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MULTISUMMABILITY AND ORDINARY MEROMORPHIC DIFFERENTIAL EQUATIONS

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Abstract. In this expository paper we consider various approaches to multisummability. We apply it to nonlinear ODE's and give a somewhat modified proof of multisummability of formal solutions of ODE's with levels 1 and 2 via Écalle's method involving convolution equations.

1. Introduction. Multisummability has been introduced by Écalle. His definition can be found in [Eca92]. It is useful if one wants to give analytic meaning to divergent formal series solutions of differential and difference equations and to formal diffeomorphisms. Multisummability has been explained in several ways in [MR91, Bal92, MR92, Ram93, Bal94, Mal95, Bal00]. Here we will give an overview of these definitions of multisummability following mainly the exposition given by Malgrange in [Mal95]. After that we show how this notion can be applied to ODE's with 2 levels. The general case with several levels has been treated in [Bra92, Bal94, RS94]. Here we give a somewhat modified version of the proof in [Bra92] using the original method of Écalle involving convolution equations.

The organization of the paper is as follows. First we give some properties of Laplace and Borel transforms. After that we give several equivalent definitions of multisummability and finally show its applicability to ODE's with two levels.

1.1. Laplace and Borel transforms. I will denote an open interval (α, β) of $\mathbb{R}$ and $|I| = \beta - \alpha$. We consider sectors $S(I) := \{x \in \mathbb{C}^* : \arg x \in I\}$. If $I' = (\alpha_1, \beta_1)$ with $\alpha < \alpha_1 < \beta_1 < \beta$ we write $I' \Subset I$. By $\Delta(0, r)$ we mean a disc in $\mathbb{C}$ of radius $r > 0$ centered at the origin. A neighborhood of 0 in $S(I)$ is a set $U \subset S(I)$ such that for all $I' \Subset I$ there exists $r > 0$ such that $S(I') \cap \Delta(0, r) \subset U$. If f is a function defined on a neighborhood of 0 in $S(I)$, then f has asymptotic expansion $\hat{f}(x) = \sum_{n=0}^{\infty} a_n x^n$ as $x \to 0$

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in U if for all $I' \Subset I$ there exists $r > 0$ such that for all $N \in \mathbb{N}^*$ there exists $C_N > 0$ such that $|f(x) - \sum_{n=0}^{N-1} a_n x^n| \leq C_N |x|^N$ for all $x \in S(I') \cap \Delta(0,r)$. Then we write $f(x) \sim \widehat{f}(x)$, $x \to 0$ in $S(I)$. $\mathcal{A}(I)$ will denote the space of all such functions. If $f \in \mathcal{A}(I)$ then also $f' \in \mathcal{A}(I)$ with as asymptotic expansion the formal series $\hat{f}$ differentiated term by term.

If $k > 0$, then $\mathcal{A}^{\leq k}(I)$ will denote the space of functions $f \in \mathcal{A}(I)$ such that f is analytic in the complete sector $S(I)$ and for all $I' \Subset I$ there exist $A > 0, B > 0$ such that $|f(x)| \leq A \exp(B|x|^k)$ for all $x \in S(I')$.

Laplace transform $\mathcal{L}$. Suppose $f \in t^\nu \mathcal{A}^{\leq 1}(I)$, $I = (\alpha, \beta)$, $\nu > -1$. If $\theta \in I$ we define $(\mathcal{L}f)(x) = \int_0^{\infty:\theta} e^{-t/x} f(t)\, dt$, where the path of integration is the ray $\arg t = \theta$, $|\arg x - \theta| < \pi/2$ and $|x|$ sufficiently small positive. Varying $\theta \in I$ we obtain an analytic function $\mathcal{L}f \in x^{\nu+1}\mathcal{A}(I_+)$, where $I_+ := (\alpha - \frac{\pi}{2}, \beta + \frac{\pi}{2})$. We have $(\mathcal{L}t^\nu)(x) = \Gamma(\nu+1)x^{\nu+1}$ and if $f(x) \sim \widehat{f}(x)$, $x \to 0$ in $S(I)$ as above, then $(\mathcal{L}f)(x) \sim \widehat{\mathcal{L}}\widehat{f}(x)$ as $x \to 0$ in $S(I_+)$, where $\widehat{\mathcal{L}}\widehat{f}(x)$ is the formal Laplace transform of $\widehat{f}$ obtained by applying $\mathcal{L}$ to each term of the formal series $\widehat{f}$.

Furthermore, $\mathcal{L} : t^\nu \mathcal{A}^{\leq 1}(I) \to x^{\nu+1}\mathcal{A}(I_+)$ is an isomorphism, where I, I_+, ν are as above. The inverse of the Laplace transform $\mathcal{L}$ is the Borel transform $\mathcal{B}$. So $(\mathcal{B}x^{\nu+1})(t) = t^\nu/\Gamma(\nu+1)$. If a is a constant we define $\mathcal{B}a = 0$. Moreover, if $f, g \in x^{\nu+1}\mathcal{A}(I_+)$, then $\mathcal{B}(fg) = \mathcal{B}f * \mathcal{B}g$, where $(F * G)(t) := \int_0^t F(t-s)G(s)\, ds$.

An integral representation of $\mathcal{B}f$ can be given as follows: Assume $f \in x^\nu \mathcal{A}(I)$, $I = (\alpha, \beta)$, $|I| > \pi$. Let $I' = [\alpha', \beta'] \Subset I$, $|I'| > \pi$. Then f is analytic in $D := S(I') \cap \Delta(r)$ for some $r > 0$. Let γ be the contour in $\mathbb{C}$ from 0 along the ray $\arg x = \beta'$ to some point $x_1 \in D$, then along the circle $|x| = |x_1|$ in negative sense till its intersection x_2 with the ray $\arg x = \alpha'$ and finally from x_2 along this ray to 0. Then

$$(\mathcal{B}f)(t) := \frac{1}{2\pi i} \int_\gamma e^{t/x} f(x)\, d(x^{-1}) \quad \text{if } \arg t \in (\alpha' + \tfrac{\pi}{2}, \beta' - \tfrac{\pi}{2}).$$

Varying α', β' we obtain $\mathcal{B}f \in t^\nu \mathcal{A}^{\leq 1}(I^*)$, where $I^* := (\alpha + \frac{\pi}{2}, \beta - \frac{\pi}{2})$. Moreover, if $f(t) \sim \widehat{f}$ as $t \to 0$ in $S(I)$, then $(\mathcal{B}f)(x) \sim (\widehat{\mathcal{B}}\widehat{f})(x)$ as $x \to 0$ in $S(I^*)$, where $(\widehat{\mathcal{B}}\widehat{f})$ is obtained by applying $\mathcal{B}$ to each term in the formal series $\widehat{f}$.

