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ASYMPTOTIC EXPANSIONS OF SOLUTIONS TO THE FIFTH PAINLEVÉ EQUATION IN NEIGHBOURHOODS OF SINGULAR AND NONSINGULAR POINTS OF THE EQUATION

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Abstract. Applying methods of plane Power Geometry we are looking for the asymptotic expansions of solutions to the fifth Painlevé equation in the neighbourhood of its singular and nonsingular points.

1. Introduction. The results obtained. We consider an ordinary differential equation of order n of the form

$$P(z, w, w', \dots, w^{(n)}) = 0, \tag{1}$$

where z is an independent, w is a dependent complex variable, P is a polynomial of its variables.

By means of two-dimensional Power Geometry [2, 4, 1] we obtain asymptotic expansions of solutions to the equation (1) in the neighbourhood of $z = 0$ and $z = \infty$. We are looking for the expansions of the form

$$w = c_r(z)z^r + \sum_{s\in\mathbf{K}} c_s(z)z^s, \tag{2}$$

where $c_r(z), c_s(z), r, s \in \mathbb{C}$, $\mathbf{K} \subset \{s : \operatorname{Re} s > \operatorname{Re} r\}$ for the expansions in the neighbourhood of $z = 0$ and $\mathbf{K} \subset \{s : \operatorname{Re} s < \operatorname{Re} r\}$ for the expansions in the neighbourhood of $z = \infty$; the set $\mathbf{K}$ is countable.

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We obtain the following six types of expansions of solutions to (2):
Type 1. $c_r(z)$ and $c_s(z)$ are constant (*power expansions*).
Type 2. $c_r(z)$ is constant, $c_s(z)$ are polynomials in $\log z$ (*power-logarithmic*).
Type 3. $c_r(z)$ and $c_s(z)$ are power series in $\log z$ (*complicated expansions*).
Type 4. $r, s \in \mathbb{R}$, $c_r(z)$ is a finite sum of powers of z^i with complex coefficients and $c_s(z)$ are power series over z^i (*half-exotic*). Here and below $i = \sqrt{-1}$.
Type 5. $r, s \in \mathbb{R}$, $c_r(z)$ and $c_s(z)$ are series in z^i, and c_r is a sum of countable number of terms, the set of power exponents of z^i in c_r is bounded either from above or from below (*exotic expansions*).
Type 6. $r, s \in \mathbb{R}$, $c_r(z)$ and $c_s(z)$ are series in z^i, and c_r is a sum of countable number of terms, the set of power exponents of z^i in c_r is bounded neither from above nor from below (*super-exotic expansions*).

If the expansion in the text below is not said to be convergent we cannot say anything about its convergence and it can be interpreted as a formal asymptotic expansion of solution.

Typical examples of set $\mathbf{K}$ for expansions of types 1–3 in case $z \to 0$ are given in Fig. 1. Exponents of z for the other types of expansions are given in Fig. 2 (for half-exotic expansions), in Fig. 3 (for exotic expansions), in Fig. 4 (for super-exotic expansions). As was mentioned before, $r, s \in \mathbb{R}$, but in the figures we also take into account the exponents of z^i in sums $c_s(z)$ and $c_r(z)$.

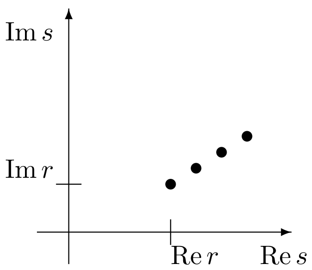

Fig. 1

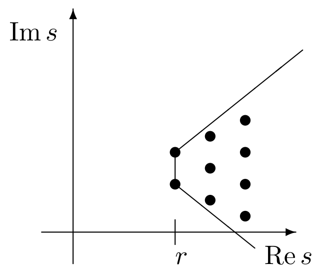

Fig. 2

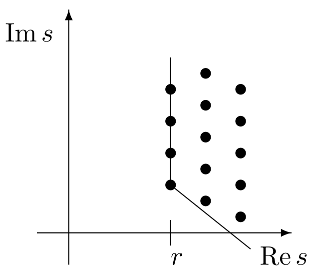

Fig. 3

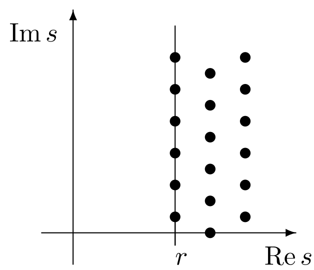

Fig. 4

We consider the fifth Painlevé equation (P_5)

$$w'' = \left(\frac{1}{2w} + \frac{1}{w-1}\right)(w')^2 - \frac{w'}{z} + \frac{(w-1)^2}{z^2}\left(\alpha w + \frac{\beta}{w}\right) + \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1}\,, \tag{3}$$

where $\alpha, \beta, \gamma, \delta$ are complex parameters, z is an independent, w is a dependent complex variable. The fifth Painlevé equation (3) has two singular points: $z = 0$ and $z = \infty$. The aim of the present work is to find all asymptotic expansions of solutions to the P_5 equation of six types described above near the singular points of the equation and to find all asymptotic expansions of solutions to the equation near its nonsingular point $z = z_0$, $z_0 \neq 0$, $z_0 \neq \infty$.

