FUNCTION SPACES VIII
BANACH CENTER PUBLICATIONS, VOLUME 79
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2008

ON THE H-PROPERTY AND ROTUNDITY OF CESÀRO DIRECT SUMS OF BANACH SPACES

SAARD YOUYEN
Department of Mathematics, Faculty of Science, Naresuan University
Pitsanuloke, 55000, Thailand
E-mail: saard_youyen@hotmail.com

SUTHEP SUANTAI
Department of Mathematics, Faculty of Science, Chiang Mai University
Chiang Mai, 50200, Thailand
E-mail: scmti005@chiangmai.ac.th

Abstract. In this paper, we define the direct sum $(\oplus_{i=1}^n X_i)_{\mathrm{ces}_p}$ of Banach spaces $X_1, \dots, X_n$ and consider it equipped with the Cesàro p-norm when $1 \le p < \infty$. We show that $(\oplus_{i=1}^n X_i)_{\mathrm{ces}_p}$ has the H-property if and only if each X_i has the H-property, and $(\oplus_{i=1}^n X_i)_{\mathrm{ces}_p}$ has the Schur property if and only if each X_i has the Schur property. Moreover, we also show that $(\oplus_{i=1}^n X_i)_{\mathrm{ces}_p}$ is rotund if and only if each X_i is rotund.

1. Introduction. The geometric properties of direct sums of Banach spaces has been studied by many mathematicians (see [5, 9]). It is well-known that the direct sum $(\oplus_{i=1}^n X_i)_2$ of normed spaces X_i $(i = 1, 2, \dots, n)$ equipped with the 2-norm $\|\cdot\|_2$ given by

$$\|(x_1, x_2, \dots, x_n)\|_2 = \sqrt{\sum_{i=1}^n \|x_i\|^2}$$

is rotund if and only if each X_i is rotund and $(\oplus_{i=1}^n X_i)_2$ is uniformly rotund if and only if each X_i is uniformly rotund (see [6]). Let $X_1, X_2, \dots, X_n$ be Banach spaces and $p \in [1, \infty]$. We use $(\oplus_{i=1}^n X_i)_p$ to denote the product space $\oplus_{i=1}^n X_i$ equipped with the norm $\|(x_1, x_2, \dots, x_n)\|_p = (\sum_{i=1}^n \|x_i\|^p)^{\frac{1}{p}}$ $(1 \le p < \infty)$ and $\|(x_1, x_2, \dots, x_n)\|_\infty = \max_{1 \le i \le n} \|x_i\|.$,

2000 *Mathematics Subject Classification*: 47H10, 47H09.

Key words and phrases: direct sum of Banach spaces, H-property, rotundity, Schur property.
Research of Suthep Suantai was supported by Thailand Researrch Fund.
The paper is in final form and no version of it will be published elsewhere.

In 1984, Landes [3, 4] showed that if X_1 and X_2 has weak normal structure (WNS), then $(X_1 \oplus X_2)_1$ need not have WNS.

In 2001 Marino, Pietramala and Xu [5] showed that if X_1 and X_2 has property (K) and the non-strict Opial property, then for each $p \in [1, \infty)$, $(X_1 \oplus X_2)_p$ has both property (K) and the non-strict Opial property.

The concept of Ψ-direct sum of Banach spaces X and Y equipped with the norm $\|(x, y)\|_\Psi = \|(\|x\|, \|y\|)\|_\Psi$ for $x \in X$ and $y \in Y$ was introduced by Saito and Kato. Note that the Ψ direct sum $X \oplus_\Psi Y$ is a generalization of the p-direct sum $(X \oplus Y)_p$, and they proved that $X \oplus_\Psi Y$ is strictly convex if and only if X and Y are strictly convex and Ψ is strictly convex. Building on this result, Saito and Kato [7] also proved that $X \oplus_\Psi Y$ is uniformly convex if X and Y are uniformly convex and Ψ is strictly convex.

For a Banach space X, we denote by $S(X)$ and $B(X)$ the unit sphere and unit ball of X, respectively. A point $x_0 \in S(X)$ is called

a) an *extreme point* of the unit ball of X if for $y, z \in S(X)$ the equation $2x_0 = y + z$ implies $y = z$,

b) an *H-point* if for any sequence (x_n) in X such that $\|x_n\| \to 1$ as $n \to \infty$, the weak convergence of (x_n) to x_0 (write $x_n \xrightarrow{w} x_0$) implies that $\|x_n - x_0\| \to 0$.