The Laplace transform may be extended to the *incomplete Laplace transform* as follows: Let $f \in t^\nu \mathcal{A}(I)$, $\nu > -1$ and $I' \Subset I$. Then f is analytic in $D := S(I') \cap \Delta(0, r_0)$ for some $r_0 > 0$. Let $\tau \in D$ and define $(\mathcal{L}^{(\tau)}f)(x) = \int_0^\tau e^{-t/x} f(t)\, dt$. This is an analytic function in $\mathbb{C}^*$ and $\mathcal{L}^{(\tau)}f \in x^{\nu+1}\mathcal{A}(I_\tau)$, where $I_\tau := (\arg\tau - \frac{\pi}{2}, \arg\tau + \frac{\pi}{2})$ and if $f \sim \widehat{f} = t^\nu \sum_0^\infty a_n t^n$ in $S(I)$, then $\mathcal{L}^{(\tau)}f \sim \widehat{\mathcal{L}}\widehat{f}$ as $x \to 0$ in $S(I_\tau)$.

Consider τ_1 and τ_2 in D. Suppose $0 \leq \arg\tau_1 - \arg\tau_2 \leq \pi - \epsilon$ and let $r = \min(|\tau_1|, |\tau_2|)$. Then

$$\left|(\mathcal{L}^{(\tau_1)}f - \mathcal{L}^{(\tau_2)}f)(x)\right| \leq A \exp\bigl(-r\sin(\epsilon/2)|x|^{-1}\bigr)$$

for $x \in S(I'')$, where $I'' := (\arg\tau_1 - \frac{\pi-\epsilon}{2}, \arg\tau_2 + \frac{\pi-\epsilon}{2})$ and A is a positive constant. This motivates the following definition: for $k > 0$ set

$$\begin{aligned}\mathcal{A}^{\leq -k}(I) := \{g :\ & g \text{ analytic in a neighborhood of 0 in } S(I) \text{ and if } I' \Subset I, \text{ then} \\ & \text{there exist } A, B, \rho > 0 \text{ such that } |g(x)| \leq A\exp(-B|x|^{-k}) \text{ for all } x \in S(I') \cap \Delta(0,\rho)\}.\end{aligned}$$

So in the previous case we have $\mathcal{L}^{(\tau_1)}f - \mathcal{L}^{(\tau_2)}f \in \mathcal{A}^{\leq -k}(I'')$. This gives rise to the notion of *k-precise quasi-function* of Ramis (cf. [Ram93]).

Let I be an open interval, $k > 0$, $\nu > -1$ and J an index-set. Let $\{I_j\}_{j\in J}$ be an open cover of I, $f_j \in x^\nu \mathcal{A}(I_j)$ such that $f_j - f_h \in \mathcal{A}^{\leq -k}(I_j \cap I_h)$ if $I_j \cap I_h \neq \emptyset$. Then $\{f_j, I_j\}_{j\in J}$ is a k-precise quasi-function on I and all f_j have the same asymptotic expansion. If $\{g_j, I'_j\}_{j\in H}$ is also a k-precise quasi-function on I, then this one is equivalent to the previous one if $f_j - g_h \in \mathcal{A}^{\leq -k}(I_j \cap I'_h)$ if $I_j \cap I'_h \neq \emptyset$. The corresponding set of equivalence classes is denoted by $\mathcal{A}/\mathcal{A}^{\leq -k}(I)$. In [Mal95] this is interpreted in terms of sheaves.

In particular, if $f \in t^\nu \mathcal{A}(I)$, $\nu > -1$, then the incomplete Laplace transforms $\mathcal{L}^{(\tau)}f$ give rise to a 1-precise quasi-function $\tilde{\mathcal{L}}f : \tilde{\mathcal{L}}f \in x^{\nu+1}\mathcal{A}/\mathcal{A}^{\leq -1}(I_+)$, where if $I = (\alpha, \beta)$, then $I_+ = (\alpha - \frac{\pi}{2}, \beta + \frac{\pi}{2})$.

Now we have: $\tilde{\mathcal{L}}$ is an isomorphism from $x^\nu \mathcal{A}(I)$ to $x^{\nu+1}\mathcal{A}/\mathcal{A}^{\leq -1}(I_+)$ with I, I_+ as above. The inverse is a modified Borel transform $\tilde{\mathcal{B}}$. Malgrange gave an integral representation for $\tilde{\mathcal{B}}$ in [Mal95].

If $p \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then $\tilde{\mathcal{L}} : \mathcal{A}^{\leq p}(I) \to x(\mathcal{A}/\mathcal{A}^{\leq -q})(I_+)$ is an isomorphism if I and I_+ are as above (cf. [Mal95]).

Laplace and Borel transforms of arbitrary order. Let $k > 0$ and $(\rho_k f)(x) := f(x^{1/k})$. Define $\mathcal{L}_k = \rho_k^{-1} \circ \mathcal{L} \circ \rho_k$ and $\mathcal{B}_k = \rho_k^{-1} \circ \mathcal{B} \circ \rho_k$ and similarly with $\mathcal{L}, \mathcal{B}$ replaced by $\tilde{\mathcal{L}}, \tilde{\mathcal{B}}$. Then $(\mathcal{L}_k t^\nu)(x) = \Gamma(1 + \nu/k)x^{\nu+k}$, $(\mathcal{B}_k x^{\nu+k})(t) = \frac{t^\nu}{\Gamma(1+\nu/k)}$ if $\nu > -k$. If a is a constant we define $\mathcal{B}_k a = 0$. From the results on $\mathcal{L}$ and $\mathcal{B}$ we have the following

THEOREM 1.1. *Let $k > 0$, $\nu > -k$, $I = (\alpha, \beta)$ and $I_+ = (\alpha - \frac{\pi}{2k}, \beta + \frac{\pi}{2k})$. Then*