REMARK. Here and below the coefficients c_s and c_{sj}, $j = 1, \ldots, 10$, which are not said to be arbitrary are uniquely determined and can be found as solutions of a system of linear equations with non-zero determinant.

THEOREM 1.1. *In the neighbourhood of* $z = 0$ *there exist the following families of asymptotic expansions corresponding to the edge* $G_3^{(1)}$ *and to the vertex* $G_3^{(0)} = (0, 3)$ *obtained from the corresponding families of expansions of solutions to the sixth Painlevé equation*:

$$\mathcal{A}_0: \qquad w = 1 + c_r z^r + \sum_{s \in \mathbf{K}} c_s z^s, \tag{4}$$

where $\operatorname{Re} r \in (0, 1)$, $\mathbf{K} = \{s : s = r + lr + m(1 - r),\ l, m \in \mathbb{Z},\ l, m \geq 0,\ l + m > 0\}$, $c_r \in \mathbb{C} \setminus \{0\}$ *is an arbitrary constant. This family of asymptotic expansions exists for all values of parameters of the equation.*

$$\mathcal{B}_0^{\tau}: \qquad w = 1 - z^{\rho}\Big(c_{\rho} + \sum_{k=1}^{\infty} \tilde{c}_k z^{k\rho}\Big) + \sum_{s \in \mathbf{K}} c_s z^s, \tag{5}$$

where $\rho \neq 0$, $i\rho \in \mathbb{R}$ *is an arbitrary constant,* $\mathbf{K} = \{s : \operatorname{Re} s \geq 0,\ \rho + l\rho + m(1-\rho),\ l, m \geq 0,\ l + m > 0,\ l, m \in \mathbb{Z}\}$, $\tau = \operatorname{sgn}(\operatorname{Im} \rho)$. *This family of asymptotic expansions exists if* $|\alpha| + |\beta| \neq 0$.

The families $w^{-1}(z)$ *for the families* $\mathcal{B}_0^{+}$ *and* $\mathcal{B}_0^{-}$ *are the same.*

Let us define $\theta_1 = \sqrt{2\beta} - \sqrt{-2\alpha}$ *and* $\theta_2 = \sqrt{2\beta} + \sqrt{-2\alpha}$.

$$\mathcal{B}_j^{\tau},\ j = 1, 2: \qquad w = (-1)^{j+1}\sqrt{-\beta/\alpha} + \sum_{s \in \mathbf{K}} c_{sj} z^s = \frac{(-1)^{j+1}\sqrt{-\beta/\alpha}}{1 + C_1 z^{\tau\theta_j}} + \sum_{s \in \mathbf{K}} c_{sj} z^s, \tag{6}$$

where $\mathbf{K} = \{s : \operatorname{Re} s \geq 1,\ l + m\tau\theta_j,\ l, m \geq 0,\ l + m > 0,\ l, m \in \mathbb{Z}\}$, C_1 *is an arbitrary constant. The expansions exist if* $\alpha \cdot \beta \neq 0$, $\operatorname{Re} \theta_j = 0$, $j = 1, 2$.

If $\operatorname{Re}\big(\sqrt{2\beta} + (-1)^j\sqrt{-2\alpha}\big) \neq 0$ *we put* $k_j = \theta_j \cdot \operatorname{sgn}(\operatorname{Re} \theta_j)$.

$$\mathcal{B}_j,\ j = 1, 2: \qquad w = (-1)^{j+1}\sqrt{-\beta/\alpha} + \sum_{s} c_{sj} z^s, \tag{7}$$

$s \in \{s : s = l + mk_j,\ l, m \geq 0,\ l + m > 0,\ l, m \in \mathbb{Z}\}$, $c_{k_j j}$ *is an arbitrary constant. The expansions exist if* $\alpha \cdot \beta \neq 0$, $\operatorname{Re} \theta_j \neq 0$, $\theta_j \notin \mathbb{Z}$, $j = 1, 2$.

If $C_1 = 0$ *the set of power exponents in the expansion* (6) *is a subset of* $\mathbb{Z}$ *and this family coincides with the family* (7) *with* $c_{k_j j} = 0$, *i.e. with* $m = 0$.

$$\mathcal{B}_j,\ j = 1, 2: \qquad w = (-1)^{j+1}\sqrt{-\beta/\alpha} + \sum_{s=1}^{\infty} c_{sj}(\ln z) z^s, \tag{8}$$

where c_{sj} *are constant for* $s < k_j$, $c_{k_jj} = a_{k_jj} + b_{k_jj}\ln z$, a_{k_jj} *is an arbitrary constant,* b_{k_jj} *is constant and uniquely defined, the coefficients* c_{sj} *for* $s > k_j$ *are polynomials in* $\ln z$ *with uniquely determined coefficients. The expansions exist if* $\alpha\cdot\beta \neq 0$, $\operatorname{Re}\theta_j \neq 0$, $\theta_j \in \mathbb{Z}\setminus\{0\}$, $j = 1,2$. *Also if* $\alpha\cdot\beta \neq 0$, $\alpha+\beta = 0$ *there exists a family* $\mathcal{B}_2$ *defined by the formulae* (8) *or* (7).

