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SUMMABILITY OF FIRST INTEGRALS OF A C^ω-NON-INTEGRABLE RESONANT HAMILTONIAN SYSTEM

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Abstract. This article studies the summability of first integrals of a C^ω-non-integrable resonant Hamiltonian system motivated by [BT] and [GZ]. The first integrals are expressed in terms of formal exponential transseries and their Borel sums (cf. [B] and [C]). Smooth Liouville integrability and a relation to the Birkhoff transformation are discussed from the point of view of the summability.

1. Introduction. In this article we shall study the summability of first integrals of a resonant C^ω-non-integrable Hamiltonian system. Let $q = (q_1, \dots, q_n) \in \mathbb{R}^n$, $p = (p_1, \dots, p_n) \in \mathbb{R}^n$ (or $\mathbb{C}^n$, $n \geq 2$). For a Hamiltonian function $H = H(q, p)$ we consider a Hamiltonian system

$$\dot{q} = \nabla_p H, \quad \dot{p} = -\nabla_q H, \tag{1}$$

or a Hamiltonian vector field

$$\chi_H := \{H, \cdot\} = \sum_{j=1}^{n} \Big(\frac{\partial H}{\partial p_j} \frac{\partial}{\partial q_j} - \frac{\partial H}{\partial q_j} \frac{\partial}{\partial p_j} \Big), \tag{2}$$

where $\{\cdot, \cdot\}$ denotes the Poisson bracket.

The function ϕ is called a first integral of χ_H if $\chi_H \phi = 0$. Eq. (1) is said to be C^ω-*Liouville integrable* if there exist n first integrals, $\phi_j \in C^\omega$ $(j = 1, \dots, n)$ which are functionally independent on an open dense set and Poisson commuting, i.e., $\{\phi_j, \phi_k\} = 0$, $\{H, \phi_k\} = 0$ for all $j, k = 1, 2, \dots, n$. If $\phi_j \in C^\infty$ $(j = 1, \dots, n)$, then we say that (1) is C^∞-*Liouville integrable.*

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We assume, for the sake of simplicity, that the Taylor expansion of H at the origin starts from terms of order 2, $H = H_2 + H_3 + \ldots$, where H_j is of homogeneous degree j. Let λ_j $(j = 1, 2, \ldots, n)$ be the eigenvalues of the bilinear form corresponding to H_2. We assume that (1) has a resonance dimension 1, namely $\lambda_1 = 0$.

In the paper [BT] Bolsinov and Taimanov showed that there exists a Hamiltonian related to geodesic flow on a Riemannian manifold which is C^∞-Liouville integrable and not C^ω-Liouville integrable. They showed that the C^ω-non-integrability is closely related to the monodromy structure of a Poincaré map, while they proved the C^∞-integrability by constructing functionally independent first integrals concretely. The study of such an operator is continued by Gorni and Zampieri in [GZ]. They showed that the Hamiltonian

$$H = -q_2 p_2 \partial_{q_1} r + (r^2 + q_2 \partial_{q_2} r) p_1, \quad r = q_1^2 + q_2^2,$$

is C^∞-integrable and not C^ω-integrable in some neighborhood of the origin of $(q_1, q_2, p_1, p_2) \in \mathbb{R}^4$. We note that $H_2 = 0$, namely the Hamiltonian has a resonance dimension 2. The proof of C^∞-integrability was made by construction of first integrals.

Heuristically speaking, the construction of n-functionally independent first integrals is equivalent to that of n-parameter family of solutions of (1), which can be expanded in a so-called exponential-log series. In view of this we express first integrals in exponential power series and apply the summability method, to show the integrability of (1) in a sector. We then construct smooth first integrals of (1) in terms of Borel-summed exponential power series. We also study the generating function of a Birkhoff normalizing transformation of (1) from the viewpoint of our Borel-summed integrals.

This paper is organized as follows. In Section 2 we construct functionally independent formal first integrals and in Section 3 we show the Borel summability of first integrals. In Section 4 we study the relation between the Borel-summed first integrals and the generating function of the symplectic transformation which transforms our Hamiltonian vector field to a resonant normal form. In the last section we briefly state C^ω-non-integrability and C^∞-integrability of our operator. The proofs of the theorems in the last section will be published in a future paper.

2. Construction of formal first integrals. In the sequel we change a little bit the notation in order to indicate the resonance variables q_1 and p_1. We write the variables in the form $(q_1, q_2, q_3, \ldots, q_n) = (q_1, q)$, $(p_1, p_2, p_3, \ldots, p_n) = (p_1, p)$. Let the Hamiltonian $H := H_0 + H_1$ be given by

$$H_0 = q_1^{2\sigma} p_1 + \sum_{j=2}^{n} \lambda_j q_j p_j, \tag{3}$$

$$H_1 = \sum_{j=2}^{n} q_j^2 B_j(q_1, q_1^{2\sigma} p_1, q), \quad q = (q_2, \ldots, q_n), \tag{4}$$

where $B_j(q_1, t, q)$ are holomorphic at the origin with respect to $(q_1, t, q) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^{n-1}$.

We assume

$$B_j = B_j(q_1, q_1^{2\sigma} p_1, q) = B_{j,0}(q_1, q) + q_1^{2\sigma} p_1 B_{j,1}(q_1, q), \tag{5}$$

where $B_{j,0}$ and $B_{j,1}$ are analytic at $q_1 = 0$, $q = 0$, $2 \le j \le n$. Moreover, we suppose

$$\lambda_j \ (j = 2, 3, \dots, n) \text{ are linearly independent over } \mathbb{Z}. \tag{6}$$

We will construct a formal first integral in exponential power series (cf. [B], [C]). Define

$$E_c \equiv E_c(q_1) := \exp\Big(\frac{c q_1^{-2\sigma+1}}{2\sigma - 1}\Big) \tag{7}$$

and construct the formal first integral v in the form

$$v = \sum_{\alpha \ge 0} v^{(\alpha)}(q_1, p_1, q, p) E^{\alpha}, \tag{8}$$

where $E^{\alpha} = E_{\lambda_2}^{\alpha_2} \cdots E_{\lambda_n}^{\alpha_n}$, and $v^{(\alpha)}(q_1, p_1, q, p)$ is a formal power series of q_1, q, p_1 and p. We say that v is a *formal integral* of χ_H if $\chi_H v = 0$ as a formal power series.

