for $k = 0, 1, \ldots, n-1$ and $n = 0, 1, \ldots$, then the condition (1.10) also holds and therefore Theorem 1 is a corollary of Theorem 4. Analogously we can derive Theorem 2 from Theorem 5.

If in the assumptions of the Theorem 5 we take $\omega(|t|) = O(|t|^p)$ with $p = 0$, then we have immediately the following corollary

COROLLARY 1. *Under the assumptions of Theorem 3 we have*

$$\|T_n(f) - f\|_C = O\left(\sum_{k=0}^{n} a_{nk}\omega\left(f, \frac{1}{k+1}\right)\right).$$

REMARK 2. It is easy to see that from the above estimate the estimate (1.11) follows by (1.9).

3. Lemmas.

To prove our theorems we need the following lemmas.

LEMMA 1. *If $\alpha_n := (a_{nk})_{k=0}^{\infty}$ belongs to $HBVS$, then for $\frac{1}{n} \le u \le \pi$*

$$|A_n(u)| \le \frac{\pi^2 (K(\alpha_n) + 1)^2 + \pi}{u} A_{n,u^{-1}}, \tag{3.1}$$

where $u^{-1} := \max\{1, [u^{-1}]\}$.

Proof. An elementary calculation shows that

$$|A_n(u)| \le \left|\sum_{k=0}^{n-u^{-1}} a_{nk} \frac{\sin\left(k+\frac{1}{2}\right)u}{\sin\frac{1}{2}u}\right| + \frac{\pi}{u}\sum_{k=n-u^{-1}+1}^{n} a_{nk} = \left|\sum\nolimits_1\right| + \frac{\pi}{u}A_{n,u^{-1}}.$$

Applying the Abel transformation and using (1.8), we get

$$\begin{aligned}\left|\sum\nolimits_1\right| &\le \sum_{k=0}^{n-u^{-1}-1} |a_{nk} - a_{nk+1}| \left(\frac{\sin\frac{k+1}{2}u}{\sin\frac{1}{2}u}\right)^2 + a_{n,n-u^{-1}}\left(\frac{\sin\frac{n-u^{-1}+1}{2}u}{\sin\frac{1}{2}u}\right)^2 \\ &\le \frac{\pi^2}{u^2}\sum_{k=0}^{n-u^{-1}-1} |a_{nk} - a_{nk+1}| + \frac{\pi^2}{u^2}a_{n,n-u^{-1}} \le \frac{\pi^2(K(\alpha_n)+1)}{u^2}a_{n,n-u^{-1}}.\end{aligned}$$

If $\alpha_n := \{a_{nk}\}_{k=0}^{\infty}$ belongs to $HBVS$ then, by (1.8)

$$a_{n\mu} - a_{nm} \le |a_{n\mu} - a_{nm}| \le \sum_{k=\mu}^{m-1} |a_{nk} - a_{nk+1}| \le K(\alpha_n)\, a_{nm}.$$

for any $m \ge \mu \ge 0$, whence we have

$$a_{n\mu} \le (K(\alpha_n) + 1)\, a_{nm}. \tag{3.2}$$

From this, we obtain

$$|A_n(u)| \le \frac{\pi^2 (K(\alpha_n) + 1)^2}{u} \sum_{k=n-u^{-1}+1}^{n} a_{nk} + \frac{\pi}{u}A_{n,u^{-1}}$$

$$\leq \frac{\pi^2 (K(\alpha_n)+1)^2+\pi}{u} A_{n,u^{-1}}$$

and our proof is complete. ■

Lemma 2. *If $\alpha_n := (a_{nk})_{k=0}^{\infty}$ belongs to $RBVS$, then for $f \in C_{2\pi}$*

$$\|T_n(f)-f\|_C \leq 8(K(\alpha_n)+1)(2K(\alpha_n)+1)\sum_{k=0}^{n} a_{nk}E_k(f), \tag{3.3}$$

where $E_n(f)$ denotes the best approximation of the function f by trigonometric polynomials of order at most n.