$\mathcal{B}_3$:

$$w = \varphi_0 + \sum_{m=1}^{\infty} \varphi_m z^m,$$

$$\varphi_0 = 1 - \frac{2}{\alpha+\beta}\,\frac{1}{\ln^2 z} + \frac{c_{-3}}{\ln^3 z} + \sum_{s=4}^{\infty}\frac{c_{-s}}{\ln^s z} = \frac{2(\alpha+\beta)}{(\alpha+\beta)^2(\ln z + C_3)^2 + 2\alpha}\,, \tag{9}$$

where c_{-3}, $C_3 \in \mathbb{C}\setminus\{0\}$ *are arbitrary constants, here and below* $\varphi_m(z)$, $\varphi_{mj}(z)$ *are series in decreasing powers of* $\ln z$. *The expansions exist if* $\alpha\beta \neq 0$, $\alpha+\beta \neq 0$.

$\mathcal{B}_j,\ j = 4,5$:

$$w = \varphi_{0j} + \sum_{m=1}^{\infty} \varphi_{mj} z^m,$$

$$\varphi_{0j} = 1 + (-1)^j\frac{1}{-\alpha}\,\frac{1}{\ln z} + \frac{c_{-2j}}{\ln^2 z} + \sum_{s=3}^{\infty}\frac{c_{-sj}}{\ln^s z} = \frac{(-1)^j}{\sqrt{-2\alpha}\ln z + C_4}\,, \tag{10}$$

where $c_{-2j} = C_4 \in \mathbb{C}\setminus\{0\}$. *The expansions exist if* $\alpha\beta \neq 0$, $\alpha+\beta \neq 0$.

$\mathcal{B}_6, \mathcal{B}_6^\tau$:

$$w = 1 + c_\rho z^\rho + \sum_{s\in\mathbf{K}} c_s z^s,$$

$$\mathbf{K} = \{s : s = \rho + l\rho + m,\ l,m \geq 0,\ l+m > 0,\ l,m \in \mathbb{Z}\},\ \ \rho = \pm\sqrt{-2\alpha}, \tag{11}$$

where $c_\rho \in \mathbb{C}\setminus\{0\}$. *If* $\operatorname{Re}\rho > 0$ *the family is of Type* 1, *we denote it by* $\mathcal{B}_6$, *if* $\operatorname{Re}\rho = 0$ *we obtain two families of Type* 5 *denoted by* $\mathcal{B}_6^\tau$, $\tau = \operatorname{sgn}(\operatorname{Im}\rho)$. *The families exist if* $\alpha \neq 0$, $\beta = 0$.

$\mathcal{B}_{8,9}$:

$$w = 1 + c_\rho z^\rho + \sum_{s\in\mathbf{K}} c_s z^s, \tag{12}$$

where ρ *is an arbitrary imaginary constant,* $\rho^2 = 2\beta$, $\operatorname{Im}\rho > 0$ *for the family* $\mathcal{B}_8$, $\operatorname{Im}\rho < 0$ *for the family* $\mathcal{B}_9$, $\mathbf{K} = \{s : \operatorname{Re}s \geq 1,\ s = \rho + l(1-\rho) + m,\ l,m \geq 0,\ l+m > 0,\ l,m \in \mathbb{Z}\}$, c_ρ *is an arbitrary non-zero constant. The families exist if* $\alpha = 0$, $\beta \neq 0$.

$\mathcal{B}_{10}$:

$$w = c_0 + \sum_{s=1}^{\infty} c_s z^s, \tag{13}$$

where $c_0 \notin \{0,1\}$ *is an arbitrary constant. The families exist if* $\alpha = 0$, $\beta = 0$.

$\mathcal{A}_1$:

$$w = 1 + c_r z^r + \sum_{s} c_s z^s, \tag{14}$$

$r = -\operatorname{sgn}(\operatorname{Re}(\sqrt{-2\beta}))\sqrt{-2\beta}$, $s \in \{s : s = r - lr + m,\ l,m \geq 0,\ l+m > 0,\ l,m \in \mathbb{Z}\}$, c_r *is an arbitrary non-zero constant. The expansions exist if* $\alpha = 0$, $\beta \neq 0$.

The family $\mathcal{A}_2$ *is obtained by the symmetry* (23) *from the family* $\mathcal{A}_1$, *it exists if* $\beta = 0$.

The families of expansions listed in the theorem below also can be found in [4].

THEOREM 1.2. *In the neighbourhood of $z = 0$ there exist the following eight families of expansions, corresponding to the edge $G_4^{(1)}$:*

$$\mathcal{H}_1: \qquad w = 1 - \frac{2\delta}{\gamma} z + \sum_{s \in \mathbf{K}} c_s z^s,$$

where $a = \operatorname{sgn}\big(\operatorname{Re}(\gamma/\sqrt{-2\delta})\big)\gamma/\sqrt{-2\delta}$, $\mathbf{K} = \{s : s = l + m + ma,\ l, m \in \mathbb{Z},\ l, m \geq 0,\ l + m > 0\}$, c_{a+1} is an arbitrary constant. The family exists if $\gamma\delta \neq 0$, $\gamma^2/2\delta = -p^2$, $p \in \mathbb{R} \setminus \mathbb{N}$. If $a \in \mathbb{Q}$ the expansion converges according to Theorem 1.7.2 [4].

