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EULER'S INTEGRAL TRANSFORMATION FOR SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS WITH IRREGULAR SINGULARITIES

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Abstract. Dettweiler and Reiter formulated Euler's integral transformation for Fuchsian systems of differential equations and applied it to a definition of the middle convolution. In this paper, we formulate Euler's integral transformation for systems of linear differential equations with irregular singularities. We show by an example that the confluence of singularities is compatible with Euler's integral transformation.

1. Introduction. Euler's integral transformation is given by

$$f(x) \mapsto \int_C f(t)(t-x)^{\nu-1}\,dt, \tag{1}$$

where ν is a complex number and C is an appropriate cycle of the integral. It is also called the Riemann–Liouville integral for specialized cycles. Euler's integral transformation has been studied classically, and several applications for ordinary differential equations were explained in Ince's textbook [6]. A typical example is given by the integral representation for solutions of Gauss hypergeometric equation.

Euler's integral transformation is still actively studied, especially in connection with the middle convolution. The middle convolution was originally introduced by Katz in his book *Rigid Local Systems* [7], and Dettweiler and Reiter [1, 2] defined a middle convolution for systems of Fuchsian differential equations written as

$$\frac{dY}{dz} = \Big(\frac{A^{(1)}}{z-t_1} + \ldots + \frac{A^{(r)}}{z-t_r}\Big)Y, \tag{2}$$

where $A^{(1)}, \ldots, A^{(r)}$ are matrices of size $n \times n$. Note that Eq. (2) has singularities at

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$\{t_1, \dots, t_r, \infty\}$, all of which are regular. For a tuple of matrices $\langle A^{(1)}, \dots, A^{(r)}\rangle$ and $\mu \in \mathbf{C}$, we define a tuple of $nr \times nr$ matrices $\langle \tilde{A}^{(1)}, \dots, \tilde{A}^{(r)}\rangle$ by

$$\tilde{A}^{(1)} = \begin{pmatrix} A^{(1)}+\mu I_n & A^{(2)} & \dots & A^{(r)} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}, \quad \tilde{A}^{(2)} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ A^{(1)} & A^{(2)}+\mu I_n & \dots & A^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix},$$
$$\dots\dots \qquad \tilde{A}^{(r)} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{(1)} & A^{(2)} & \dots & A^{(r)}+\mu I_n \end{pmatrix}, \tag{3}$$

where I_n is the identity matrix of size n, and the correspondence of the tuples of matrices is called convolution. The following proposition shows that convolution is compatible with Euler's integral transformation of solutions of differential equations.

PROPOSITION 1.1 ([2]). *Assume that* $Y = \begin{pmatrix} y_1(z) \\ \vdots \\ y_n(z) \end{pmatrix}$ *is a solution of Eq.* (2). *Let* $\mu \in \mathbf{C}$ *and* γ *be an appropriate cycle of integrals. Then the function* U *defined by*

$$U = \begin{pmatrix} U^{(1)}(z) \\ U^{(2)}(z) \\ \vdots \\ U^{(r)}(z) \end{pmatrix}, \quad U^{(i)}(z) = \begin{pmatrix} \int_\gamma (w-t_i)^{-1} y_1(w)(z-w)^\mu \, dw \\ \vdots \\ \int_\gamma (w-t_i)^{-1} y_n(w)(z-w)^\mu \, dw \end{pmatrix} \tag{4}$$

satisfies the system of differential equations

$$\frac{dU}{dz} = \Big(\frac{\tilde{A}^{(1)}}{z-t_1} + \dots + \frac{\tilde{A}^{(r)}}{z-t_r} \Big) U. \tag{5}$$

In general, irreducibility of the tuple is not inherited by convolution. Middle convolution is defined by taking a suitable quotient of the convolution matrices, whose divisor is given explicitly (see [1, 2]).

In this paper, we study Euler's differential transformation for systems of linear differential equations with irregular singularities, which are written as

$$\frac{dY}{dz} = \Big(-\sum_{j=1}^{m_0} A_j^{(0)} z^{j-1} + \sum_{i=1}^{r} \sum_{j=0}^{m_i} \frac{A_j^{(i)}}{(z-t_i)^{j+1}} \Big) Y. \tag{6}$$

Convolution and middle convolution for Eq. (6) was introduced in [11], which was motivated by Kawakami's work [8]. A main purpose of the present paper is to give a proof of the compatibility of convolution and Euler's differential transformation for Eq. (6).

Note that Euler's differential transformation for single differential equations of higher order has been studied by Oshima [9] and Hiroe [5], and that their formulation differs from ours.

This paper is organized as follows. In Section 2, we review a definition of convolution for linear systems of differential equations, and present a theorem which is an analogue of Proposition 1.1. In Section 3, we prove the main theorem. In Section 4, we show an example where the convolution matrices are compatible with the confluence of singularities, which is the basis for our definition of convolution.