A Banach space X is said to be *rotund* if every point of $S(X)$ is an extreme point of $B(X)$. It is well-known that X is rotund if and only if $\|\frac{x+y}{2}\| < 1$ whenever $x, y \in S(X)$ with $x \neq y$. If every point in $S(X)$ is an H-point of $B(X)$, then X is said to have the *H-property.*

For $p \in [1, \infty)$, the *Cesàro sequence space* ces_p is defined as the space of all real sequences $x = (x(j))_{i=1}^\infty$ such that

$$\|x\|_p = \left(\sum_{n=1}^\infty \left(\frac{1}{n} \sum_{i=1}^n |x(i)| \right)^p \right)^{1/p} < \infty \quad \text{and} \quad \|x\|_\infty = \sup_{n \in \mathbb{N}} \frac{1}{n} \sum_{i=1}^n |x(i)| < \infty.$$

For $n \in \mathbb{N}$, ces_p^n is the space $\mathbb{R}^n$ equipped with the norm

$$\|x\|_p = \left(\sum_{k=1}^n \left(\frac{1}{k} \sum_{i=1}^k |x(i)| \right)^p \right)^{1/p}.$$

It is well-known that ces_p $(1 < p < \infty)$ is rotund, and so is the space ces_p^n. For $p \in [1, \infty]$, we use $(\oplus_{i=1}^n X_i)_{\mathrm{ces}_p}$ to denote the product $\oplus_{i=1}^n X_i$ equipped with the Cesàro p-norm $\|(x_1, x_2, \ldots, x_n)\|_{\mathrm{ces}_p} = (\sum_{k=1}^n (\frac{1}{k} \sum_{i=1}^k \|x_i\|)^p)^{1/p}$ and $\|(x_1, x_2, \ldots, x_n)\|_{\mathrm{ces}_\infty} = \max_{1 \le k \le n} \frac{1}{k} \sum_{i=1}^k \|x_i\|$.

2. Main results. We first show that$(\oplus_{i=1}^n X_i)_{\mathrm{ces}_p}$ has the Schur property if and only if each X_i has the Schur property. To do this, we need the following lemmas.

LEMMA 2.1. *Let $X_1, X_2, \ldots, X_n$ be Banach spaces and $p \in [1, \infty)$, and let $(x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)})_{k=1}^\infty$ be a sequence in $(\oplus_{i=1}^n X_i)_{\mathrm{ces}_p}$. Then $(x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}) \to (0, 0, \ldots, 0)$ as $k \to \infty$ if and only if $x_i^{(k)} \to 0$ as $k \to \infty$ for all $i = 1, 2, \ldots, n$.*

Proof. Since

$$\|(x_1^{(k)},\dots,x_n^{(k)})\|_{\mathrm{ces}_p}=\left(\|x_1^{(k)}\|^p+\cdots+\left(\frac{\|x_1^{(k)}\|+\|x_2^{(k)}\|+\cdots+\|x_n^{(k)}\|}{n}\right)^p\right)^{1/p},$$

it follows that $(x_1^{(k)},x_2^{(k)},\dots,x_n^{(k)})\to(0,0,\dots,0)$ as $k\to\infty$ if and only if $x_i^{(k)}\to 0$ as $k\to\infty$ for all $i=1,2,\dots,n$. ∎

Lemma 2.2. *Let $X_1,X_2,\dots,X_n$ be Banach spaces and let $f_i\in X_i^*$ $(i=1,2,\dots,n)$. For each $i\in\{1,2,\dots,n\}$ define $f_i':\oplus_{i=1}^n X_i\to R$ by $f_i'(x_1,x_2,\dots,x_n)=f_i(x_i)$. Then $f_i'\in(\oplus_{i=1}^n X_i)^*_{\mathrm{ces}_p}$ for each $i=1,2,\dots,n$.*