$$\mathcal{H}_2: \qquad w = 1 - \frac{2\delta}{\gamma} z + \sum_{s=1}^{\infty} c_s z^s,$$

where c_s, $1 \leq s \leq a$, are constants and c_s, $s \geq a+2$, are polynomials in $\log z$ with uniquely defined coefficients, $c_{a+1} = A \log z + C$, where C is an arbitrary constant. $\mathcal{H}_2$ exists if $\gamma\delta \neq 0$, $\gamma^2/2\delta = -n^2$, $n \in \mathbb{N}$.

$$\mathcal{H}_3: \qquad w = 1 - \frac{\delta}{\gamma} z - \frac{\gamma}{2} (\ln z + C)^2 z + \sum_{p=2}^{\infty} \varphi_p z^p,$$

where C is an arbitrary constant, φ_p are series in decreasing powers of $\log z$ with uniquely defined coefficients.

$$\mathcal{H}_1^\tau: \qquad w = 1 + \Big(-\frac{2\delta}{\gamma} + C z^{i\tau\gamma/\sqrt{2\delta}}\Big) z + \sum_{\operatorname{Re} s > 1} c_s z^s, \quad \gamma^2/\delta \in \mathbb{R}_+,\ \tau = \pm 1,\ C \neq 0;$$

$$\mathcal{H}_4: \qquad w = 1 + \Big(c_r z^{ir} - \frac{\gamma}{r^2} + \frac{\gamma^2 - 2\delta r^2}{4 c_r r^4} z^{-ir}\Big) z + \sum_{\operatorname{Re} s > 1} c_s z^s,$$

where $r \in \mathbb{R} \setminus \{0\}$, $c_r \in \mathbb{C}$, r and c_r are arbitrary constants.

The families $\mathcal{H}_1$, $\mathcal{H}_1^\tau$, $\mathcal{H}_2$ and $\mathcal{H}_3$ are one-parametric, the family $\mathcal{H}_4$ is two-parametric.

If $\gamma \neq 0$, $\delta = 0$, then the families $\mathcal{H}_3$ and $\mathcal{H}_4$ exist (we should substitute $\delta = 0$ in the corresponding formulae). If $\gamma = 0$, $\delta \neq 0$, there exist two families of expansions

$$\mathcal{H}_j^{(1)},\ j = 5, 6: \qquad w = 1 + (-1)^j \sqrt{-2\delta} (\ln z + C) z + \sum_{p=2}^{\infty} \varphi_p z^p,$$

where C is an arbitrary constant, φ_p are series in decreasing powers of $\log z$ with uniquely defined coefficients. Also the family $\mathcal{H}_4$ exists (we should substitute $\gamma = 0$ in the corresponding formulae).

THEOREM 1.3. *If $\alpha\beta\delta \neq 0$ in the neighbourhood of $z = \infty$ there exist five asymptotic expansions of solutions to the equation (3) (power expansions):*

$$\mathcal{D}_k: \qquad w = (-1)^k \sqrt{\frac{\beta}{\delta}} \frac{1}{z} + \Big(-\frac{2\beta}{\delta} + (-1)^k \frac{\gamma}{2\delta} \sqrt{\frac{\beta}{\delta}}\Big) \frac{1}{z^2} + \sum_{s=3}^{\infty} \frac{c_{sk}}{z^s},$$

$$\mathcal{E}_1: \qquad w = -1 + \frac{2\gamma}{\delta z} + \sum_{s=2}^{\infty} \frac{c_s}{z^s},$$

$$\mathcal{F}_k: \qquad w = (-1)^k \sqrt{-\frac{\delta}{\alpha}} z + 2 + (-1)^k \frac{1}{2} \frac{\gamma}{\sqrt{-\alpha\delta}} + \sum_{s=1}^{\infty} \frac{c_{s,k}}{z^s},$$

where c_s, c_{sk} are uniquely defined complex constants, $k = 1, 2$.

THEOREM 1.4. *If $\alpha = 0$, $\delta \neq 0$ in the neighbourhood of $z = \infty$ there exist two families $\mathcal{V}_+$, $\mathcal{V}_-$. They are defined by the formulae*:

$$\mathcal{V}_\sigma : \qquad w = Cz^{(1-\sigma\gamma/\sqrt{-2\delta})} \exp\Big\{\sigma\sqrt{-2\delta}\, z + \sum_{s=2}^{\infty} c_s \,\frac{z^{-s+1}}{-s+1}\Big\}, \tag{15}$$

where C is an arbitrary constant, $\sigma = \pm 1$, $\mathrm{Re}(\sigma\sqrt{-\delta}z) > 0$.