Proof. Let $m=m_n \geq 0$ be such that $2^m \leq n+1 < 2^{m+1}$ for $n=0,1,\dots$, then applying the Abel transformation we get

$$T_n(f;x)-f(x)=\sum_{k=0}^{n} a_{nk}(S_k(f;x)-f(x)) = a_{n0}(S_0(f;x)-f(x))$$

$$+\sum_{k=1}^{m-1}\sum_{i=2^{k-1}}^{2^k-1} a_{ni}(S_i(f;x)-f(x)) + \sum_{k=2^{m-1}}^{n} a_{nk}(S_k(f;x)-f(x))$$

$$= a_{n0}(S_0(f;x)-f(x)) + \sum_{k=1}^{m-1}\Bigg(\sum_{i=2^{k-1}}^{2^k-2}(a_{ni}-a_{ni+1})\sum_{l=2^{k-1}}^{i}(S_l(f;x)-f(x))$$

$$+a_{n,2^k-1}\sum_{i=2^{k-1}}^{2^k-1}(S_i(f;x)-f(x))\Bigg)$$

$$+\sum_{k=2^{m-1}}^{n-1}(a_{nk}-a_{nk+1})\sum_{l=2^{m-1}}^{k}(S_l(f;x)-f(x)) + a_{nn}\sum_{k=2^{m-1}}^{n}(S_k(f;x)-f(x))$$

$$= a_{n0}(\sigma_{00}(f;x)-f(x))$$

$$+\sum_{k=1}^{m-1}\sum_{i=2^{k-1}}^{2^k-2}\Big((a_{ni}-a_{ni+1})\left(i-2^{k-1}+1\right)\left(\sigma_{i,i-2^{k-1}}(f;x)-f(x)\right)$$

$$+a_{n,2^k-1}2^{k-1}\left(\sigma_{2^k-1,2^{k-1}-1}(f;x)-f(x)\right)\Big)$$

$$+\sum_{k=2^{m-1}}^{n-1}(a_{nk}-a_{nk+1})\left(k-2^{m-1}+1\right)\left(\sigma_{k,k-2^{m-1}}(f;x)-f(x)\right)$$

$$+a_{nn}\left(n-2^{m-1}+1\right)\left(\sigma_{n,n-2^{m-1}}(f;x)-f(x)\right),$$

where

$$\sigma_{n,m}(f;x)=\frac{1}{m+1}\sum_{k=n-m}^{n}(S_k(f;x)-f(x)) \quad (0\leq m\leq n)$$

is the de la Vallée-Poussin mean. Using the well known inequality [6]

$$\|\sigma_{n,m}(f)-f\|_C \leq 2\frac{n+1}{m+1}E_{n-m}(f),$$

we obtain

$$\left\|T_n(f)-f\right\|_C \leq 2a_{n0}E_0(f)$$

$$+\sum_{k=1}^{m-1}\Big(2E_{2^{k-1}}(f)\sum_{i=2^{k-1}}^{2^k-2}\left|a_{ni}-a_{ni+1}\right|(i+1)+4a_{n,2^k-1}2^{k-1}E_{2^{k-1}}(f)\Big)$$

$$+2E_{2^{m-1}}(f)\sum_{k=2^{m-1}}^{n-1}\left|a_{nk}-a_{nk+1}\right|(k+1)+2a_{nn}(n+1)E_{2^{m-1}}(f).$$

By (1.7) we have

$$\left\|T_n(f)-f\right\|_C \leq 2a_{n0}E_0(f)$$

$$+\sum_{k=1}^{m-1}\left(2K(\alpha_n)\left(2^k-1\right)a_{n,2^{k-1}}E_{2^{k-1}}(f)+4a_{n,2^k-1}2^{k-1}E_{2^{k-1}}(f)\right)$$

$$+2nK(\alpha_n)a_{n,2^{m-1}}E_{2^{m-1}}(f)+2(n+1)a_{nn}E_{2^{m-1}}(f).$$

If $\alpha_n := \{a_{nk}\}_{k=0}^{\infty}$ belongs to $RBVS$ then, by (1.7)

$$a_{nn}-a_{nm} \leq \left|a_{nm}-a_{nn}\right| \leq \sum_{k=m}^{n-1}\left|a_{nk}-a_{nk+1}\right|$$