1. *$\mathcal{L}_k$ is an isomorphism from $t^\nu \mathcal{A}^{\leq k}(I)$ to $x^{\nu+k}\mathcal{A}(I_+)$ with inverse $\mathcal{B}_k$.*
2. *$\tilde{\mathcal{L}}_k$ is an isomorphism from $t^\nu \mathcal{A}(I)$ to $x^{\nu+k}\mathcal{A}/\mathcal{A}^{\leq -k}(I_+)$ with inverse $\tilde{\mathcal{B}}_k$.*
3. *If $p, q, k > 0$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{k}$, then $\tilde{\mathcal{L}}_k$ is an isomorphism from $t^\nu \mathcal{A}^{\leq p}(I)$ to $x^{\nu+k}\mathcal{A}/\mathcal{A}^{\leq -q}(I_+)$ with inverse $\tilde{\mathcal{B}}_k$.*

REMARK 1.1. Écalle introduced the acceleration operator $\mathbb{A}_{l,k}$ which may be defined as $\mathbb{A}_{l,k} = \tilde{\mathcal{B}}_l \circ \tilde{\mathcal{L}}_k$, where $l > k > 0$. From the assertions 3 with $q = l$, $p = \kappa := (k^{-1} - l^{-1})^{-1}$, $\nu > -k$ and 2 in Theorem 1.1 it follows that it is an isomorphism from $t^\nu \mathcal{A}^{\leq \kappa}(I)$ to $t^{\nu+k-l}\mathcal{A}(I')$, where if $I = (\alpha, \beta)$, then $I' = (\alpha - \frac{\pi}{2\kappa}, \beta + \frac{\pi}{2\kappa})$.

1.2. Gevrey properties and *k*-summability. Let $\widehat{f} = \sum_{n=0}^\infty a_n x^n$ be a formal series. Suppose there exist $s > 0$, $A > 0$ and $B > 0$ such that $|a_n| \leq AB^n\Gamma(1 + ns)$ for all $n \in \mathbb{N}$. Then $\widehat{f}$ is called a *Gevrey series of order s* and the class of such series with s fixed is denoted by $\mathbb{C}[[x]]_s$.

Suppose $\widehat{f} = \sum_{n=0}^\infty a_n x^n$ and there exists $f \in \mathcal{A}(I)$ such that for all $I' \Subset I$ there exist positive C and r with the property that for all $N \in \mathbb{N}^*$ we have

$$\left| f(x) - \sum_{n=0}^{N-1} a_n x^n \right| \leq C^N \Gamma(1 + Ns)|x|^N \quad \text{for all } x \in S(I') \cap \Delta(0, r).$$

Then f is said to be a *Gevrey function of order s on* $S(I)$, denoted by $f \sim_s \widehat{f}$ on $S(I)$. It follows that $\widehat{f} \in \mathbb{C}[[x]]_s$. The set of Gevrey functions of order s on $S(I)$ is denoted by $\mathcal{A}_{(s)}(I)$. Let $\mathcal{A}^{<0}(I)$ denote the subset of functions $f \in \mathcal{A}(I)$ with asymptotic expansion 0.

Then one may show

$$\mathcal{A}_{(s)}(I) \cap \mathcal{A}^{<0}(I) = \mathcal{A}^{\le -k}(I), \tag{1}$$

where here and in the following always $k = 1/s$. Furthermore $\mathcal{A}^{\le -k}(I) = 0$ if $|I| > \pi/k$ and therefore the map $\mathcal{A}_{(s)}(I) \to \mathbb{C}[[x]]_s$ is injective if $|I| > s\pi$.

If $\widehat{f} \in \mathbb{C}[[x]]_s$ and $\widehat{f} = \sum_{n=0}^{\infty} a_n x^n$, then $t^{k-1}\widehat{\mathcal{B}}_k \widehat{f} = \sum_{n=1}^{\infty} a_n \frac{t^{n-1}}{\Gamma(n/k)} \in \mathbb{C}\{t\}$ with sum $\tilde{\phi}$. Let $\phi = t^{1-k}\tilde{\phi}$. Then we will say that ϕ is the sum of $\widehat{\mathcal{B}}_k \widehat{f} \in t^{1-k}\mathbb{C}\{t\}$. From assertion 2 of Theorem 1.1 it follows that $f := a_0 + \tilde{\mathcal{L}}_k(\phi) \in \mathcal{A}/\mathcal{A}^{\le -k}(\mathbb{R})$. Also one may show

$$\mathbb{C}[[x]]_s \simeq \mathcal{A}/\mathcal{A}^{\le -k}(\mathbb{R}) = \mathcal{A}_{(s)}/\mathcal{A}^{\le -k}(\mathbb{R}). \tag{2}$$

Let $\widehat{f} \in \mathbb{C}[[x]]_s$, $|I| > s\pi$, $k = 1/s$. Then $\widehat{f}$ is said to be *k-summable on I* if there exists $f \in \mathcal{A}_{(s)}(I)$ such that $f \sim_s \widehat{f}$ on $S(I)$. This k-sum is unique because of (1) and $\mathcal{A}^{\le -k}(I) = 0$.

An equivalent definition is as follows: Let ϕ be the sum of $\widehat{\mathcal{B}}_k \widehat{f} \in t^{1-k}\mathbb{C}\{t\}$ as above. Assume that ϕ has an analytic continuation such that $\phi \in t^{1-k}\mathcal{A}^{\le k}(I')$, where $I' = (\alpha', \beta')$. Then $f := a_0 + \mathcal{L}_k \phi$ exists in a neighborhood of 0 in $S(I)$ where $I = (\alpha' - \frac{\pi}{2k}, \beta' + \frac{\pi}{2k})$ and is said to be the k-sum of $\widehat{f}$ on $S(I)$. Here again $f \in \mathcal{A}_{(s)}(I)$ and $f \sim_s \widehat{f}$ on $S(I)$.

1.3. Definitions of multisummability. First we consider as example the notion of $(1,2)$-summability and give three equivalent definitions.

Suppose $\widehat{f} = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{C}[[x]]_1$. Then the definition of $(1,2)$-summability given by Malgrange and Ramis (cf. [MR92]) reads: let $f_0 \in \mathcal{A}/\mathcal{A}^{\le -1}(\mathbb{R})$ correspond to $\widehat{f}$ (cf.(2)) and assume that there exist $f_1 \in \mathcal{A}/\mathcal{A}^{\le -2}(I_1)$ and $f_2 \in \mathcal{A}(I_2)$ with $I_2 \subset I_1$, $|I_j| > \pi/j$ for $j = 1,2$ such that $f_j|_{I_{j+1}} = f_{j+1} \bmod \mathcal{A}^{\le -(j+1)}$ for $j = 0,1$. Then $\widehat{f}$ is said to be $(1,2)$-summable on (I_1, I_2) with sum (f_1, f_2). This sum is uniquely determined on (I_1, I_2) and $f_j \sim \widehat{f}$ on $S(I_j)$, $j = 1,2$.