If $\beta = 0$, $\delta \neq 0$, there exist two families of expansions $\mathcal{U}_1$, $\mathcal{U}_2$, which can be obtained from $\mathcal{V}_1$, $\mathcal{V}_2$ by the symmetry (23). *If $\delta = 0$, $\gamma \neq 0$ there exist two families for $\alpha = 0$ and two more for $\beta = 0$. They are of an analogous form.*

THEOREM 1.5. *If $\alpha\beta\gamma \neq 0$, $\delta = 0$ in the neighbourhood of $z = \infty$, there exist five asymptotic expansions of solutions to the equation* (3) (*power expansions*):

$$\mathcal{D}_k,\ k = 3,4 : \qquad w = (-1)^k \sqrt{-\frac{\beta}{\gamma}}\,\frac{1}{\sqrt{z}} + \frac{\beta}{\gamma}\,\frac{1}{z} + \sum_{s=3}^{\infty} \frac{c_{s,k}}{z^{s/2}}\,,$$

$$\mathcal{E}_2 : \qquad w = 1,$$

$$\mathcal{F}_k,\ k = 3,4 : \qquad w = (-1)^k \sqrt{-\frac{\gamma}{\alpha}}\,\sqrt{z} + 1 + \sum_{s=1}^{\infty} \frac{c_{s,k}}{z^{s/2}}\,,$$

where c_s, c_{sk} are uniquely defined complex constants.

THEOREM 1.6. *In the neighbourhood of the nonsingular point $z = z_0$ of the equation* (3) *there exist* 10 *families of asymptotic expansions of its solutions*:

$$\mathcal{O}_j,\ j = 1,2 : \qquad w = (-1)^j\,\frac{\sqrt{-2\beta}}{z_0}\,(z - z_0) + \sum_{s=2}^{\infty} c_{sj}(z - z_0)^s; \tag{16}$$

they exist if $\beta \neq 0$.

$$\mathcal{O}_j,\ j = 3,4 : \qquad w = (-1)^j\,\frac{z_0}{\sqrt{2\alpha}(z - z_0)} + \sum_{s=0}^{\infty} c_{sj}(z - z_0)^s; \tag{17}$$

they exist if $\alpha \neq 0$.

$$\mathcal{O}_5 : \qquad w = \sum_{s=0}^{\infty} c_s(z - z_0)^s, \tag{18}$$

where $c_0, c_1 \in \mathbb{C}$ are arbitrary constants, $c_0 \neq 0$, $c_0 \neq 1$. It exists for all values of parameters of the equation.

$$\mathcal{O}_j,\ j = 6,7 : \qquad w = 1 + (-1)^j\sqrt{-2\delta}\,(z - z_0) + \sum_{s=2}^{\infty} c_{sj}(z - z_0)^s,$$

these expansions exist if $\delta \neq 0$.

$$\mathcal{O}_8 : \qquad w = 1 - \frac{\gamma}{2z_0}\,(z - z_0)^2 + \sum_{s=4}^{\infty} c_s(z - z_0)^s,$$

where c_4 is an arbitrary constant. The expansion exists when $\gamma \neq 0$, $\delta = 0$.

$$\mathcal{O}_9 : \qquad w = \sum_{s=-2}^{\infty} c_s(z - z_0)^s, \tag{20}$$

where c_{-2} is an arbitrary constant. The expansion exists when $\alpha = 0$.

$$\mathcal{O}_{10}: \qquad w = \sum_{s=2}^{\infty} c_s (z - z_0)^s, \tag{21}$$

where c_2 is an arbitrary constant. The expansion exists when $\beta = 0$.

2. Methods and algorithms of Power Geometry. Now we introduce the main notions of plane Power Geometry.

Let us be given an ordinary differential equation of the form (1), we call the left part of this equation a *differential sum*. To each differential monomial $a(z,w)$ in the differential sum (1) we put in correspondence its *two-dimensional vector power exponent* $Q(a(z,w)) = (q_1, q_2)$ according to the following rule:

$$Q(cz^r w^s) = (r, s);$$
$$Q\Big(\frac{d^l w}{dz^l}\Big) = (-l, 1);$$
$$Q(a(z,w)b(z,w)) = Q(a(z,w)) + Q(b(z,w)).$$

The set of all vector power exponents of differential monomials in the differential sum $P(z,w)$ (1) is called *a support of the differential sum* $P(z,w)$ and is denoted by $S(P)$. The convex hull $\Gamma(P)$ of a support $S(P)$ is called *a polygon of the differential sum* $P(z,w)$, the boundary $\partial\Gamma(P)$ of $\Gamma(P)$ consists of vertices $\Gamma_j^{(0)}$ and edges $\Gamma_j^{(1)}$.

To find asymptotic forms of solutions to the equation (1) we work with the *truncated equations*, corresponding to the vertices (and edges) of the polygon of the differential sum $P(z,w)$. These truncated equations contain all the terms of the equation, the power exponents of which belong to this vertex (or edge) of the polygon of the differential sum $P(z,w)$. Solutions of the truncated equations are called the *truncated solutions*.

Let us consider an edge having an external normal (n_1, n_2). *The normal cone of the edge* is a ray $\lambda(n_1, n_2)$, where $\lambda > 0$. If we consider a vertex belonging to the edges having external normals (n_{11}, n_{21}) and (n_{21}, n_{22}) we define *the normal cone of the vertex* as $\lambda_1(n_{11}, n_{21}) + \lambda_2(n_{12}, n_{22})$, where $\lambda_1 > 0$ and $\lambda_2 > 0$. If a normal cone of a vertex or of an edge intersects with a part of a plane where $n_1 > 0$, solution to the truncated equation corresponding to it can give asymptotic form of solutions to the equation under consideration in the neighbourhood of infinity, if a normal cone intersects with $n_1 < 0$, the situation is the same (replacing neighbourhood of infinity by the neighbourhood of zero).