2. Convolution. Let $\mathbf{A} = \langle A^{(0)}_{m_0}, \dots, A^{(0)}_1, A^{(1)}_{m_1}, \dots, A^{(1)}_0, \dots, A^{(r)}_{m_r}, \dots, A^{(r)}_0 \rangle$ be a tuple of matrices acting on the finite-dimensional vector space V ($\dim V = n$) attached to the system of differential equations (6). We denote Eq.(6) by $D_{\mathbf{A}}$. Set

$$V' = (V^{(0)}_{m_0} \oplus \dots \oplus V^{(0)}_1) \oplus \bigoplus_{i=1}^{r} (V^{(i)}_{m_i} \oplus \dots \oplus V^{(i)}_0) \quad (V^{(i)}_j = V \ \forall i, j). \tag{7}$$

We fix $\mu \in \mathbf{C}$ and define the convolution matrices $\tilde{A}^{(i)}_j$ $(i = 0, \dots, r,\ j = \delta_{i,0}, \dots m_i)$ acting on V' by

$$\begin{pmatrix} u^{(0)}_{m_0} \\ \vdots \\ u^{(0)}_1 \\ u^{(1)}_{m_1} \\ \vdots \\ u^{(r)}_0 \end{pmatrix} = \tilde{A}^{(i)}_j \begin{pmatrix} v^{(0)}_{m_0} \\ \vdots \\ v^{(0)}_1 \\ v^{(1)}_{m_1} \\ \vdots \\ v^{(r)}_0 \end{pmatrix}, \tag{8}$$

where $v^{(i')}_{j'}, u^{(i')}_{j'} \in V^{(i')}_{j'} = V$ $(i' = 0, \dots, r,\ j' = \delta_{i',0}, \dots m_{i'})$ and $u^{(i')}_{j'}$ are given by

$$u^{(i')}_{j'} = \begin{cases} \mu v^{(i')}_{j'-j} & i' = i,\ j' > j, \\ \displaystyle\sum_{i''=0}^{r} \sum_{j''=\delta_{i'',0}}^{m_{i''}} A^{(i'')}_{j''} v^{(i'')}_{j''} & i = 0,\ i' = i,\ j' = j, \\ \displaystyle\mu v^{(i')}_0 + \sum_{i''=0}^{r} \sum_{j''=\delta_{i'',0}}^{m_{i''}} A^{(i'')}_{j''} v^{(i'')}_{j''} & i \neq 0,\ i' = i,\ j' = j, \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

Namely,

$$\tilde{A}^{(0)}_j = \left(\begin{array}{ccccccccc} & & \left.\begin{matrix} \mu I_n & & \\ & \ddots & \\ & & \mu I_n \end{matrix}\right\}_{m_0 - j} & & & & & & \\ A^{(0)}_{m_0} & \dots & A^{(0)}_2 & A^{(0)}_1 & A^{(1)}_{m_1} & \dots & A^{(1)}_0 & A^{(2)}_{m_2} \ \dots & A^{(r)}_0 \\ & & & & & & & & \\ \end{array} \right), \tag{10}$$

$$\begin{array}{c}\tilde{A}^{(i)}_j = \\ (i \geq 1)\end{array} \left(\begin{array}{ccccccccc}
 & & & & & & \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} m_0+(m_1+1)+\dots+(m_{i-1}+1) & & \\
 & & & \mu I_n & & & \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} m_i - j & & \\
 & & & & \ddots & & & & \\
 & & & & & \mu I_n & & & \\
A^{(0)}_{m_0} & \cdots & A^{(i-1)}_{\delta_{i,0}} & A^{(i)}_{m_i} & \dots & A^{(i)}_1 & A^{(i)}_0 + \mu I_n & A^{(i+1)}_{m_{i+1}} \quad \dots & A^{(r)}_0 \\
 & & & & & & & & \\
 & & & & & & & &
\end{array}\right). \tag{11}$$

We denote the tuple of convolution matrices by $c_\mu(\mathbf{A}) = \langle \tilde{A}^{(0)}_{m_0}, \dots, \tilde{A}^{(0)}_1, \tilde{A}^{(1)}_{m_1}, \dots, \tilde{A}^{(r)}_0 \rangle$. We can then show that the convolution corresponds to Euler's integral transformation on linear differential equations, which is analogous to Proposition 1.1.

THEOREM 2.1. *Assume that* $Y = \begin{pmatrix} y_1(z) \\ \vdots \\ y_n(z) \end{pmatrix}$ *is a solution of* $D_{\mathbf{A}}$, *i.e.*

$$\frac{dY}{dz} = \Big(- \sum_{j=1}^{m_0} A^{(0)}_j z^{j-1} + \sum_{i=1}^{r} \sum_{j=0}^{m_i} \frac{A^{(i)}_j}{(z-t_i)^{j+1}} \Big) Y. \tag{12}$$

(i) *Let* $\mu \in \mathbf{C}$ *and* γ *be a path such that* $[r(w) y_l(w)(z-w)^\mu]_{w \in \partial\gamma} = 0$ *for* $r(w) = 1/(w-t_i)^j$ $(j = 0, \dots, m_i,\ i = 1, \dots, r)$, $r(w) = w^j$ $(j = 0, \dots, m_0 - 1)$ *and* $l = 1, \dots, n$, *where* $\partial\gamma$ *is the boundary of* γ. *Then the function* U *defined by*

$$U = \begin{pmatrix} U^{(0)}_{m_0}(z) \\ \vdots \\ U^{(0)}_1(z) \\ U^{(1)}_{m_1}(z) \\ \vdots \\ U^{(1)}_0(z) \\ U^{(2)}_{m_2}(z) \\ \vdots \\ \vdots \\ U^{(r)}_0(z) \end{pmatrix}, \quad \begin{array}{l} U^{(0)}_j(z) = - \begin{pmatrix} \int_\gamma w^{j-1} y_1(w)(z-w)^\mu \, dw \\ \vdots \\ \int_\gamma w^{j-1} y_n(w)(z-w)^\mu \, dw \end{pmatrix}, \\ \\ U^{(i)}_j(z) = \begin{pmatrix} \int_\gamma (w-t_i)^{-j-1} y_1(w)(z-w)^\mu \, dw \\ \vdots \\ \int_\gamma (w-t_i)^{-j-1} y_n(w)(z-w)^\mu \, dw \end{pmatrix} \\ \qquad (i \neq 0) \end{array} \tag{13}$$