Proof. It is easy to see that f_i' is linear. We will show that f_i' is continuous at zero. To do this, suppose that $(x_1^{(k)},x_2^{(k)},\dots,x_n^{(k)})\in(\oplus_{i=1}^n X_i)_{\mathrm{ces}_p}$ such that $(x_1^{(k)},x_2^{(k)},\dots,x_n^{(k)})\to(0,0,\dots,0)$. By lemma 2.1, $x_i^{(k)}\to 0$ as $k\to\infty$, hence $f_i(x_i^{(k)})\to 0$ as $k\to\infty$. It follows that $f_i'(x_1^{(k)},x_2^{(k)},\dots,x_n^{(k)})\to 0$ as $k\to\infty$. Hence f_i' is continuous at zero. Therefore $f_i'\in(\oplus_{i=1}^n X_i)^*_{\mathrm{ces}_p}$. ∎

Lemma 2.3. *Let $X_1,X_2,\dots,X_n$ be Banach spaces and let $(x_1^{(k)},x_2^{(k)},\dots,x_n^{(k)})_{k=1}^\infty$ be a sequence in $(\oplus_{i=1}^n X_i)_{\mathrm{ces}_p}$ and let $(x_1,\dots,x_n)\in(\oplus_{i=1}^n X_i)_{\mathrm{ces}_p}$. If $(x_1^{(k)},x_2^{(k)},\dots,x_n^{(k)})\xrightarrow{w}(x_1,x_2,\dots,x_n)$ as $k\to\infty$, then $x_i^{(k)}\xrightarrow{w}x_i$ as $k\to\infty$ for each $i=1,2,\dots,n$.*

Proof. Let $f_i\in X_i^*$ $(i=1,2,\dots,n)$. Define $f_i':\oplus_{i=1}^n X_i\to R$ by $f_i'(x_1,x_2,\dots,x_n)=f_i(x_i)$. By Lemma 2.2, f_i' is a bounded linear functional on $\oplus_{i=1}^n X_i$, so $f_i'(x_1^{(k)},\dots,x_n^{(k)})\longrightarrow f_i'(x_1,x_2,\dots,x_n)$. Thus $f_i(x_i^{(k)})\to f_i(x_i)$ as $k\to\infty$, hence $x_i^{(k)}\xrightarrow{w}x_i$ as $k\to\infty$ for all $i=1,2,\dots,n$. ∎

Lemma 2.4. *Let $X_1,X_2,\dots,X_n$ be Banach spaces and $p\in[1,\infty)$. Then X_i is isometrically isomorphic to a subspace of $(\oplus_{i=1}^n X_i)_{\mathrm{ces}_p}$.*

Proof. For each $i=1,2,\dots,n$, let $X_i'=\{(0,\dots,0,x_i,0,\dots,0)\in(\oplus_{j=1}^n X_j):x_i\in X_i\}$. It is clear that X_i' is a subspace of $\oplus_{i=1}^n X_i$. We define $T_i:X_i\to X_i'$ by

$$T_i(x_i)=(0,\dots,0,\alpha_i x_i,0,\dots,0)\quad\text{where}\quad\alpha_i=\left(\frac{1}{\sum_{j=i}^n(\frac{1}{j})^p}\right)^{\frac{1}{p}}.$$

Then T_i is linear and

$$\|T_ix\|=\|(0,\dots,0,\alpha_ix,0,\dots,0)\|=\left(\left(\left\|\frac{\alpha_ix}{i}\right\|\right)^p+\left(\left\|\frac{\alpha_ix}{i+1}\right\|\right)^p+\cdots+\left(\left\|\frac{\alpha_ix}{n}\right\|)^p\right)^{\frac{1}{p}}$$

$$=\left(\|\alpha_ix\|^p\left[\frac{1}{i^p}+\frac{1}{(i+1)^p}+\cdots+\frac{1}{n^p}\right]\right)^{\frac{1}{p}}=\|x\|,$$

hence $T_i:X_i\to X_i'$ is isometrically isomorphic from X_i onto X_i'. ∎

Theorem 2.5. *Let $X_1,X_2,\dots,X_n$ be Banach spaces and $p\in[1,\infty)$. Then $(\oplus_{i=1}^n X_i)_{\mathrm{ces}_p}$ has the Schur property if and only if each X_i has the Schur property.*

Proof. Necessity is obvious, since each X_i is isometrically isomorphic to a subspace of $(\oplus_{i=1}^n X_i)_{\mathrm{ces}_p}$ and every subspace of a normed space with the Schur property has also the Schur property.

Sufficiency. Suppose that each X_i has the Schur property for $i = 1, 2, \ldots, n$.