More detailed description of the Power Geometry methods can be found in [2, 4, 1].

3. The fifth Painlevé equation. Let us represent the P_5 equation (3) in the form of a differential sum (a polynomial in z, w, w', w''), i.e. multiply it by $z^2 w(w-1)$ and put all the terms of the equation to the right part of the equation:

$$f(z,w) \stackrel{\text{def}}{=} -z^2 w(w-1)w'' + z^2\Big(\frac{3}{2}w - \frac{1}{2}\Big)(w')^2 - zw(w-1)w' + (w-1)^3(\alpha w^2 + \beta) + \gamma z w^2 (w-1) + \delta z^2 w^2 (w+1) = 0. \tag{22}$$

The P_5 equation (3) is invariant under the substitution

$$w \mapsto \frac{1}{w}, \tag{23}$$

the parameters of the equation transform under this substitution according to the following rule:

$$(\alpha, \beta, \gamma, \delta) \mapsto (-\beta, -\alpha, -\gamma, \delta). \tag{24}$$

We will use this fact later.

The polygon $\Gamma(f)$ for the equation (22) in the case $\alpha\beta\gamma\delta \neq 0$ is shown in Fig. 5. The edges $\Gamma_j^{(1)}$, $j = 1, 2, 3$, give the expansions in the neighbourhood of $z = \infty$, the edge $\Gamma_4^{(1)}$ gives the expansions near $z = 0$. The vertices of the polygon $\Gamma(f)$ correspond to algebraic equations, i.e. the equations not containing any derivatives of w. Such algebraic equations corresponding to the vertices consist of the only term of the form $cz^{q_1}w^{q_2}$, these equations have only trivial solutions $z = 0$ and $w = 0$.

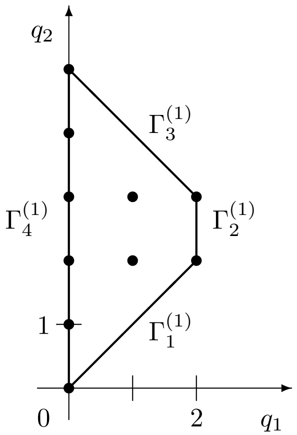

Fig. 5

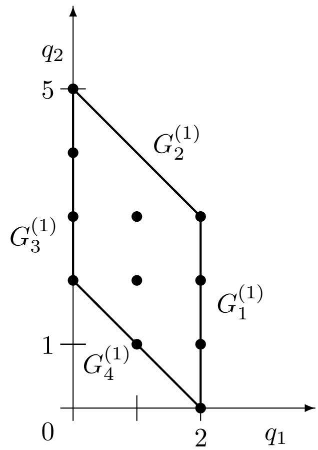

Fig. 6

4. Asymptotic expansions of solutions in the neighbourhood of zero. The following truncated equation corresponds to the edge $\Gamma_4^{(1)}$

$$-z^2w(w-1)w'' + z^2\Big(\frac{3}{2}\,w - \frac{1}{2}\Big)(w')^2 - zw(w-1)w' + (w-1)^3(\alpha w^2 + \beta) = 0. \tag{25}$$

We are looking for the solution of the equation (25) in the form $w = c$, $c = \text{const}$, $c \neq 0$. We obtain the equation

$$(c-1)^3(\alpha c^2 + \beta) = 0. \tag{26}$$

If $\alpha = 0$ and $\beta = 0$ every c satisfies equation (26), if $\alpha = 0$, $\beta \neq 0$, equation (26) has a solution $c = 1$, if $\alpha \neq 0$, equation (26) has solutions $c = 1$ and $c_{1,2} = \pm\sqrt{-\beta/\alpha}$. The differential operator (the first variation of (25)) on the substitution $z = c$ is equal to

$$\mathcal{L} = -z^2c(c-1)\,\frac{d^2}{dz^2} - c(c-1)z\,\frac{d}{dz} + (c-1)^2(5\alpha c^2 + \beta).$$

For $c \neq 0$ the operator $\mathcal{L}$ vanishes only at $c = 1$. Thus $w = 1$ is a special solution of the equation (25).

We substitute $w = 1 + \tilde{w}$ into (3) and obtain an equation $g(z, \tilde{w}) = 0$ with polygon $\tilde{\Gamma} = \Gamma(g)$ having edges $G_j^{(1)}$, $j = 1, 2, 3, 4$, and vertices (see Fig. 6). As we consider the case $z \to 0$, the appropriate truncated solutions can be given by equations corresponding to vertex $(0, 2)$ and edges $G_3^{(1)}, G_4^{(1)}$.

The truncated equation corresponding to the vertical edge $G_3^{(1)}$ can be transformed to the truncated equation corresponding to a vertical edge of the polygon of the sixth Painlevé equation [4], the asymptotic forms of solutions to the fifth Painlevé equation corresponding to the vertical edge of the polygon can be obtained from those of the vertical edge of the polygon of the sixth Painlevé equation. Moreover, as the sets of the power exponents of the polygons of both equations have the same bases, the structure of the expansions and the set $\mathbf{K}$ in (2) remains the same in the expansions of solutions to the fifth and sixth Painlevé equations. The same situation holds to the equation corresponding to a vertex $(0, 2)$ of $\Gamma(f)$. So we obtain Theorem 1.1.