satisfies the system of differential equations $D_{c_\mu(\mathbf{A})}$, *i.e.*

$$\frac{dU}{dz} = \Big(- \sum_{j=1}^{m_0} \tilde{A}^{(0)}_j z^{j-1} + \sum_{i=1}^{r} \sum_{j=0}^{m_i} \frac{\tilde{A}^{(i)}_j}{(z-t_i)^{j+1}} \Big) U. \tag{14}$$

(ii) *Let $n \in \mathbf{Z}_{\geq 0}$. The function U defined by*

$$U = \begin{pmatrix} U^{(0)}_{m_0}(z) \\ \vdots \\ U^{(0)}_{1}(z) \\ U^{(1)}_{m_1}(z) \\ \vdots \\ U^{(r)}_{0}(z) \end{pmatrix}, \qquad \begin{aligned} U^{(0)}_j(z) &= \frac{d^n}{dz^n}(-z^{j-1}Y), \\ U^{(i)}_j(z) &= \frac{d^n}{dz^n}\Big(\frac{Y}{(z-t_i)^{j+1}}\Big) \quad (i \neq 0) \end{aligned} \tag{15}$$

satisfies the system of differential equations $D_{c_{-n-1}(\mathbf{A})}$.

We prove this theorem in Section 3.

We now discuss the path γ appearing in the theorem. If we take γ_t to be a closed path that encloses the point $w = t$ counter-clockwisely, then the Pochhammer contour $[\gamma_z, \gamma_{t_i}] = \gamma_z \gamma_{t_i} \gamma_z^{-1} \gamma_{t_i}^{-1}$ for $i = 0, \dots, r$ ($t_0 = \infty$) satisfies the condition of Theorem 2.1, because the branching of the function $r(w) y_l(w)(z-w)^{\mu}$ in the variable w is canceled after analytic continuation along the contour. We may have other paths which reflect the irregularity of the singularity when the function $Y(w)$ has a direction of exponential decay about an irregular singularity $w = t_i$. Then the contour which starts from the point $w = t_i$ goes in the opposite direction of the exponential decay about $w = t_i$, moves around the point $w = z$ and returns to $w = t_i$ from the direction of exponential decay (see Fig. A), also satisfies the condition of Theorem 2.1. For example, if $Y(w) =$

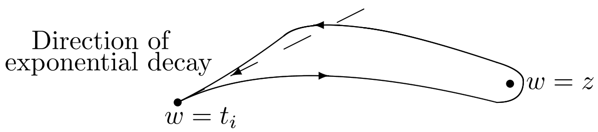

Fig. A.

$\exp(1/(w-t)^2)$, then the angle θ of the direction of exponential decay about $w = t$ is chosen as $-3\pi/4 < \theta < -\pi/4$ or $\pi/4 < \theta < 3\pi/4$.

We remark that if $m_0 = 0$ then the system of differential equations $D_{c_\mu(\mathbf{A})}$ may also be expressed in a generalized Okubo normal form $(zI_{n(m_1+\ldots+m_r+r)} - T)\frac{dU}{dz} = BU$ (T, B—constant matrices) by using Eq. (17). Kawakami [8] studied Euler's integral transformation for differential equations in generalized Okubo normal form.

The middle convolution $mc_\mu(\mathbf{A})$ is defined by taking a suitable quotient of convolution matrices, whose divisor is given explicitly (see [11]). Yamakawa [13] introduced another definition of middle convolution in the case $m_0 \leq 1$ by using Harnad duality [4] and the geometric invariant theory. I believe that Yamakawa's middle convolution essentially coincides with ours, although the correspondence should be made explicit. In our definition, the relationship between middle convolution and Harnad's dual system is not clear. Theories of dual isomonodromic deformations were developed in [10, 12], and it is also important to study middle convolution from the viewpoint of isomonodromic deformations (see [3] for the case of Fuchsian differential equations).

3. Proof of the theorem. The system $D_{c_\mu(\mathbf{A})}$ may be written as

$$U=\begin{pmatrix} U^{(0)}_{m_0}(z)\\ \vdots \\ U^{(0)}_1(z)\\ U^{(1)}_{m_1}(z)\\ \vdots\\ U^{(r)}_0(z)\end{pmatrix},\quad \begin{aligned} \frac{dU^{(0)}_j(z)}{dz}&=-z^{j-1}\sum_{i'=0}^{r}\sum_{j'=\delta_{i',0}}^{m_i}A^{(i')}_{j'}U^{(i')}_{j'}(z)-\mu\sum_{j'=1}^{j-1}z^{j'-1}U^{(0)}_{j-j'}(z),\\ \frac{dU^{(i)}_j(z)}{dz}&=\frac{\sum_{i'=0}^{r}\sum_{j'=\delta_{i',0}}^{m_i}A^{(i')}_{j'}U^{(i')}_{j'}(z)}{(z-t_i)^{j+1}}+\mu\sum_{j'=0}^{j}\frac{U^{(i)}_{j-j'}(z)}{(z-t_i)^{j'+1}},\quad i\neq 0,\end{aligned} \tag{16}$$