Let $(x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}), (x_1, x_2, \ldots, x_n) \in (\oplus_{i=1}^n X_i)_{\mathrm{ces}_p}$ such that $(x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}) \xrightarrow{w} (x_1, x_2, \ldots, x_n)$. By Lemma 2.3, we have $x_i^{(k)} \xrightarrow{w} x_i$ as $k \to \infty$. Since X_i has the Schur property, $x_i^{(k)} \to x_i$ as $k \to \infty$. That is, $\|x_i^{(k)} - x_i\| \to 0$ as $k \to \infty$ for each $i = 1, 2, \ldots, n$. Since

$$\begin{aligned}
&\|(x_1^{(k)}, \ldots, x_n^{(k)}) - (x_1, \ldots, x_n)\|_{\mathrm{ces}_p} \\
&\qquad = \|(x_1^{(k)} - x_1, x_2^{(k)} - x_2, \ldots, x_n^{(k)} - x_n)\|_{\mathrm{ces}_p} \\
&\qquad = \left(\|x_1^{(k)} - x_1\|^p + \left(\frac{\|x_1^{(k)} - x_1\| + \|x_2^{(k)} - x_2\|}{2} \right)^p + \ldots \right. \\
&\qquad\quad \left. + \left(\frac{\|x_1^{(k)} - x_1\| + \|x_2^{(k)} - x_2\| + \cdots + \|x_n^{(k)} - x_n\|}{n} \right)^p \right)^{\frac{1}{p}},
\end{aligned}$$

it follows that $\|(x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}) - (x_1, x_2, \ldots, x_n)\|_{\mathrm{ces}_p} \to 0$.

Thus $(x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}) \to (x_1, x_2, \ldots, x_n)$ as $k \to \infty$. Hence $(\oplus_{i=1}^n X_i)_{\mathrm{ces}_p}$ has the Schur property. ∎

If $X_1, X_2, \ldots, X_n$ are Banach spaces and p $\in [1, \infty)$, we will show that $(\oplus_{i=1}^n X_i)_{\mathrm{ces}_p}$ has the H-property if and only if each X_i has the H-property. To do this, it is enough to show only that $(X_1 \oplus X_2)_{\mathrm{ces}_p}$ has the H-property if and only if X_1 and X_2 has the H-property.

THEOREM 2.6. *Let X_1 and X_2 be Banach spaces and $p \in [1, \infty)$. Then $(X_1 \oplus X_2)_{\mathrm{ces}_p}$ has the H-property if and only if X_1 and X_2 have the H-property.*

Proof. Necessity follows from the fact that each X_i is isometrically isomorphic with a subspace of $(X_1 \oplus X_2)_{\mathrm{ces}_p}$(Lemma 2.4) and every subspace of the space having the H-property has also the H-property.

Sufficiency. Let $(x_1^{(k)}, x_2^{(k)}), (x_1, x_2) \in S(X_1 \oplus X_2)_{\mathrm{ces}_p}$ such that $(x_1^{(k)}, x_2^{(k)}) \xrightarrow{w} (x_1, x_2)$ as $k \to \infty$. By Lemma 2.3, we have $x_i^{(k)} \xrightarrow{w} x_i$ as $k \to \infty$ for each $i = 1, 2$. Next we shall show that $\|x_i^{(k)}\| \to \|x_i\|$ as $k \to \infty$ for $i = 1, 2$. We have $\|x_i\| \le \liminf_{k\to\infty} \|x_i^{(k)}\|$. We will show that $\limsup_{k\to\infty} \|x_i^{(k)}\| \le \|x_i\|$ for i=1,2. If not, we get that $\limsup_{k\to\infty} \|x_1^{(k)}\| > \|x_1\|$ or $\limsup_{k\to\infty} \|x_2^{(k)}\| > \|x_2\|$.

CASE 1: $\limsup_{k\to\infty} \|x_1^{(k)}\| > \|x_1\|$. Then there exists a subsequence (m_k) of (k) such that $\|x_1^{(m_k)}\| > \|x_1\| + \epsilon_1$ for some $\epsilon_1 > 0$ for all $k \in \mathbb{N}$. Now we consider $\limsup_{k\to\infty} \|x_2^{(m_k)}\|$.