As we consider the remaining edge $G_4^{(1)}$ we obtain the results formulated in Theorem 1.2 (in fact, these results can also be obtained from one of the truncated equations corresponding to a polygon of the third Painlevé equation).

More detailed description of our results obtained concerning the expansions in the neighbourhood of $z = 0$ can be found in [8].

5. Asymptotic expansions of solutions in the neighbourhood of infinity. Now we pass to the analysis of the expansions of solutions to the fifth Painlevé equation in the neighbourhood of infinity. As was written above, only the edges $\Gamma_j^{(1)}$, $j = 1, 2, 3$ (Fig. 5) give the expansions in the neighbourhood of $z = \infty$. If $\alpha\beta\delta \neq 0$ we obtain the following asymptotic expansions of solution to the P_5 equation. For all the expansions in this case the set $\mathbf{K}$ is a subset of $\mathbb{Z}$. All these expansions have been known before [9]. The expansions obtained are listed in Theorem 1.3.

If $\alpha\beta\gamma \neq 0$, $\delta = 0$ a polygon of the equation is shown in Fig. 7, only the edges $\tilde{\Gamma}_j^{(1)}$, $j = 1, 2, 3$, give the expansions in the neighbourhood of $z = \infty$, truncated equations corresponding to the vertices remain algebraic and, as was explained above, do not give any expansions of solutions. For all the expansions in this case the set $\mathbf{K}$ is a subset of $\frac{1}{2}\mathbb{Z} = \{\frac{1}{2}k,\ k \in \mathbb{Z}\}$. The expansions obtained are listed in Theorem 1.4.

For every expansion of the ten listed before, except the exact solution $\mathcal{E}_2$, there exist two corresponding exponential additions of the form $b(z)Ce^{\varphi(z)}$, where C is an arbitrary constant, $\varphi'(z)$, $b(z)$ are power expansions [5]. For example, for the expansion $\mathcal{E}_1$ these exponential additions have the following form

$$w = Cz^{-1/2} \exp\Big\{(-1)^m \sqrt{-\frac{\delta}{2}}\, z + \sum_{s=1}^{\infty} \frac{b_{sm}}{z^s}\Big\},$$

where b_{sm} are uniquely defined complex constants, C is an arbitrary constant. Here and below we fix a branch of complex square root such that $\sqrt{1} = 1$, we obtain that as $|z| \to \infty$ these exponential expansions are small in the part of the complex plane $(-1)^m \operatorname{Re} \sqrt{-\delta/2}\, z < 0$.

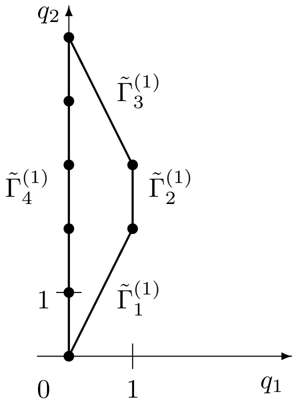

Fig. 7

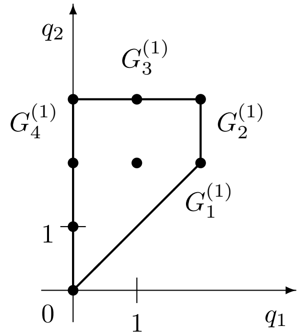

Fig. 8

If $\alpha = 0$, $\beta\gamma\delta \neq 0$, a polygon of the equation is shown in Fig. 8, only the edges $G_j^{(1)}$, $j = 1, 2, 3$, give the expansions in the neighbourhood of $z = \infty$, truncated equations corresponding to the appropriate vertices remain algebraic and do not give any expansions of solutions. Expansions corresponding to the non-horizontal edges remain the same, so we should pay attention only to a horizontal edge $G_3^{(1)}$.

The edge $G_3^{(1)}$ corresponds to the truncated equation

$$-z^2w^2w'' + \frac{3}{2}\,z^2w(w')^2 - zw^2w' + \beta w^3 + \gamma zw^3 + \delta z^2w^3 = 0. \tag{27}$$

We perform a logarithmic substitution to equation (27), i.e. substitute $\zeta = (\ln w)'$ to it:

$$-z^2\zeta' + \frac{1}{2}\,z^2\zeta^2 - z\zeta + \beta + \gamma z + \delta z^2 = 0,$$

we find appropriate asymptotic forms of solutions to this equation and we obtain that solutions of the fifth Painlevé equation have the families defined by formula (15), where C is an arbitrary constant, c_{sm} are uniquely defined constants, $m = 1, 2$. As the edge $G_3^{(1)}$ is an upper one for the polygon of the fifth Painlevé equation, $w \to \infty$, so these expansions exist for z, i.e. $(-1)^m \operatorname{Re}(\sqrt{-\delta}z) > 0$. The formula (15) does not work in case $\delta = 0$, but the analogous calculations can also be performed in case $\delta = 0$ [5].

If $\delta = \gamma = 0$ the fifth Painlevé equation (22) can be solved directly [4].