and is equivalent to

$$\begin{aligned} &\frac{dU^{(0)}_j(z)}{dz}-z\frac{dU^{(0)}_{j-1}(z)}{dz}=-\mu U^{(0)}_{j-1}(z), && j=2,\dots,m_0,\\ &\frac{dU^{(0)}_1(z)}{dz}=-\sum_{i'=0}^{r}\sum_{j'=\delta_{i',0}}^{m_i}A^{(i')}_{j'}U^{(i')}_{j'}(z), &&\\ &(z-t_i)\,\frac{dU^{(i)}_j(z)}{dz}-\frac{dU^{(i)}_{j-1}(z)}{dz}=\mu U^{(i)}_j(z), && i\neq 0,\ j=1,\dots,m_i,\\ &(z-t_i)\frac{dU^{(i)}_0(z)}{dz}=\mu U^{(i)}_0(z)+\sum_{i'=0}^{r}\sum_{j'=\delta_{i',0}}^{m_i}A^{(i')}_{j'}U^{(i')}_{j'}(z), && i\neq 0.\end{aligned} \tag{17}$$

The following lemma is a generalization of [2, Lemma 4.3(i), Lemma 4.2].

Lemma 3.1.

(i) *If Y is a solution of $D_{\mathbf{A}}$, then the function U defined by*

$$U=\begin{pmatrix} U^{(0)}_{m_0}(z)\\ \vdots\\ U^{(r)}_0(z)\end{pmatrix},\qquad \begin{aligned} U^{(0)}_j(z)&=-z^{j-1}Y, && j=1,\dots,m_0,\\ U^{(i)}_j(z)&=\frac{Y}{(z-t_i)^{j+1}}, && i\neq 0,\ j=0,\dots,m_i,\end{aligned} \tag{18}$$

is a solution of $D_{c_{-1}(\mathbf{A})}$.

(ii) *If U is a solution of $D_{c_\mu(\mathbf{A})}$, then dU/dz is a solution of $D_{c_{\mu-1}(\mathbf{A})}$.*

(iii) *Let $U=\begin{pmatrix} U^{(0)}_{m_0}(z)\\ \vdots\\ U^{(r)}_0(z)\end{pmatrix}$ be a solution of $D_{c_{\mu_1}(\mathbf{A})}$ and let γ be a cycle such that $[r(w)U^{(i)}_j(w)(z-w)^{\mu_2-1}]_{w\in\partial\gamma}=0$ for*

$$r(w)=\begin{cases} w-t_i, & i=1,\dots,r,\ j=0,\dots,m_i,\\ w, & i=0,\ j=1,\dots,m_0-1\\ 1, & i=0,1,\dots,r,\ j=\delta_{0,i},\dots,m_i-1+\delta_{0,i}.\end{cases}$$

Then the function defined by

$$\tilde{U} = \begin{pmatrix} \tilde{U}^{(0)}_{m_0}(z) \\ \vdots \\ \tilde{U}^{(r)}_0(z) \end{pmatrix}, \qquad \tilde{U}^{(i)}_j(z) = \int_\gamma U^{(i)}_j(w)(z-w)^{\mu_2 - 1}\, dw \tag{19}$$

is a solution of $D_{c_{\mu_1+\mu_2}(\mathbf{A})}$.

Proof. (i) We show Eq. (17) in the case $\mu = -1$. The relations

$$\begin{aligned} \frac{dU^{(0)}_j(z)}{dz} - z\frac{dU^{(0)}_{j-1}(z)}{dz} &= -(-1)U^{(0)}_{j-1}(z), \quad j = 2, \dots, m_0, \\ (z - t_i)\frac{dU^{(i)}_j(z)}{dz} - \frac{dU^{(i)}_{j-1}(z)}{dz} &= -U^{(i)}_j(z), \quad i \neq 0,\ j = 1, \dots, m_i, \end{aligned} \tag{20}$$

follow immediately from the definition. In the case $i \neq 0$, we have

$$\begin{aligned} (z - t_i)\frac{dU^{(i)}_0(z)}{dz} &= -\frac{Y}{z - t_i} + \frac{dY}{dz} = -U^{(i)}_0(z) - \sum_{j'=1}^{m_0} A^{(0)}_{j'} z^{j'-1} Y \\ &+ \sum_{i'=1}^{r}\sum_{j'=0}^{m_i} A^{(i')}_{j'} \frac{Y}{(z - t_{i'})^{j'}} = -U^{(i)}_0(z) + \sum_{i'=0}^{r}\sum_{j'=\delta_{i',0}}^{m_i} A^{(i')}_{j'} U^{(i')}_{j'}(z). \end{aligned} \tag{21}$$

The relation $\dfrac{dU^{(0)}_1(z)}{dz} = -\displaystyle\sum_{i'=0}^{r}\sum_{j'=\delta_{i',0}}^{m_i} A^{(i')}_{j'} U^{(i')}_{j'}(z)$ is shown similarly.

(ii) The statement is proved by differentiating Eq. (17).