The original definition of Écalle (cf. [Eca85, Eca92, MR91]) is closely related to the previous one. Let ϕ_1 be the sum of $\widehat{\mathcal{B}}\widehat{f} \in \mathbb{C}\{t\}$. Assume that ϕ_1 can be analytically continued to $\phi_1 \in \mathcal{A}^{\le 2}(I_1')$, where $I_1' = (\alpha_1', \beta_1')$. Then using Remark 1.1 we have $\phi_2 := \mathbb{A}_{2,1}\phi_1 \in t^{-1}\mathcal{A}(I_1'')$, where $I_1'' = (\alpha_1' - \frac{\pi}{4}, \beta_1' + \frac{\pi}{4})$.

Assume that ϕ_2 can be analytically extended to $\phi_2 \in t^{-1}\mathcal{A}^{\le 2}(I_2')$, where $I_2' = (\alpha_2', \beta_2') \subset I_1''$. Then $f_2 := a_0 + \mathcal{L}_2\phi_2 \in \mathcal{A}(I_2)$ with $I_2 = (\alpha_2, \beta_2)$, $\alpha_2 = \alpha_2' - \frac{\pi}{4}$, $\beta_2 = \beta_2' + \frac{\pi}{4}$ and $|I_2| > \pi/2$. Then $\widehat{f}$ is said to be $(1,2)$-summable with sum f_2 on a neighborhood of 0 in $S(I_2)$. The sum defined in this way is uniquely associated with I_1' and I_2, and $f_2 \sim \widehat{f}$ in this sector $S(I_2)$. The relation with the definition of Malgrange and Ramis is given by $f_1 := a_0 + \tilde{\mathcal{L}}_1\phi_1 \in \mathcal{A}/\mathcal{A}^{\le -2}(I_1)$, where $I_1 := (\alpha_1, \beta_1)$, $\alpha_1 = \alpha_1' - \frac{\pi}{2}$, $\beta_1 = \beta_1' + \frac{\pi}{2}$ (cf. Theorem 1.1).

Another definition due to Balser (cf. [Bal92]) reads: $\widehat{f}$ is $(1,2)$-summable on (I_1, I_2) iff $\widehat{f} = \widehat{h_1} + \widehat{h_2}$, where $\widehat{h_j}$ is j-summable on I_j, $j = 1,2$, and $I_2 \subset I_1$, $|I_j| > \frac{\pi}{j}$, $j = 1,2$. Balser also gave a definition through iterated Laplace transforms in [Bal94] which is useful in numerical calculations. An inductive definition also has been given by Balser in [Bal94] and by Tougeron (cf. [Mal95]).

Next we consider shortly the general case.

Let $0 < m_1 < \ldots < m_r$ and $\widehat{f} \in \mathbb{C}[[x]]_{1/m_1}$. Then $\widehat{f}$ corresponds to a unique f_0 in $\mathcal{A}/\mathcal{A}^{\leq -m_1}(\mathbb{R})$ by (2). Let I_j, $j = 1, \ldots, r$, be intervals with $|I_j| > \pi/m_j$ and $I_j \subset I_{j-1}$, $j = 2, \ldots, r$. Assume there exist $f_j \in \mathcal{A}/\mathcal{A}^{\leq -m_{j+1}}(I_j)$, $j = 1, \ldots, r$ with $m_{r+1} = \infty$, and therefore $f_r \in \mathcal{A}(I_r)$, such that $f_j|_{I_{j+1}} = f_{j+1} \bmod \mathcal{A}^{\leq -m_{j+1}}$, $j = 0, \ldots, r-1$. Then $\widehat{f}$ is said to be $(m_1, \ldots, m_r)$-*summable on* $(I_1, \ldots, I_r)$ *with multisum* $(f_1, \ldots, f_r)$. This sum is unique.

Écalle's definition runs as follows using the same notation as above. Let ϕ_1 be the sum of $\widehat{\mathcal{B}}_{m_1}\widehat{f} \in t^{1-m_1}\mathbb{C}\{t\}$. Let $\kappa_j^{-1} := m_j^{-1} - m_{j+1}^{-1}$, $m_{r+1} = \infty$, $I_j = (\alpha_j, \beta_j)$, $|I_j| > \pi/m_j$ and $I'_j := (\alpha_j + \frac{\pi}{2m_j}, \beta_j - \frac{\pi}{2m_j}) =: (\alpha'_j, \beta'_j)$.

$\widehat{f}$ is said to be $(m_1, \ldots, m_r)$-*summable on the set of intervals* $(I_1, \ldots, I_r)$ if the following holds: Suppose for $j = 1, \ldots, r$ consecutively: ϕ_j has an analytic extension $\phi_j \in t^{1-m_j}\mathcal{A}^{\leq \kappa_j}(I'_j)$ and if $j < r$ define $\phi_{j+1} = \mathbb{A}_{m_{j+1}, m_j}\phi_j$ (cf. Remark 1.1) and so $\phi_{j+1} \in t^{1-m_{j+1}}\mathcal{A}(I'_{j+})$, where $I'_{j+} := (\alpha'_j - \frac{\pi}{2\kappa_j}, \beta'_j + \frac{\pi}{2\kappa_j})$. Finally, $f_r = a_0 + \mathcal{L}_{m_r}\phi_r \in \mathcal{A}(I_r)$ is the $(m_1, \ldots, m_r)$-sum of $\widehat{f}$ associated with $(I_1, \ldots, I_r)$. Here the elements f_j in the definition of Malgrange and Ramis are given by $\tilde{\mathcal{L}}_{m_j}\phi_j$.

The definition of Balser reads: $\widehat{f}$ is said to be $(m_1, \ldots, m_r)$-*summable on* $(I_1, \ldots, I_r)$ if there exist $\widehat{g}_j$, $j = 1, \ldots, r$, such that $\widehat{f} = \sum_{j=1}^r \widehat{g}_j$ and g_j is k_j-summable on I_j, where the intervals I_j are as above.

The definitions of Balser and Tougeron mentioned before in the case of $(1,2)$-summability also may be extended to the general case.