CASE 1.1: $\limsup_{k\to\infty} \|x_2^{(m_k)}\| > \|x_2\|$. Then there exists a subsequence (m'_k) of (m_k) such that $\|x_2^{(m'_k)}\| > \|x_2\| + \epsilon_2$ for some $\epsilon_2 > 0$ for all $k \in \mathbb{N}$. Hence, we have

$$\begin{aligned}
1 = \|(x_1^{(m'_k)}, x_2^{(m'_k)})\|_{\mathrm{ces}_p} &= \left(\|x_1^{(m'_k)}\|^p + \left(\frac{\|x_1^{(m'_k)}\| + \|x_2^{(m'_k)}\|}{2} \right)^p \right)^{\frac{1}{p}} \\
> \left((\|x_1\| + \epsilon_1)^p + \left(\frac{\|x_1\| + \epsilon_1 + \|x_2\| + \epsilon_2}{2} \right)^p \right)^{\frac{1}{p}} &> \left(\|x_1\|^p + \left(\frac{\|x_1\| + \|x_2\|}{2} \right)^p \right)^{\frac{1}{p}} = 1,
\end{aligned}$$

which is a contradiction.

CASE 1.2: $\limsup_{k\to\infty}\|x_2^{(m_k)}\|\le\|x_2\|$. Since

$$\|x_2\|\le\liminf_{k\to\infty}\|x_2^{(m_k)}\|\le\limsup_{k\to\infty}\|x_2^{(m_k)}\|\le\|x_2\|,$$

we get that $\lim_{k\to\infty}\|x_2^{(m_k)}\|=\|x_2\|$. Therefore, there exists $k_o\in\mathbb{N}$ for each $k\ge k_o$, $\|x_2\|-\frac{\epsilon_1}{2}\le\|x_2^{(m_k)}\|$. Hence, for each $k\ge k_o$ we have

$$\begin{aligned}1=\|(x_1^{(m_k)},x_2^{(m_k)})\|_{\mathrm{ces}_p}&=\left(\|x_1^{(m_k)}\|^p+\left(\frac{\|x_1^{(m_k)}\|+\|x_2^{(m_k)}\|}{2}\right)^p\right)^{\frac{1}{p}}\\&>\left((\|x_1\|+\epsilon_1)^p+\left(\frac{\|x_1\|+\epsilon_1+\|x_2\|-\frac{\epsilon_1}{2}}{2}\right)^p\right)^{\frac{1}{p}}\\&=\left((\|x_1\|+\epsilon_1)^p+\left(\frac{\|x_1\|+\|x_2\|+\frac{\epsilon_1}{2}}{2}\right)^p\right)^{\frac{1}{p}}\\&>\left(\|x_1\|^p+\left(\frac{\|x_1\|+\|x_2\|}{2}\right)^p\right)^{\frac{1}{p}}=1,\end{aligned}$$

which is a contradiction.

CASE 2: $\limsup_{k\to\infty}\|x_2^{(k)}\|>\|x_2\|$. The proof of this case is analogous to that of case 1 which leads to a contradiction.

Hence we obtain that $\limsup_{k\to\infty}\|x_i^{(k)}\|\le\|x_i\|$ for all $i=1,2$. This implies $\|x_i^{(k)}\|\to\|x_i\|$ for each $i=1,2$. Since X_i has the H property, we have $x_i^{(k)}\to x_i$ as $k\to\infty$. By lemma 2.1, we get that $\|(x_1^{(k)},x_2^{(k)})-(x_1,x_2)\|_{\mathrm{ces}_p}\to 0$. ∎

THEOREM 2.7. *Let $X_1,X_2,\ldots,X_n$ be Banach spaces and $p\in(1,\infty)$. Then $(\oplus_{i=1}^n X_i)_{\mathrm{ces}_p}$ is rotund if and only if each X_i is rotund.*

Proof. If $(\oplus_{i=1}^n X_i)_{\mathrm{ces}_p}$ is rotund, then each X_i is also rotund since X_i is isometrically isomorphic to a subspace of $(\oplus_{i=1}^n X_i)_{\mathrm{ces}_p}$. Conversely, assume that each X_i is rotund. Let $(x_1,x_2,\ldots,x_n)$ and $(y_1,y_2,\ldots,y_n)$ be different elements in $S(\oplus_{i=1}^n X_i)_{\mathrm{ces}_p}$. The proof will be finished if we show that $\|\frac{1}{2}(x_1+y_1,x_2+y_2,\ldots,x_n+y_n)\|<1$. Notice that $(\|x_1\|,\|x_2\|,\ldots,\|x_n\|)$and $(\|y_1\|,\|y_2\|,\ldots,\|y_n\|)\in S(\mathrm{ces}_p^n)$. If $\|x_i\|\neq\|y_i\|$ for some $i=1,2,\ldots,n$, then it follows from the rotundity of ces_p^n that