(iii) Let $U(z)$ be a solution of $D_{c_{\mu_1}(\mathbf{A})}$. If $i \neq 0$ and $j \geq 1$, then

$$\begin{aligned} 0 &= \Big[\big\{(w - t_i)U^{(i)}_j(w) - U^{(i)}_{j-1}(w)\big\}(z-w)^{\mu_2 - 1}\Big]_{w \in \partial\gamma} \\ &= \int_\gamma \frac{d}{dw}\Big(\big\{(w - t_i)U^{(i)}_j(w) - U^{(i)}_{j-1}(w)\big\}(z-w)^{\mu_2 - 1}\Big)\, dw \\ &= \int_\gamma \Big\{(w - t_i)\,\frac{dU^{(i)}_j(w)}{dw} - \frac{dU^{(i)}_{j-1}(w)}{dw} + U^{(i)}_j(w)\Big\}(z-w)^{\mu_2 - 1}\, dw \\ &\quad - (\mu_2 - 1)\int_\gamma \big\{(-(z-w) + z - t_i)U^{(i)}_j(w) - U^{(i)}_{j-1}(w)\big\}(z-w)^{\mu_2 - 2}\, dw. \end{aligned} \tag{22}$$

It follows from Eq. (17) that the last term of Eq. (22) is equivalent to

$$\begin{aligned} \mu_1 \int_\gamma U^{(i)}_j(w)(z-w)^{\mu_2 - 1}\, dw &+ \mu_2 \int_\gamma U^{(i)}_j(w)(z-w)^{\mu_2 - 1}\, dw \\ &- (\mu_2 - 1)\int_\gamma \big\{(z - t_i)U^{(i)}_j(w) - U^{(i)}_{j-1}(w)\big\}(z-w)^{\mu_2 - 2}\, dw. \end{aligned} \tag{23}$$

Hence we have

$$\begin{aligned} (z - t_i)\,\frac{d\tilde{U}^{(i)}_j(z)}{dz} - \frac{d\tilde{U}^{(i)}_{j-1}(z)}{dz} &= (\mu_2 - 1)\int_\gamma \big\{(z - t_i)U^{(i)}_j(w) - U^{(i)}_{j-1}(w)\big\}(z-w)^{\mu_2 - 2} dw \\ &= (\mu_1 + \mu_2)\int_\gamma U^{(i)}_j(w)(z-w)^{\mu_2 - 1}\, dw = (\mu_1 + \mu_2)\tilde{U}^{(i)}_j(z). \end{aligned} \tag{24}$$

Similarly we have

$$\begin{aligned}
0 &= \Big[\big\{(w-t_i)U_0^{(i)}(w)\big\}(z-w)^{\mu_2-1}\Big]_{w\in\partial\gamma} \\
&= \int_\gamma \frac{d}{dw}\Big(\big\{(w-t_i)U_0^{(i)}(w)\big\}(z-w)^{\mu_2-1}\Big)\,dw \\
&= \int_\gamma \Big\{\mu_1 U_0^{(i)}(w) + \sum_{i'=0}^{r}\sum_{j'=\delta_{i',0}}^{m_i} A_{j'}^{(i')}U_{j'}^{(i')}(w)\Big\}(z-w)^{\mu_2-1}\,dw \\
&\qquad + \mu_2\int_\gamma U_0^{(i)}(w)(z-w)^{\mu_2-1}\,dw - (\mu_2-1)\int_\gamma (z-t_i)U_0^{(i)}(w)(z-w)^{\mu_2-2}\,dw.
\end{aligned} \tag{25}$$

Hence

$$\begin{aligned}
(z-t_i)\,\frac{d\tilde{U}_0^{(i)}(z)}{dz} &= (\mu_2-1)\int_\gamma (z-t_i)U_0^{(i)}(w)(z-w)^{\mu_2-2}\,dw \\
&= (\mu_1+\mu_2)\int_\gamma U_0^{(i)}(w)(z-w)^{\mu_2-1}\,dw + \int_\gamma \sum_{i'=0}^{r}\sum_{j'=\delta_{i',0}}^{m_i} A_{j'}^{(i')}U_{j'}^{(i')}(w)(z-w)^{\mu_2-1}\,dw \\
&= (\mu_1+\mu_2)\tilde{U}_0^{(i)}(z) + \sum_{i'=0}^{r}\sum_{j'=\delta_{i',0}}^{m_i} A_{j'}^{(i')}\tilde{U}_{j'}^{(i')}(z).
\end{aligned} \tag{26}$$

Therefore, we have the differential equations for $\tilde{U}_j^{(i)}(z)$ $(i\neq 0)$. The differential equations for $\tilde{U}_j^{(0)}(z)$, that is,

$$\begin{aligned}
-z\frac{d\tilde{U}_{j-1}^{(0)}(z)}{dz} + \frac{d\tilde{U}_j^{(0)}(z)}{dz} &= -(\mu_1+\mu_2)\tilde{U}_{j-1}^{(0)}(z), \\
\frac{d\tilde{U}_1^{(0)}(z)}{dz} &= -\sum_{i'=0}^{r}\sum_{j'=\delta_{i',0}}^{m_i} A_{j'}^{(i')}\tilde{U}_{j'}^{(i')}(z),
\end{aligned} \tag{27}$$

are obtained similarly from the following equalities:

$$\begin{aligned}
0 &= \Big[\big\{-wU_{j-1}^{(0)}(w) + U_j^{(0)}(w)\big\}(z-w)^{\mu_2-1}\Big]_{w\in\partial\gamma} \\
&= -\mu_1\int_\gamma U_{j-1}^{(0)}(w)(z-w)^{\mu_2-1}\,dw - \mu_2\int_\gamma U_{j-1}^{(0)}(w)(z-w)^{\mu_2-1}\,dw \\
&\qquad + (\mu_2-1)\int_\gamma \big\{zU_{j-1}^{(0)}(w) - U_j^{(0)}(w)\big\}(z-w)^{\mu_2-2}\,dw,
\end{aligned} \tag{28}$$