The definitions above have been formulated for scalar functions but they may be extended in an obvious way to functions with values in $\mathbb{C}^n$.

2. ODE's with 2 levels. In this section we consider the system

$$\operatorname{diag}\{xI^{(1)}, x^2 I^{(2)}\}x\,\frac{dy}{dx} = \Lambda y + xg(x,y), \tag{3}$$

where $I^{(j)}$ denotes the identity matrix of dimension $n_j \in \mathbb{N}$ and $n = n_1 + n_2$, $y \in \mathbb{C}^n$, $\Lambda = \operatorname{diag}\{\lambda_1, \ldots, \lambda_n\}$, and g is analytic at (0,0) in $\mathbb{C} \times \mathbb{C}^n$. We also assume that Λ is invertible. Because of the powers 1 and 2 of x in the left-hand side of (3) this equation is said to have levels 1 and 2. The equation has a formal solution $\widehat{y} = \sum_{j=1}^\infty c_j x^j$ as may be verified easily. Moreover, from the recurrence relations for the coefficients one may deduce that this series is Gevrey of order 1. If $y = (y_1, \ldots, y_n)$, $y_j \in \mathbb{C}$, then we define $y^{(1)} := (y_1, \ldots, y_{n_1})$, $y^{(2)} := (y_{n_1+1}, \ldots, y_n)$ and similarly for Λ and g. Then

THEOREM 2.1. *The formal solution* $\widehat{y}$ *of* (3) *is* $(1,2)$-*summable on* (I_1, I_2), *where* $I_1 = (\alpha_1 - \pi/2, \beta_1 + \pi/2)$ *with* $\alpha_1 < \beta_1$ *and* $\lambda_j \notin S(\alpha_1, \beta_1)$ *for* $j = 1, \ldots, n_1$, *and* $I_2 = ((\alpha_2 - \pi/2)/2, (\beta_2 + \pi/2)/2)$ *with* $\alpha_2 < \beta_2$, $I_2 \subset I_1$ *and* $\lambda_j \notin S(\alpha_2, \beta_2)$ *for* $j = n_1 + 1, \ldots, n$.

REMARK 2.1. The conditions on I_1 and I_2 may be reformulated in terms of Stokes rays. At these rays there is a change of growth order of solutions of (3). For example in case $g = 0$ there are solutions $\exp(-\lambda_j/x)e_j$ if $j \leq n_1$ and $\exp(-\lambda_j/(2x^2))e_j$ if $j > n_1$. Here there is a change of growth order at the rays $\arg x = \tau_{j\pm} = \arg(\lambda_j) \pm \pi/2$, $j \leq n_1$ and $\arg x = \tau_{j\pm} = (\arg(\lambda_j) \pm \pi/2)/2$, $j > n_1$, and these rays are the Stokes rays of level 1 and 2 respectively. Then the conditions on I_1 and I_2 may be reformulated as $S(I_h)$ does not contain any pair of Stokes rays τ_{j+} and τ_{j-} of level h and $I_2 \subset I_1$, $|I_h| > \pi/h$ for $h = 1, 2$.

We give a proof of the theorem via convolution equations as in [Bra92]. First we apply the formal Borel transform to (3). Let $\widehat{\mathcal{B}}\widehat{y} = \phi$, $G = \widehat{\mathcal{B}}\big(xg(x, \widehat{y}(x))\big)$. Then ϕ is a convergent power series in a disc $\Delta(0, \rho_0)$ with $\rho_0 > 0$, its sum will also be denoted by ϕ. Then $f_0 := \tilde{\mathcal{L}}\phi \in \mathcal{A}_{(1)}/\mathcal{A}^{\leq -1}(\mathbb{R})$ (cf. (2)) and f_0 is a solution of (3) $\bmod \mathcal{A}^{<0}$, and in view of (1) also $\bmod \mathcal{A}^{\leq -1}$. From the Taylor expansion $g(x, y) = \sum_{j \succeq 0} g_j(x) y^j$ we deduce that $G(t, \phi) = \sum G_j(t) * \phi^{*j}(t)$, where $G_j = \mathcal{B}(xg_j(x))$. Since $\mathcal{B}(x) = 1$, $\mathcal{B}(x^2 \frac{dy}{dx}) = t\phi(t)$, we obtain from the formal Borel transform of (3),

$$
\begin{aligned}
(tI^{(1)} - \Lambda^{(1)})\phi^{(1)} &= G^{(1)}(t, \phi), \\
1 * (t\phi^{(2)}) &= \Lambda^{(2)}\phi^{(2)} + G^{(2)}(t, \phi)
\end{aligned} \tag{4}
$$

which can be written as $\phi = T\phi$, where

$$
\begin{aligned}
(T\phi)^{(1)} &:= (tI^{(1)} - \Lambda^{(1)})^{-1} G^{(1)}(t, \phi), \\
(T\phi)^{(2)} &:= (\Lambda^{(2)})^{-1}\big[\big(1 * (t\phi^{(2)})\big) - G^{(2)}(t, \phi)\big].
\end{aligned} \tag{5}
$$

We already have the analytic solution ϕ in $\Delta(0, \rho_0)$. Next we solve (5) on the sector $S(I)$, where $I := (\alpha_1, \beta_1)$, α_1 and β_1 as in Theorem 2.1. We will show

Proposition 2.1. *There exists a unique analytic solution ϕ of (4) on $S(I) \cup \Delta(0, \rho_0)$ and $\phi \in \mathcal{A}^{\leq 2}(I)$.*

We delay the proof to Subsection 2.1. From this proposition and Theorem 1.1 it follows that

$$f_1 := \tilde{\mathcal{L}}_1\phi \in \mathcal{A}/\mathcal{A}^{\leq -2}(I_1).$$

Moreover, f_1 is solution of (3) $\bmod \mathcal{A}^{\leq -2}$. Here I_1 is as in Theorem 2.1.