$$\leq \sum_{k=m}^{\infty}\left|a_{nk}-a_{nk+1}\right| \leq K(\alpha_n)a_{nm}$$

holds for any $n \geq m \geq 0$, whence we have

$$a_{nn} \leq (K(\alpha_n)+1)a_{nm}. \tag{3.4}$$

From this

$$\left\|T_n(f)-f\right\|_C \leq 2a_{n0}E_0(f)$$

$$+(2K(\alpha_n)+1)\Big(4\sum_{k=1}^{m-1}a_{n,2^{k-1}}2^{k-1}E_{2^{k-1}}(f)+2(n+1)a_{n,2^{m-1}}E_{2^{m-1}}(f)\Big).$$

By (3.4)

$$(n+1)a_{n,2^{m-1}}E_{2^{m-1}}(f) \leq 2^{m+1}a_{n,2^{m-1}}E_{2^{m-1}}(f)$$

$$\leq 4a_{n,2^{m-1}}\sum_{k=2^{m-2}+1}^{2^{m-1}}E_k(f) \leq 4(K(\alpha_n)+1)\sum_{k=2^{m-2}+1}^{2^{m-1}}a_{nk}E_k(f)$$

and for $k \geq 2$

$$a_{n,2^{k-1}}2^{k-1}E_{2^{k-1}}(f) \leq 2a_{n,2^{k-1}}\sum_{i=2^{k-2}+1}^{2^{k-1}}E_i(f) \leq 2(K(\alpha_n)+1)\sum_{i=2^{k-2}+1}^{2^{k-1}}a_{ni}E_i(f).$$

Therefore

$$\left\|T_n(f)-f\right\|_C \leq 2a_{n0}E_0(f)+4(2K(\alpha_n)+1)a_{n1}E_1(f)$$

$$+8(K(\alpha_n)+1)(2K(\alpha_n)+1)\Big(\sum_{k=2}^{m-1}\sum_{i=2^{k-2}+1}^{2^{k-1}}a_{ni}E_i(f)+\sum_{k=2^{m-2}+1}^{2^{m-1}}a_{nk}E_k(f)\Big)$$

$$\leq 8\left(K\left(\alpha_n\right)+1\right)\left(2K\left(\alpha_n\right)+1\right)\sum_{k=0}^{2^{m-1}} a_{nk}E_k\left(f\right)$$

$$\leq 8\left(K\left(\alpha_n\right)+1\right)\left(2K\left(\alpha_n\right)+1\right)\sum_{k=0}^{n} a_{nk}E_k\left(f\right)$$

and our proof is complete. ∎

Lemma 3. *If $\alpha_n := (a_{nk})_{k=0}^{\infty}$ belongs to $RBVS$, then*

$$\int_0^{\pi} \left|A_n\left(t\right)\right| dt \leq 4K\left(\alpha_n\right)\left(K\left(\alpha_n\right)+1\right). \tag{3.5}$$

Proof. Let $m = m_n \geq 0$ be such that $2^m \leq n+1 < 2^{m+1}$ for $n = 0, 1, \ldots,$ then applying the Abel transformation we get

$$A_n\left(t\right) = \sum_{k=0}^{n}\left(a_{nk}-a_{nk+1}\right)\left(k+1\right)F_k\left(t\right) = \left(a_{n0}-a_{n1}\right)F_0\left(t\right)$$

$$+\sum_{k=1}^{m-1}\sum_{l=2^{k-1}}^{2^k-1}\left(a_{nl}-a_{nl+1}\right)\left(l+1\right)F_l\left(t\right)+\left(n+1\right)F_n\left(t\right)\sum_{k=2^{m-1}}^{n}\left(a_{nk}-a_{nk+1}\right),$$

where $F_n(t)$ is the Fejér kernel. Since the kernel $F_n\left(t\right)$ is positive, using (1.7) we can show that