$$\begin{aligned}
0 = \Big[U_1^{(0)}(w)(z-w)^{\mu_2-1}\Big]_{w\in\partial\gamma} &= -\int_\gamma \Big\{\sum_{i'=0}^{r}\sum_{j'=\delta_{i',0}}^{m_i} A_{j'}^{(i')}U_{j'}^{(i')}(w)\Big\}(z-w)^{\mu_2-1}\,dw \\
&\qquad - (\mu_2-1)\int_\gamma U_1^{(0)}(w)(z-w)^{\mu_2-2}\,dw. \quad \blacksquare
\end{aligned} \tag{29}$$

We now prove Theorem 2.1. If Y is a solution of $D_{\mathbf{A}}$, then it follows from Lemma 3.1(i) that the function U defined by Eq. (18) is a solution of $D_{c_{-1}(\mathbf{A})}$. Hence, we obtain Theorem 2.1(i) from Lemma 3.1(iii) by setting $\mu_1 = -1$ and $\mu_2 = \mu + 1$. Note that we can also confirm that the condition of the cycle γ in Lemma 3.1(iii) follows from the condition in Theorem 2.1(i).

Theorem 2.1(ii) is obtained by applying Lemma 3.1(ii) repeatedly to $D_{c_{-1}(\mathbf{A})}$.

4. Confluence of singularities. We show that the definition of the convolution matrices is compatible with the confluence of singularities using an example.

Let $\langle B_1^{(0)}, B_2^{(0)}, B_3^{(0)}, B_1^{(1)}, B_2^{(1)}, B_3^{(1)}\rangle$ be a tuple of $n\times n$ matrices. Write

$$\frac{dY}{dz} = \Big(\sum_{i=1}^{3} \frac{B_i^{(0)}}{z-1/\epsilon_i} + \sum_{i=1}^{3} \frac{B_i^{(1)}}{z-\epsilon'_i}\Big)Y, \tag{30}$$

and consider a confluence of singularities. We define the matrices $A_3^{(0)}, A_2^{(0)}, A_1^{(0)}$ and $A_2^{(1)}, A_1^{(1)}, A_0^{(1)}$ by

$$\begin{aligned} \sum_{i=1}^{3} \frac{B_i^{(0)}}{z-1/\epsilon_i} &= -\frac{A_3^{(0)}z^2 + A_2^{(0)}z + A_1^{(0)}}{(1-\epsilon_1 z)(1-\epsilon_2 z)(1-\epsilon_3 z)}, \\ \sum_{i=1}^{3} \frac{B_i^{(1)}}{z-\epsilon'_i} &= \frac{A_2^{(1)} + A_1^{(1)}z + A_0^{(1)}z^2}{(z-\epsilon'_1)(z-\epsilon'_2)(z-\epsilon'_3)}. \end{aligned} \tag{31}$$

Let P and P' be the $3n\times 3n$ matrices defined by

$$P = \begin{pmatrix} \epsilon_1\epsilon_2\epsilon_3 I_n & -\epsilon_1(\epsilon_2+\epsilon_3)I_n & \epsilon_1 I_n \\ \epsilon_1\epsilon_2\epsilon_3 I_n & -\epsilon_2(\epsilon_1+\epsilon_3)I_n & \epsilon_2 I_n \\ \epsilon_1\epsilon_2\epsilon_3 I_n & -\epsilon_3(\epsilon_1+\epsilon_2)I_n & \epsilon_3 I_n \end{pmatrix}, \; P' = \begin{pmatrix} \epsilon'_2\epsilon'_3 I_n & -(\epsilon'_2+\epsilon'_3)I_n & I_n \\ \epsilon'_1\epsilon'_3 I_n & -(\epsilon'_1+\epsilon'_3)I_n & I_n \\ \epsilon'_1\epsilon'_2 I_n & -(\epsilon'_1+\epsilon'_2)I_n & I_n \end{pmatrix}. \tag{32}$$

Then the matrices $A_3^{(0)}, \dots, A_0^{(1)}$ may be written as

$$(A_3^{(0)}\ A_2^{(0)}\ A_1^{(0)}) = (B_1^{(0)}\ B_2^{(0)}\ B_3^{(0)})P, \qquad (A_2^{(1)}\ A_1^{(1)}\ A_0^{(1)}) = (B_1^{(1)}\ B_2^{(1)}\ B_3^{(1)})P'. \tag{33}$$

Conversely we have

$$B_1^{(0)} = \frac{A_3^{(0)} + \epsilon_1 A_2^{(0)} + \epsilon_1^2 A_1^{(0)}}{\epsilon_1(\epsilon_1-\epsilon_2)(\epsilon_1-\epsilon_3)}, \ \dots, \ B_3^{(1)} = \frac{A_2^{(1)} + \epsilon'_3 A_1^{(1)} + {\epsilon'_3}^2 A_0^{(1)}}{(\epsilon'_3-\epsilon'_1)(\epsilon'_3-\epsilon'_2)}. \tag{34}$$