Next we consider an acceleration of ϕ. From $f_1 = \tilde{\mathcal{L}}_1\phi$ and Remark 1.1 we deduce that $\tilde{\mathcal{B}}_2 f_1 = \tilde{\mathcal{B}}_2 \circ \tilde{\mathcal{L}}_1\phi = \mathbb{A}_{2,1}\phi \in t^{-1}\mathcal{A}(I')$, where $I' := (\alpha_1 - \pi/4, \beta_1 + \pi/4)$. Let $\psi := \rho_2 \circ \mathbb{A}_{2,1}\phi = \rho_2 \circ \tilde{\mathcal{B}}_2 f_1 = \tilde{\mathcal{B}} u_1$ if $u_1 := \rho_2(f_1)$. From (3) it follows that u_1 satisfies $\bmod \mathcal{A}^{\leq -1}$

$$2x^2 \frac{du}{dx} = \operatorname{diag}\{\sqrt{x}\, I^{(1)}, I^{(2)}\}\big(\Lambda u + \sqrt{x}\, g(\sqrt{x}, u)\big). \tag{6}$$

We apply the Borel transform to this equation. From $\rho_2^{-1}\psi = \mathbb{A}_{2,1}\phi \in t^{-1}\mathcal{A}(I')$ it follows that there exists $\rho > 0$ such that ψ exists in $U := \Delta(0, \rho) \cap S(I_+)$, where $I_+ := (2\alpha_1 - \pi/2, 2\beta_1 + \pi/2)$ and ψ satisfies

$$
\begin{gathered}
2t\psi(t) = \operatorname{diag}\Big\{\frac{1}{\sqrt{\pi t}} * (\Lambda\psi)^{(1)}, (\Lambda\psi)^{(2)}\Big\} + F(t, \psi), \text{ where} \\
F^{(1)}(t, \psi) := \big(\mathcal{B}\big(xg^{(1)}(\sqrt{x}, u(x))\big)\big)(t), \quad F^{(2)}(t, \psi) := \big(\mathcal{B}\big(\sqrt{x}\, g^{(2)}(\sqrt{x}, u(x))\big)\big)(t),
\end{gathered} \tag{7}
$$

where we used $(\mathcal{B}\sqrt{x})(t) = \frac{1}{\sqrt{\pi t}}$. Here g has a Taylor expansion

$$g(x, y) = \sum_{j \succeq 0} g_j(x) y^j = \sum_{j \succeq 0} \sum_{m=0}^{\infty} g_{j,m} x^m y^j \tag{8}$$

convergent for $|x| \leq r_1$, $|y| \leq r_2$ for some positive r_1, r_2 and

$$|g_j(x)| \leq M r_2^{|j|}, \qquad |g_{j,m}| \leq M r_2^{|j|} r_1^{-m}, \tag{9}$$

for some $M > 0$. Hence

$$F(\psi) = \sum_{j\geq 0} F_j * \psi^{*j}, \quad \text{where}$$
$$F_j^{(1)}(t) = \sum_{m=0}^{\infty} g_{j,m}^{(1)} \frac{t^{m/2}}{\Gamma((m/2)+1)}, \quad F_j^{(2)}(t) = \sum_{m=0}^{\infty} g_{j,m}^{(2)} \frac{t^{(m-1)/2}}{\Gamma((m+1)/2)}, \tag{10}$$

the series being convergent because of (9). So $t^{1/2}F_j(t)$ is an entire function of $t^{1/2}$ for all j.

We rewrite (7) as

$$\psi = P(\psi) := D_0\{D_1(\psi) + F(\psi)\}, \qquad \text{where}$$
$$D_0 := \operatorname{diag}\bigl\{(2t)^{-1}I^{(1)}, (2t-\Lambda^{(2)})^{-1}\bigr\}, \quad D_1(\psi) = \operatorname{diag}\bigl\{[(\pi t)^{-1/2} * (\Lambda\psi)]^{(1)}, 0^{(2)}\bigr\}. \tag{11}$$

So ψ satisfies (11) and we extend ψ in the following two lemmas.

Lemma 2.1. *The solution ψ can be analytically continued on $\tilde S := S(\tilde I)$, where $\tilde I = (\alpha_2, \beta_2) \subset I_+$ and $\tilde S$ does not contain any eigenvalue $\lambda_j/2$, $j = n_1+1, \ldots, n$, of $\Lambda^{(2)}/2$.*

Lemma 2.2. *$\psi \in t^{-1/2}\mathcal{A}^{\leq 1}(\tilde I)$.*

We will delay the proofs to Subsections 2.2 and 2.3. The last lemma implies $\mathbb{A}_{2,1}\phi = \rho_2^{-1}\psi \in t^{-1}\mathcal{A}^{\leq 2}(I')$, $I' := (\alpha_2/2, \beta_2/2)$ and Theorem 2.1 follows.

2.1. Proof of Proposition 2.1. It is sufficient to prove this proposition in the case that $\beta_1 - \alpha_1 \leq \pi/2 - \epsilon$, $\epsilon > 0$, since if $I' \Subset I$, then I' is the union of a finite number of these more special intervals and the solutions on the corresponding sectors glue to a unique analytic function because they already coincide on $\Delta(0, \rho_0)$. For the proof in the case $\beta_1 - \alpha_1 \leq \pi/2 - \epsilon$ we consider $\theta_0 := (\alpha_1 + \beta_1)/2$, $c \neq 0$, $c = |c|e^{-2i\theta_0}$, and the space $C(I)$ of continuous functions f on $S(I)$ which are analytic in the interior of $S(I)$ and such that $\|f\| := \sup_{t\in S(I)} |e^{-ct^2} f(t)| < \infty$. Now Proposition 2.1 is a consequence of

Lemma 2.3. *The operator T introduced in* (5) *defines a contraction on the ball $B_2 := \{f \in C(I) : \|f\| \leq \delta\}$ for sufficiently large $|c|$ if $\delta > 2M$ with M as in* (9)*.*

For the proof of this lemma we use some properties of the space $C(I)$ with $I = I'$ as above.

Lemma 2.4. *If $f, g \in C(I)$, then*

1. *$f * g \in C(I)$ and $\|f * g\| \leq \bigl(\pi/(|c|\sin\epsilon)\bigr)^{1/2}\|f\|\cdot\|g\|$.*
2. *$\|e^{p|t|}\| = e^{p^2/(4|c|\sin\epsilon)}$ if $p \in \mathbb{R}$.*
3. *$\|1 * (tf)\| \leq \|f\|/(2|c|\sin\epsilon)$.*

Proof. We have $|(f*g)(t)| \leq \|f\|\cdot\|g\|R(t)$ with $R(t) = |e^{ct^2}| * |e^{ct^2}|$. If $t = re^{i\theta} \in S(I)$, $r \geq 0$, $\theta \in \mathbb{R}$ and $c_0 := |c|\cos(2\theta - 2\theta_0)$, then $c_0 \geq c_1 := |c|\sin\epsilon$ and $R(t) \leq \int_0^r e^{c_0((r-\sigma)^2+\sigma^2)}\,d\sigma = e^{c_0r^2}\int_0^r e^{-c_0\sigma(2r-\sigma)}\,d\sigma \leq e^{c_0r^2}\int_0^\infty e^{-c_0\sigma^2}\,d\sigma = \frac{1}{2}e^{c_0r^2}(\frac{\pi}{c_0})^{1/2} \leq |e^{ct^2}|(\frac{\pi}{c_1})^{1/2}$ and item 1. follows.