$$\int_0^{\pi}\left|A_n\left(t\right)\right|dt \leq \int_0^{\pi}\Big\{\left|a_{n0}-a_{n1}\right|F_0\left(t\right)+\sum_{k=1}^{m-1}\sum_{l=2^{k-1}}^{2^k-1}\left|a_{nl}-a_{nl+1}\right|\left(l+1\right)F_l\left(t\right)$$

$$+\left(n+1\right)F_n\left(t\right)\sum_{k=2^{m-1}}^{n}\left|a_{nk}-a_{nk+1}\right|\Big\}dt$$

$$=\left|a_{n0}-a_{n1}\right|\int_0^{\pi}F_0\left(t\right)dt+\sum_{k=1}^{m-1}\sum_{l=2^{k-1}}^{2^k-1}\left|a_{nl}-a_{nl+1}\right|\left(l+1\right)\int_0^{\pi}F_l\left(t\right)dt$$

$$+\left(n+1\right)\sum_{k=2^{m-1}}^{n}\left|a_{nk}-a_{nk+1}\right|\int_0^{\pi}F_n\left(t\right)dt$$

$$=\frac{1}{2}\Big\{\left|a_{n0}-a_{n1}\right|+\sum_{k=1}^{m-1}\sum_{l=2^{k-1}}^{2^k-1}\left|a_{nl}-a_{nl+1}\right|\left(l+1\right)$$

$$\left(n+1\right)\sum_{k=2^{m-1}}^{n}\left|a_{nk}-a_{nk+1}\right|\Big\}$$

$$\leq\frac{K(\alpha_n)}{2}\Big(a_{n0}+\sum_{k=1}^{m-1}2^k a_{n,2^{k-1}}+\left(n+1\right)a_{n,2^{m-1}}\Big).$$

By (3.4)

$$\left(n+1\right)a_{n,2^{m-1}}\leq 2^{m+1}a_{n,2^{m-1}}\leq 8\left(K(\alpha_n)+1\right)\sum_{k=2^{m-2}+1}^{2^{m-1}}a_{nk}$$

and for $k \geq 2$

$$2^k a_{n,2^{k-1}} \leq 4\left(K(\alpha_n)+1\right) \sum_{l=2^{k-2}+1}^{2^{k-1}} a_{nl}.$$

Thus we obtain the desired estimate

$$\int_0^\pi |A_n(t)|\, dt \leq \frac{K(\alpha_n)}{2}\Big(a_{n0}+2a_{n1}+4\left(K(\alpha_n)+1\right)\sum_{k=1}^{m-1}\sum_{l=2^{k-2}+1}^{2^{k-1}} a_{nl}$$

$$+8\left(K(\alpha_n)+1\right)\sum_{k=2^{m-2}+1}^{2^{m-1}} a_{nk}\Big)$$

$$\leq 4K(\alpha_n)\left(K(\alpha_n)+1\right)\sum_{k=0}^{2^{m-1}} a_{nk}$$

$$\leq 4K(\alpha_n)\left(K(\alpha_n)+1\right)\sum_{k=0}^{n} a_{nk} = 4K(\alpha_n)\left(K(\alpha_n)+1\right). \quad \blacksquare$$

4. Proofs of the theorems.

In this section we shall prove our theorems.

4.1. *Proof of Theorem 4.* Setting

$$R_n(x,y) = R_n(x) - R_n(y) = \frac{1}{2\pi}\int_0^\pi \left(\phi_x(t)-\phi_y(t)\right) A_n(t)\, dt$$

and

$$R_n(x) = T_n(f;x) - f(x) = \frac{1}{2\pi}\int_0^\pi \phi_x(t)\, A_n(t)\, dt$$

we get

$$|R_n(x,y)| \leq \frac{1}{2\pi}\int_0^\pi |\phi_x(t)-\phi_y(t)|\, |A_n(t)|\, dt$$

and

$$|R_n(x)| \leq \frac{1}{2\pi}\int_0^\pi |\phi_x(t)|\, |A_n(t)|\, dt \leq \frac{1}{\pi}\int_0^\pi \omega(f;t)\, |A_n(t)|\, dt.$$