Taking the limit $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon'_1, \epsilon'_2, \epsilon'_3 \to 0$ in Eq. (30) while keeping $A_3^{(0)}, \dots, A_0^{(1)}$ fixed, we obtain

$$\frac{dY}{dz} = \Big(-\sum_{j=1}^{3} A_j^{(0)} z^{j-1} + \sum_{j=0}^{2} \frac{A_j^{(1)}}{z^{j+1}}\Big)Y. \tag{35}$$

According to Dettweiler and Reiter, the $6n\times 6n$ convolution matrices $\tilde{B}_1^{(0)}, \dots, \tilde{B}_3^{(1)}$ in the tuple $\langle B_1^{(0)}, \dots, B_3^{(1)}\rangle$ are defined as

$$\tilde{B}_1^{(0)} = \begin{pmatrix} B_1^{(0)}+\lambda I_n & B_2^{(0)} & B_3^{(0)} & B_1^{(1)} & B_2^{(1)} & B_3^{(1)} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\tilde{B}_2^{(0)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ B_1^{(0)} & B_2^{(0)} + \lambda I_n & B_3^{(0)} & B_1^{(1)} & B_2^{(1)} & B_3^{(1)} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\vdots$$

$$\tilde{B}_3^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 \\ B_1^{(0)} & B_2^{(0)} & B_3^{(0)} & B_1^{(1)} & B_2^{(1)} & B_3^{(1)} + \lambda I_n \end{pmatrix}.$$

Let $\dot{A}_3^{(0)}$, $\dot{A}_2^{(0)}$, $\dot{A}_1^{(0)}$, $\dot{A}_2^{(1)}$, $\dot{A}_1^{(1)}$, $\dot{A}_0^{(1)}$ be the $6n \times 6n$ matrices defined by

$$\sum_{i=1}^{3} \frac{\tilde{B}_i^{(0)}}{z - 1/\epsilon_i} = -\frac{\dot{A}_3^{(0)} z^2 + \dot{A}_2^{(0)} z + \dot{A}_1^{(0)}}{(1-\epsilon_1 z)(1-\epsilon_2 z)(1-\epsilon_3 z)},$$

$$\sum_{i=1}^{3} \frac{\tilde{B}_i^{(1)}}{z - \epsilon'_i} = \frac{\dot{A}_2^{(1)} + \dot{A}_1^{(1)} z + \dot{A}_0^{(1)} z^2}{(z-\epsilon'_1)(z-\epsilon'_2)(z-\epsilon'_3)}.$$

Set $Q = \begin{pmatrix} P & 0 \\ 0 & P' \end{pmatrix}$ and define the matrices $\tilde{A}_3^{(0)}$, $\tilde{A}_2^{(0)}$, $\tilde{A}_1^{(0)}$, $\tilde{A}_2^{(1)}$, $\tilde{A}_1^{(1)}$, $\tilde{A}_0^{(1)}$ by

$$\tilde{A}_j^{(0)} = Q^{-1} \dot{A}_j^{(0)} Q \ (j = 1, 2, 3), \quad \tilde{A}_j^{(1)} = Q^{-1} \dot{A}_j^{(1)} Q \ (j = 0, 1, 2).$$

By straightforward calculation, we have

$$\tilde{A}_3^{(0)} = \begin{pmatrix} A_3^{(0)} + \lambda\tilde{\epsilon}_3 I_n & A_2^{(0)} & A_1^{(0)} & A_2^{(1)} & A_1^{(1)} & A_0^{(1)} \\ 0 & \lambda\tilde{\epsilon}_3 I_n & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda\tilde{\epsilon}_3 I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\tilde{A}_2^{(0)} = \begin{pmatrix} 0 & -\lambda\tilde{\epsilon}_1 I_n & \lambda I_n & 0 & 0 & 0 \\ A_3^{(0)} + \lambda\tilde{\epsilon}_3 I_n & A_2^{(0)} - \lambda\tilde{\epsilon}_2 I_n & A_1^{(0)} & A_2^{(1)} & A_1^{(1)} & A_0^{(1)} \\ 0 & \lambda\tilde{\epsilon}_3 I_n & -\lambda\tilde{\epsilon}_2 I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\tilde{A}_1^{(0)} = \begin{pmatrix} 0 & \lambda I_n & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda I_n & 0 & 0 & 0 \\ A_3^{(0)} + \lambda\tilde{\epsilon}_3 I_n & A_2^{(0)} - \lambda\tilde{\epsilon}_2 I_n & A_1^{(0)} + \lambda\tilde{\epsilon}_1 I_n & A_2^{(1)} & A_1^{(1)} & A_0^{(1)} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where $\tilde{\epsilon}_1 = \epsilon_1 + \epsilon_2 + \epsilon_3$, $\tilde{\epsilon}_2 = \epsilon_1\epsilon_2 + \epsilon_1\epsilon_3 + \epsilon_2\epsilon_3$, $\tilde{\epsilon}_3 = \epsilon_1\epsilon_2\epsilon_3$, and

$$\tilde{A}_2^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ A_3^{(0)} & A_2^{(0)} & A_1^{(0)} & A_2^{(1)} + \lambda\tilde{\epsilon}'_2 I_n & A_1^{(1)} - \lambda\tilde{\epsilon}'_1 I_n & A_0^{(1)} + \lambda I_n \\ 0 & 0 & 0 & \lambda\tilde{\epsilon}'_3 I_n & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda\tilde{\epsilon}'_3 I_n & 0 \end{pmatrix},$$