Item 2. follows from $\|e^{p|t|}\| \leq \sup_{r\geq 0}\exp(pr - c_0r^2) = \exp(p^2/(4c_0))$. Finally, item 3. follows from $|1*(tf)| = \bigl|\int_0^t sf(s)\,ds\bigr| \leq \|f\|\cdot\int_0^t |se^{cs^2}\,ds| \leq \|f\|\int_0^r \sigma e^{c_0\sigma^2}\,d\sigma \leq (e^{c_0r^2}/(2c_0))\|f\|$. ∎

Now the proof of Proposition 2.1 may be given as follows: Using (8) and (9) we may estimate $G_j := \mathcal{B}(xg_j(x))$ by $|G_j(t)| \leq \sum |g_{j,m}t^m/m!| \leq Mr_2^{|j|}\exp(|t|/r_1)$. From this and item 2. in Lemma 2.4 we deduce that $\|G_j\| \leq Mr_2^{|j|}\exp\{1/(4r_1^2|c|\sin\epsilon)\} \leq (\delta/2)r_2^{|j|}$ if $|c|$ is sufficiently large. Then using items 1. and 2. of Lemma 2.4 we see that if $\phi \in B_2$, then $\|\phi^{*j}\| \leq (\pi/(|c|\sin\epsilon))^{(|j|-1)/2}\delta^{|j|}$ and $\|G_j * \phi^{*j}\| \leq (\delta/2)(\pi r_2^2\delta^2/(|c|\sin\epsilon))^{|j|/2}$ for all j. Hence $G(t,\phi) = \sum_j G_j * \phi^{*j}$ exists and maps B_2 into itself if $|c|$ is sufficiently large. From the boundedness of $|(tI^{(1)} - \Lambda^{(1)})^{-1}|$ on $S(I)$ it follows also that T maps B_2 into itself if $|c|$ is sufficiently large. Next we show that T is a contraction. For this we use that if $v, w \in C(I)$ and $l \succ 0$, then

$$\big\|(v+w)^{*l} - v^{*l}\big\| \leq |l|\big(\|v\| + \|w\|\big)^{|l|-1}\|w\|. \tag{12}$$

This may be shown by induction (cf. [Cos09, p. 175]). From this and Lemma 2.4 it follows that T defines a contraction on B_2. Hence we have a unique solution of (4) in B_2 which evidently coincides with the convergent series for ϕ on $\Delta(0,\rho_0)$ defined before. ∎

2.2. Proof of Lemma 2.1. It is sufficient to give the proof for the case $\tilde{I} \Subset I_+$.

First we give an extension of the usual convolution product due to Écalle. Choose $t_0 \in U$, where U is defined after (6), with $0 < |t_0| =: r_0 < \rho$. Choose $0 < r \leq r_0$ with $r_0 + r > \rho$ and consider $U_0 := \Delta(0,r_0) \cap \tilde{S}$ and $V := \{t \in \mathbb{C} : t - t_0 \in \tilde{S}, |t - t_0| \leq r\}$. For $t \in V$ we define $\gamma(t)$ to be the path from 0 to t consisting of the rectilinear segments $[0, t-t_0], [t-t_0, t_0], [t_0, t]$. Then $[0, t-t_0] \cup [t-t_0, t_0] \subset U_0$.

If f and g are continuous scalar functions on $U_0 \cup V$ then we define for $t \in V$: $(f * g)(t) = \int_{\gamma(t)} f(t-s)g(s)\,ds$. Then $f * g = g * f$ and if $f|_{U_0} = g|_{U_0} = 0$, then $(f * g)|_V = 0$. If f and g are analytic on the interior of $U_0 \cup V$, then $f * g$ is analytic on the interior of V and if moreover they are analytic in a neighborhood of t_0, then the extended $f * g$ is the analytic continuation of the usual $f * g$.

Let W be the space of continuous functions $f : V \to \mathbb{C}^n$ which are analytic in the interior of V and $\|f\| := \sup_{t\in V}|f(t)|$. Let ψ be the solution of (11) on U, $H(t) = \psi(t)$, $h(t) = 0$ if $t \in U_0$ and $H(t) = \psi(t_0)$ if $t \in V$, $h|_V \in W$. We want to determine h such that $\psi = H + h$ satisfies (11) on V. Hence h has to satisfy

$$h = P(H+h) - P(H) + R(H) =: \mathcal{M}(h), \quad \text{where } R(H) := P(H) - H. \tag{13}$$

From (10) and (11) it follows that

$$\begin{aligned} F(H+h) &= \sum_{l\succeq 0} F_l * (H+h)^{*l} = \sum_{j\succeq 0} q_j * h^{*j}, \\ q_j &= \sum_{m\succeq 0}\binom{j+m}{j} F_{j+m} * H^{*m}, \quad F(H) = q_0, \\ P(H+h) - P(H) &= D_0\Big(D_1(h) + \sum_{j\succ 0} q_j * h^{*j}\Big). \end{aligned} \tag{14}$$

Since $|H(t)| \leq K_0$ for some $K_0 > 0$ we have $|H^{*m}(t)| \leq K_0^{|m|}|t|^{|m|-1}/(|m|-1)!$. Since $\sqrt{t}\,F_j(t)$ is analytic in $\sqrt{t}$ the same holds for $\sqrt{t}\,q_j(t)$. From $h|_{U_0} = 0$ it follows that $h_j * h_l = 0$ and therefore $h^{*m} = 0$ if $|m| > 0$. Hence we may restrict the sum over j in the first part of (14) to $|j| \leq 1$ and in the last part to $|j| = 1$. Therefore

$\sqrt{t}\,R(H)(t)$ and $\sqrt{t}\,(P(H+h)-P(H))$ are analytic in $\sqrt{t}$. So (13) implies $\mathcal{M}(h)(t) = D_0(t)\int_{t_0}^{t} B(t-s)h(s)\,ds + R(H)$, where $\sqrt{t}\,B(t)$ is analytic in $\sqrt{t}$. So $h = \mathcal{M}(h)$ is a Volterra equation with a weak singularity and therefore $\mathcal{M}^l$ is a contraction on W for some integer l (cf. [Mik64]) and there is a unique analytic solution on V. In this way also the reasoning on p. 535 of [Bra92] may be corrected.