More detailed description of our results obtained concerning the expansions in the neighbourhood of $z = \infty$ can be found in [5].

6. Asymptotic expansions of solutions in the neighbourhood of the nonsingular point of the equation. To explore the expansions near the nonsingular point $z = z_0$ ($z_0 \neq 0$, $z_0 \neq \infty$) of the equation we perform the transformation $z = t + z_0$ which permits us to apply to the transformed equation the algorithms of Power Geometry described above.

If $\alpha\beta\gamma\delta \neq 0$ a polygon of the transformed equation is shown in Fig. 9, only the edges $\Gamma_j^{(1)}$, $j = 1, 2, 3$, and the vertices $(0, 0)$, $(-2, 2)$, $(-2, 3)$ and $(0, 5)$ belonging to these edges give the expansions in the neighbourhood of $t = 0$. The vertices $(0, 0)$ and $(0, 5)$ do not give any expansions as the truncated equations associated with them are algebraic, the vertices $(-2, 2)$ and $(-2, 3)$ do not give appropriate asymptotic forms.

The truncated equation corresponding to the edge $\Gamma_1^{(1)}$ gives two families (16) of asymptotic expansions of solutions to the P_5 equation (3).

The truncated equation corresponding to the edge $\Gamma_3^{(1)}$ gives two families (17) of asymptotic expansions of solutions to the P_5 equation (3).

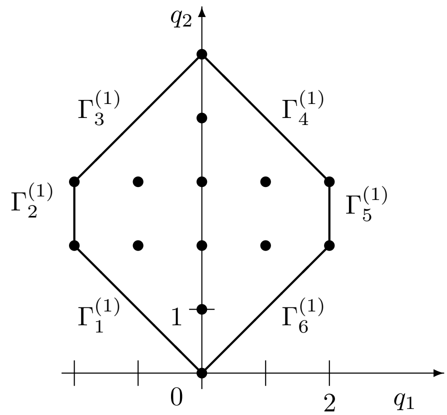

Fig. 9

The edge $\Gamma_2^{(1)}$ corresponds to the following truncated equation:

$$\hat{f}_2^{(1)}(t, w) \stackrel{\text{def}}{=} z_0^2 w w'' - \frac{1}{2}\, z_0^2 (w')^2 - z_0^2 w^2 w'' + \frac{3}{2}\, z_0^2 w (w')^2 = 0. \tag{28}$$

We are looking for the solution to the equation (28) in a form $w = c_0$, substituting $w = c_0$ to (28), we obtain an identical equality. If we substitute $w = c_0$, $c_0 \neq 0$ to the first variation $\delta \hat{f}_2^{(1)}/\delta w$ we obtain an operator $\mathcal{L}_5 = z_0^2 c_0 (1 - c_0)\, \frac{d^2}{dt^2}$, which is equal to the zero-operator iff $c_0 = 1$ (as $c_0 \neq 0$), so we should consider the case $c_0 = 1$ separately. The characteristic polynomial [4] of $\mathcal{L}_5$ is equal to $c_0(1 - c_0) z_0^2 k(k - 1)$ and has the roots $k_1 = 0$ and $k_2 = 1$. The only root $k = 1$ belongs to the set $\mathcal{K} = \{k > 0\}$, it is the only critical value. We consider this set $\mathcal{K}$ as the expansion is in increasing powers of t, so power exponents s of t in the expansion should satisfy this inequality: $\operatorname{Re} s \geq 0$, $s \neq 0$. Compatibility condition [4] is satisfied. So we obtain a family (18) of asymptotic expansions of solutions to the P_5 equation (3) and 3-dimensional Power Geometry permits us to work with a wider class of functions.

THEOREM 6.1. *Expansions of seven families $\mathcal{O}_j$, $j = 1, 2, 5, 6, 7, 8, 10$, are Taylor series and converge in the neighbourhood of $z = z_0$, expansions of the other three families $\mathcal{O}_3$, $\mathcal{O}_4$, $\mathcal{O}_9$ are Laurent series and converge in the deleted neighbourhood of $z = z_0$.*

The proof of this theorem is based on the Cauchy theorem and can be found in [6]. The families of expansions $\mathcal{O}_j$, $j = 1, 2, 3, 4, 5, 6, 7, 9, 10$, are not new and can be found in [9, 10], the family $\mathcal{O}_8$ is a new one.

More detailed description of our results obtained concerning the expansions of solutions in the neighbourhood of the nonsingular point can be found in [6].

The further exploration of expansions of solutions to the fifth Painlevé equation can be performed using methods of 3-dimensional Power Geometry [3]. The first results concerning the fifth Painlevé equation obtained using these methods can be found in [7]. We can briefly explain the difference between the expansions obtained by means of two- and three-dimensional Power Geometry: let $z \to 0$, we consider a function $\psi(z)$, we fix $z = re^{i\varphi}$. *An order of a function* as $z \to 0$ is

$$p_-(\psi(z), \varphi) = \varliminf_{r \to 0} \frac{\ln |\psi(re^{i\varphi})|}{\ln |r|}\,.$$

Two-dimensional Power Geometry permits us to obtain asymptotic forms $w = \psi(z)$ of solutions for which $p_-(\psi'(z), \varphi) = p_-(\psi(z), \varphi) - 1$.

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