$$\tilde{A}_1^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda\tilde{\epsilon}'_1 I_n & \lambda I_n & 0 \\ A_3^{(0)} & A_2^{(0)} & A_1^{(0)} & A_2^{(1)} & A_1^{(1)} - \lambda\tilde{\epsilon}'_1 I_n & A_0^{(1)} + \lambda I_n \\ 0 & 0 & 0 & \lambda\tilde{\epsilon}'_3 I_n & -\lambda\tilde{\epsilon}'_2 I_n & 0 \end{pmatrix},$$

$$\tilde{A}_0^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda I_n & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda I_n & 0 \\ A_3^{(0)} & A_2^{(0)} & A_1^{(0)} & A_2^{(1)} & A_1^{(1)} & A_0^{(1)} + \lambda I_n \end{pmatrix},$$

where $\tilde{\epsilon}'_1 = \epsilon'_1 + \epsilon'_2 + \epsilon'_3$, $\tilde{\epsilon}'_2 = \epsilon'_1\epsilon'_2 + \epsilon'_1\epsilon'_3 + \epsilon'_2\epsilon'_3$, $\tilde{\epsilon}'_3 = \epsilon'_1\epsilon'_2\epsilon'_3$. Thus, we can reconstruct a special case of the definition of convolution matrices with irregular singularities (see Eqs. (10), (11)) by taking the limit $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon'_1, \epsilon'_2, \epsilon'_3 \to 0$ $(\tilde{\epsilon}_1, \dots, \tilde{\epsilon}'_3 \to 0)$.

We now investigate the transition of solutions given by Euler's integral transformation in the process of confluence of singularities. If Y is a solution of the system of differential equations (30), the function U given by

$$U = \begin{pmatrix} \int_\gamma (w - 1/\epsilon_1)^{-1} Y(w)(z-w)^\mu \, dw \\ \int_\gamma (w - 1/\epsilon_2)^{-1} Y(w)(z-w)^\mu \, dw \\ \int_\gamma (w - 1/\epsilon_3)^{-1} Y(w)(z-w)^\mu \, dw \\ \int_\gamma (w - \epsilon'_1)^{-1} Y(w)(z-w)^\mu \, dw \\ \int_\gamma (w - \epsilon'_2)^{-1} Y(w)(z-w)^\mu \, dw \\ \int_\gamma (w - \epsilon'_3)^{-1} Y(w)(z-w)^\mu \, dw \end{pmatrix}, \tag{36}$$

where γ is an appropriate cycle, satisfies

$$\frac{dU}{dz} = \left(\sum_{i=1}^{3} \frac{\tilde{B}_i^{(0)}}{z - 1/\epsilon_i} + \sum_{i=1}^{3} \frac{\tilde{B}_i^{(1)}}{z - \epsilon'_i} \right) U, \tag{37}$$

(see Proposition 1.1 and [2]). Recall that the matrices $\tilde{A}_3^{(0)}, \dots, \tilde{A}_0^{(1)}$ are defined by transition with respect to the matrix Q. Thus the function U should transfer to $Q^{-1}U$ in the process of confluence of singularities. Since

$$Q\begin{pmatrix} \int_\gamma -d(w)w^2Y(w)(z-w)^\mu\,dw \\ \int_\gamma -d(w)wY(w)(z-w)^\mu\,dw \\ \int_\gamma -d(w)Y(w)(z-w)^\mu\,dw \\ \int_\gamma \tilde{d}(w)w^{-3}Y(w)(z-w)^\mu\,dw \\ \int_\gamma \tilde{d}(w)w^{-2}Y(w)(z-w)^\mu\,dw \\ \int_\gamma \tilde{d}(w)w^{-1}Y(w)(z-w)^\mu\,dw \end{pmatrix} = \begin{pmatrix} \int_\gamma (w-1/\epsilon_1)^{-1}Y(w)(z-w)^\mu\,dw \\ \int_\gamma (w-1/\epsilon_2)^{-1}Y(w)(z-w)^\mu\,dw \\ \int_\gamma (w-1/\epsilon_3)^{-1}Y(w)(z-w)^\mu\,dw \\ \int_\gamma (w-\epsilon'_1)^{-1}Y(w)(z-w)^\mu\,dw \\ \int_\gamma (w-\epsilon'_2)^{-1}Y(w)(z-w)^\mu\,dw \\ \int_\gamma (w-\epsilon'_3)^{-1}Y(w)(z-w)^\mu\,dw \end{pmatrix}, \tag{38}$$

where $d(w) = 1/\prod_{i=1}^{3}(1-\epsilon_i w)$ and $d'(w) = 1/\prod_{i=1}^{3}(1-\epsilon'_i w)$, we have

$$Q^{-1}U \to \begin{pmatrix} \int_\gamma w^2Y(w)(z-w)^\mu\,dw \\ \int_\gamma wY(w)(z-w)^\mu\,dw \\ \int_\gamma Y(w)(z-w)^\mu\,dw \\ \int_\gamma w^{-3}Y(w)(z-w)^\mu\,dw \\ \int_\gamma w^{-2}Y(w)(z-w)^\mu\,dw \\ \int_\gamma w^{-1}Y(w)(z-w)^\mu\,dw \end{pmatrix} \tag{39}$$

as $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon'_1, \epsilon'_2, \epsilon'_3 \to 0$. Thus we obtain a prototype of Euler's integral transformation in Theorem 2.1.

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