Hence we have the solution $\psi = H + h$ of (11) on V. This solution coincides on $V \cap U$ with the solution ψ we started with on U and thus it is analytic on the interior of $U \cup V$. By varying t_0 we obtain an analytic solution of (11) on $\Delta(0, r_0 + r) \cap \tilde{S}$. We may repeat this procedure of analytic extension next with U replaced by $\Delta(0, r_0 + lr) \cap \tilde{S}$ for $l = 1, 2, \ldots$ consecutively and thus we obtain an analytic solution ψ of (11) on $\tilde{S}$.

2.3. Proof of Lemma 2.2. It is sufficient to show that $\psi \in t^{-1/2}\mathcal{A}^{\le 1}(I_0)$ if $I_0 \Subset \tilde{I}$. In the following we restrict s to $S(I_0)$ and let $t = |s|$. For $t > 0$ we define $v(t) = \sup_{s\in S(I_0),|s|=t} \|\psi(s)\|$, where $\|\cdot\|$ denotes the Euclidean norm. From (10), (9) and (11) it follows that $\|D_1(\psi)(s) + F(\psi)(s)\| < KF_+(v)(t)$ for some $K > 2M$, where

$$F_+(v) = \sum_{j=0}^{\infty} F_{j+} * v^{*j}, \qquad F_{j+}(t) = r_2^j \sum_{m=0}^{\infty} r_1^{-m}\,\frac{t^{(m-1)/2}}{\Gamma((m+1)/2)}\,.$$

Let $R > 0$ to be chosen later on. If $t = |s| \ge R$ then $\|D_0(s)\| \le K_0$ for some constant $K_0 > 0$. Hence $v(t) < K(F_+(v))(t)$ if $t \ge R$ by increasing K suitably. Since $\psi = \mathcal{B}\rho_2(f_1)$, where $f_1 \sim \widehat{f}$, so $f_1(x) \sim c_1 x$, it follows that $\psi(s) \sim c_1/\sqrt{\pi s}\,$, $v(t) \sim |c_1|/\sqrt{\pi t}\,$. Also $F_+(t) \ge 1/\sqrt{\pi t}\,$. Hence we may choose R and $K > |c_1|$ such that $v(t) < K(F_+(v))(t)$ also for all $t < R$ and therefore for all $t > 0$.

We use the majorant method and first consider $v_0 = KF_+(v_0)$. If $w = \mathcal{L}v_0$, then $w(x) = K(\mathcal{L}F_+(v_0))(x) = K\sum_{j=0}^{\infty}(r_2 w(x))^j \sum_{m=0}^{\infty} r_1^{-m} x^{(m+1)/2}$. So $w(x) = K\sqrt{x}\,[(1 - r_2 w(x))(1 - \sqrt{x}/r_1)]^{-1}$ and this equation has a solution w analytic in $\sqrt{x}$ in a neighborhood of 0, real-valued for $x > 0$, whereas $w(x) \sim K\sqrt{x}$ as $x \to 0$. Now $v_0 = \mathcal{B}w \in t^{-1/2}\mathcal{A}^{\le 1}$ by Theorem 1.1 and $v_0(t) \sim K/\sqrt{\pi t}$ as $t \to 0$. Since $v(t) \sim |c_1|/\sqrt{\pi t}\,$, $|c_1| < K$ we have $v(t) < v_0(t)$ for t sufficiently small. Suppose $v(t) < v_0(t)$ for all $t \in (0, t_0)$. Then $v(t_0) < K(F_+ v)(t_0) < K(F_+ v_0)(t_0) = v(t_0)$. Hence $v < v_0$ on $\mathbb{R}_+$ and consequently $v(t) = O(e^{pt})$ as $t \to \infty$ for some $p > 0$. The definition of v then implies $\psi \in t^{-1/2}\mathcal{A}^{\le 1}(I_0)$.

3. ODE's with more levels. Theorem 2.1 may be extended as follows: Consider

$$\mathrm{diag}\{x^{m_1} I^{(1)}, \ldots, x^{m_r} I^{(r)}\}\, x\,\frac{dy}{dx} = \Lambda y + x g(x, y), \tag{15}$$

where $r \in \mathbb{N}$, $m_j \in \mathbb{N}$ for $j = 1, \ldots, r$, $0 < m_1 < \ldots < m_r$, $I^{(j)}$ denotes the identity matrix of dimension $n_j \in \mathbb{N}$ and $n = n_1 + \ldots + n_r$, $y \in \mathbb{C}^n$, $\Lambda = \mathrm{diag}\{\lambda_1, \ldots, \lambda_n\}$, Λ is invertible and g is analytic at $(0,0)$ in $\mathbb{C} \times \mathbb{C}^n$. Let $\widehat{y} = \sum_{h=1}^{\infty} c_h x^h$ be a formal solution of (15). Then (cf. [Bra92, Bal94, RS94])

THEOREM 3.1. *The formal solution $\widehat{y}$ of (15) is $(m_1, \ldots, m_r)$-summable on $(I_1, \ldots, I_r)$, where $I_j = (\alpha_j, \beta_j)$ with $\beta_j - \alpha_j > \pi/m_j$ and $\lambda_h \notin S(\alpha_j + \pi/(2m_j), \beta_j - \pi/(2m_j))$ for all $h \in [n_1 + \ldots + n_{j-1} + 1, n_1 + \ldots + n_j]$, and $I_j \subset I_{j-1}, j = 1, \ldots, r$, where $I_0 = \mathbb{R}$.*

The conditions on the intervals involving the eigenvalues of Λ may be reformulated in terms of Stokes rays as in the previous theorem. Theorem 3.1 may be proven with the methods used in the proof of Theorem 2.1. Now one considers recursively the equations for $\rho_{m_j} y$, apply the Borel transform, show that it results in functions in some suitable $\mathcal{A}^{\leq\kappa}$ and utilize accelerations $\mathbb{A}_{m_{j+1},m_j}$ to go to the level m_{j+1}.

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