It is clear that

$$|\phi_x(t)-\phi_y(t)| \leq 4\omega(f;|x-y|) \tag{4.1}$$

and

$$|\phi_x(t)-\phi_y(t)| \leq 4\omega(f;|t|). \tag{4.2}$$

Hence, using (4.1), we have

$$|R_n(x,y)| \leq \frac{2C}{\pi}\omega(f;|x-y|)\left(\int_0^{\frac{\pi}{n}} + \int_{\frac{\pi}{n}}^{\pi}\right)|A_n(t)|\, dt$$

$$= \frac{2C}{\pi}\omega(f;|x-y|)\left(I_1+I_2\right). \tag{4.3}$$

It is obvious that

$$I_1 \leq \int_0^{\frac{\pi}{n}} \frac{1}{\sin \frac{1}{2} t} \sum_{k=0}^{n} a_{nk} \left| \sin \left(k + \frac{1}{2} \right) t \right| dt \leq (2n+1) \frac{\pi}{n} < 3\pi. \tag{4.4}$$

Using (3.1), we get

$$\begin{aligned} I_2 &\leq K_1 \int_{\frac{\pi}{n}}^{\pi} \frac{A_{n,t^{-1}}}{t} dt = K_1 \int_{\frac{1}{\pi}}^{\frac{n}{\pi}} \frac{A_{n,t}}{t} dt \\ &= K_1 \sum_{k=1}^{n-1} \int_{\frac{k}{\pi}}^{\frac{k+1}{\pi}} \frac{A_{n,t}}{t} dt \leq K_1 \sum_{k=1}^{n-1} \frac{A_{n,k+1}}{k} \\ &\leq 2K_1 \sum_{k=2}^{n} \frac{A_{n,k}}{k} \leq 2K_1 \sum_{k=1}^{n+1} \frac{A_{n,k}}{k}. \end{aligned} \tag{4.5}$$

From (3.2) and (1.2) we can observe that

$$\sum_{k=1}^{n+1} \frac{A_{n,k}}{k} = \sum_{k=1}^{n+1} \frac{1}{k} \sum_{r=n-k+1}^{n} a_{nr} \geq \frac{1}{K+1} \sum_{k=1}^{n+1} a_{n,n-k+1} = \frac{1}{K+1} \sum_{k=0}^{n} a_{nk} = \frac{1}{K+1}$$

and by (4.3)-(4.5), we obtain

$$|R_n(x,y)| \leq K_2 \omega(f; |x-y|) \sum_{k=1}^{n+1} \frac{A_{n,k}}{k}. \tag{4.6}$$

On the other hand, using (4.2), we have

$$\begin{aligned} |R_n(x,y)| &\leq \frac{2}{\pi} \int_0^{\pi} \omega(f;t) |A_n(t)| dt \\ &= \frac{2}{\pi} \Big(\int_0^{\frac{\pi}{n}} + \int_{\frac{\pi}{n}}^{\pi} \Big) \omega(f;t) |A_n(t)| dt = \frac{2}{\pi} (I_1' + I_2'). \end{aligned} \tag{4.7}$$

Again by (1.2)

$$I_1' \leq \omega\left(f, \frac{\pi}{n}\right) \int_0^{\frac{\pi}{n}} |A_n(t)| dt \leq 3\pi \omega\left(f, \frac{\pi}{n}\right).$$

and by (3.1) and using the monotonicity of the modulus of continuity

$$\begin{aligned} I_2' &\leq K_1 \int_{\frac{\pi}{n}}^{\pi} \omega(f;t) \frac{A_{n,t^{-1}}}{t} dt = K_1 \int_{\frac{1}{\pi}}^{\frac{n}{\pi}} \omega\left(f; \frac{1}{t}\right) \frac{A_{n,t}}{t} dt \\ &= K_1 \sum_{k=1}^{n-1} \int_{\frac{k}{\pi}}^{\frac{k+1}{\pi}} \omega\left(f; \frac{1}{t}\right) \frac{A_{n,t}}{t} dt \leq K_1 \sum_{k=1}^{n-1} \omega\left(f; \frac{\pi}{k}\right) \frac{A_{n,k+1}}{k} \\ &\leq 8K_1 \sum_{k=2}^{n} \omega\left(f; \frac{\pi}{k}\right) \frac{A_{n,k}}{k} \leq 8K_1 \sum_{k=1}^{n+1} \omega\left(f; \frac{\pi}{k}\right) \frac{A_{n,k}}{k}. \end{aligned} \tag{4.8}$$