$$\begin{aligned}&\left\|\frac{1}{2}(x_1+y_1,x_2+y_2,\ldots,x_n+y_n)\right\|\\&\qquad=\frac{1}{2}\Bigg(\|x_1+y_1\|^p+\left(\frac{\|x_1+y_1\|+\|x_2+y_2\|}{2}\right)^p+\ldots\\&\qquad\quad+\left(\frac{\|x_1+y_1\|+\|x_2+y_2\|+\cdots+\|x_n+y_n\|}{n}\right)^p\Bigg)^{\frac{1}{p}}\\&\qquad=\frac{1}{2}\|(\|x_1+y_1\|,\|x_2+y_2\|,\ldots,\|x_n+y_n\|)\|_{\mathrm{ces}_p^n}\\&\qquad\le\frac{1}{2}\|(\|x_1\|+\|y_1\|,\|x_2\|+\|y_2\|,\ldots,\|x_n\|+\|y_n\|)\|_{\mathrm{ces}_p^n}<1.\end{aligned}$$

Thus, it may be assumed that $\|x_i\| = \|y_i\|$ for all $i = 1, 2, \ldots, n$. It may be assumed that $x_i \neq y_i$ for some i. Then $\|\frac{1}{2}(x_i + y_i)\| < \|x_i\| = \|y_i\| = \frac{1}{2}(\|x_i\| + \|y_i\|)$ by the rotundity of X_i. Therefore

$$\begin{aligned}
&\left\|\frac{1}{2}(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)\right\| \\
&\qquad = \frac{1}{2}\Bigg(\|x_1 + y_1\|^p + \left(\frac{\|x_1 + y_1\| + \|x_2 + y_2\|}{2}\right)^p + \ldots \\
&\qquad\quad + \left(\frac{\|x_1 + y_1\| + \|x_2 + y_2\| + \cdots + \|x_n + y_n\|}{n}\right)^p\Bigg)^{\frac{1}{p}} \\
&\qquad = \frac{1}{2}\|(\|x_1 + y_1\|, \|x_2 + y_2\|, \ldots, \|x_n + y_n\|)\|_{\mathrm{ces}_p^n} \\
&\qquad < \frac{1}{2}\|(\|x_1\| + \|y_1\|, \|x_2\| + \|y_2\|, \ldots, \|x_n\| + \|y_n\|)\|_{\mathrm{ces}_p^n} \\
&\qquad = \frac{1}{2}\|(2\|x_1\|, 2\|x_2\|, \ldots, 2\|x_n\|)\|_{\mathrm{ces}_p^n} = 1. \blacksquare
\end{aligned}$$

Acknowledgements. The authors would like to thank the Uttaradit Rajabhat University Research Fund for their financial support.

References

[1] K. W. Anderson, *Midpoint local uniform convexity and other geometric properties of Banach spaces*, 1960.

[2] J. A. Clarkson, *Uniformly convex spaces*, Trans. Amer. Math. Soc. 40 (1936), 396–414.

[3] T. Landes, *Permanence properties of normal structure*, Pacific J. Math. 110 (1984), 125–143.

[4] T. Landes, *Normal structure and the sum-property*, Pacific J. Math. 123 (1986), 127–147.

[5] G. Marino, P. Pietramala and H.-K. Xu, *Geometrical conditions in product spaces*, Nonlinear Analysis 46 (2001), 1063–1071.

[6] R. E. Megginson, *An Introduction to Banach Space Theory*, Springer, 1998.

[7] K.-S. Saito and M. Kato, *Uniform convexity of ψ-direct sums of Banach spaces*, J. Math. Anal. Appl. 277 (2003), 1–11.

[8] M. A. Smith, *Rotundity and extremity in $l^p(X_i)$ and $L^p(\mu, X)$*, Amer. Math. Soc. 52 (1986), 143–162.

[9] T. Zhang, *The coordinatewise uniformly Kadec-Klee property in some Banach spaces*, Siberian Math. J. 44 (2003), 363–365.