From the monotonicity of the modulus of continuity by (3.2) and (1.2)

$$\sum_{k=1}^{n+1} \omega\left(f; \frac{\pi}{k}\right) \frac{A_{n,k}}{k} \geq \omega\left(f, \frac{\pi}{n+1}\right) \sum_{k=1}^{n+1} \frac{A_{n,k}}{k}$$

$$\geq \frac{1}{2}\omega\left(f,\frac{\pi}{n}\right)\frac{1}{K+1}\sum_{k=0}^{n}a_{nk}=\frac{1}{2\left(K+1\right)}\omega\left(f,\frac{\pi}{n}\right). \tag{4.9}$$

Combining (4.6)-(4.9) we obtain

$$\left|R_{n}\left(x,y\right)\right|\leq K_{3}\sum_{k=1}^{n+1}\omega\left(f;\frac{\pi}{k}\right)\frac{A_{n,k}}{k} \tag{4.10}$$

and the same estimate for $\left|R_{n}\left(x\right)\right|$ is true

$$\left|R_{n}\left(x\right)\right|\leq K_{4}\sum_{k=1}^{n+1}\omega\left(f;\frac{\pi}{k}\right)\frac{A_{n,k}}{k}. \tag{4.11}$$

Using (4.6), (1.4) and (4.10), we get

$$\sup_{x,\ y}\left\{\Delta^{\omega}R_{n}\left(x,y\right)\right\}=\sup_{x,\ y}\left\{\frac{\left|R_{n}\left(x,y\right)\right|^{\frac{p}{q}}}{\omega\left(\left|x-y\right|\right)}\left|R_{n}\left(x,y\right)\right|^{1-\frac{p}{q}}\right\}$$

$$\leq K_{5}\left\{\sum_{k=1}^{n+1}\frac{A_{nk}}{k}\right\}^{\frac{p}{q}}\left\{\sum_{k=1}^{n+1}A_{nk}\frac{\omega\left(f,\frac{1}{k}\right)}{k}\right\}^{1-\frac{p}{q}}. \tag{4.12}$$

Now collecting our partial results (4.11) and (4.12) we obtain (2.1), and this completes the proof. ∎

4.2. *Proof of Theorem 5.* Using the same notations as in the above proof, from (4.1) and (3.5) we get

$$\left|R_{n}\left(x,y\right)\right|\leq\frac{2C}{\pi}\omega\left(f;\left|x-y\right|\right)\int_{0}^{\pi}\left|A_{n}\left(t\right)\right|dt$$

$$\leq\frac{8CK\left(K+1\right)}{\pi}\omega\left(f;\left|x-y\right|\right). \tag{4.13}$$

On the other hand

$$\left|R_{n}\left(x,y\right)\right|\leq\left|T_{n}\left(f;x\right)-f\left(x\right)\right|+\left|T_{n}\left(f;y\right)-f\left(y\right)\right|$$

by (4.2) and (3.3), whence we have

$$\left|R_{n}\left(x,y\right)\right|\leq16\left(K+1\right)\left(2K+1\right)\sum_{k=0}^{n}a_{nk}E_{k}\left(f\right)\leq K_{1}\sum_{k=0}^{n}a_{nk}\omega\left(f,\frac{1}{k+1}\right). \tag{4.14}$$

Analogously, we can show that

$$\left|R_{n}\left(x\right)\right|=\left|T_{n}\left(f;x\right)-f\left(x\right)\right|\leq8\left(K+1\right)\left(2K+1\right)\sum_{k=0}^{n}a_{nk}E_{k}\left(f\right)$$

$$\leq K_{2}\sum_{k=0}^{n}a_{nk}\omega\left(f,\frac{1}{k+1}\right). \tag{4.15}$$

Finally, using the same method as in the proof of Theorem 4, (2.2) follows from (4.13)-(4.15). ∎

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