By definition we have, for $\mathcal{L} := \{H_0, \cdot\}$ and $R := \{H_1, \cdot\}$,

$$\mathcal{L} = q_1^{2\sigma} \frac{\partial}{\partial q_1} - 2\sigma q_1^{2\sigma-1} p_1 \frac{\partial}{\partial p_1} + \sum_{j=2}^{n} \lambda_j \Big(q_j \frac{\partial}{\partial q_j} - p_j \frac{\partial}{\partial p_j}\Big), \tag{9}$$

$$R = \sum_{j=2}^{n} \Big(-2q_j B_j \frac{\partial}{\partial p_j} + q_j^2 (\partial_{p_1} B_j) \frac{\partial}{\partial q_1} - q_j^2 (\partial_{q_1} B_j) \frac{\partial}{\partial p_1} - q_j^2 \nabla_q B_j \cdot \frac{\partial}{\partial p}\Big). \tag{10}$$

By using the formula

$$\partial_{p_1} B_j = B_{j,1} q_1^{2\sigma}, \quad q_1^{2\sigma} (\partial / \partial q_1) E^{\alpha} = -\Big(\sum_{j=2}^{n} \lambda_j \alpha_j\Big) E^{\alpha} = -\langle \lambda, \alpha \rangle E^{\alpha},$$

we have

$$\mathcal{L}(v^{(\alpha)} E^{\alpha}) = E^{\alpha} \bigg(q_1^{2\sigma} \frac{\partial}{\partial q_1} - 2\sigma q_1^{2\sigma-1} p_1 \frac{\partial}{\partial p_1} + \sum_{j=2}^{n} \lambda_j \Big(q_j \frac{\partial}{\partial q_j} - p_j \frac{\partial}{\partial p_j} - \alpha_j\Big)\bigg) v^{(\alpha)}, \tag{11}$$

and

$$R(v^{(\alpha)} E^{\alpha}) = E^{\alpha} \bigg(-\langle \lambda, \alpha \rangle \sum_{j=2}^{n} q_j^2 B_{j,1} + R\bigg) v^{(\alpha)}. \tag{12}$$

It follows that if v is a formal first integral of χ_H, then every $v^{(\alpha)}$ satisfies

$$\bigg(q_1^{2\sigma} \frac{\partial}{\partial q_1} - 2\sigma q_1^{2\sigma-1} p_1 \frac{\partial}{\partial p_1} + \sum_{j=2}^{n} \lambda_j \Big(q_j \frac{\partial}{\partial q_j} - p_j \frac{\partial}{\partial p_j} - \alpha_j\Big)\bigg) v^{(\alpha)} + \bigg(-\sum_{j=2}^{n} \langle \lambda, \alpha \rangle q_j^2 B_{j,1} + R\bigg) v^{(\alpha)} = 0. \tag{13}$$

Expand $v^{(\alpha)}$ into the formal power series

$$v^{(\alpha)} = \sum_{\nu, k, \ell} v_{\nu,k,\ell}^{(\alpha)}(q_1) p_1^{\nu} p^k q^{\ell}, \tag{14}$$

then insert the expansion into (13) and compare the coefficients of $p_1^{\nu} p^k q^{\ell}$. One can easily see that the first term of the left-hand side of (13) yields

$$\Big(q_1^{2\sigma} \frac{\partial}{\partial q_1} - 2\sigma q_1^{2\sigma-1} \nu + \lambda \cdot (\ell - k - \alpha)\Big) v_{\nu,k,\ell}^{(\alpha)}(q_1). \tag{15}$$

Hence we obtain the recurrence relation like

$$\Big(q_1^{2\sigma}\frac{\partial}{\partial q_1} - 2\sigma q_1^{2\sigma-1}\nu + \lambda\cdot(\ell-k-\alpha)\Big)v^{(\alpha)}_{\nu,k,\ell}(q_1) = F, \tag{16}$$

where F denotes terms which appear from the second term of the left-hand side of (13).

In order to get the detailed expression of F we first note that

$$-2q_jB_j\frac{\partial}{\partial p_j}v^{(\alpha)} = -2B_j\sum v^{(\alpha)}_{\nu,k+e_j,\ell-e_j}(q_1)p_1^{\nu}p^kq^{\ell}(k_j+1). \tag{17}$$

Expand B_j into the power series of q and compare the coefficients of $p_1^{\nu}p^kq^{\ell}$ of the right-hand side. One can see that the terms containing $v^{(\alpha)}_{\nu,k+e_j,\mu}(q_1)$, $\mu\le\ell-e_j$, appear from (17). Similar terms appear from $q_j^2\nabla_qB_j\cdot\frac{\partial}{\partial p}v^{(\alpha)}$ and $q_j^2(\partial_{q_1}B_j)\frac{\partial}{\partial p_1}v^{(\alpha)}$. In the latter case there appear terms $v^{(\alpha)}_{\nu+1,k,\mu}(q_1)$ with $\mu\le\ell-2e_j$. In the same way one can see that there appear terms containing the quantities

$$v^{(\alpha)}_{\nu,k,\mu}(q_1),\quad q_1^{2\sigma}B_{j,1}\frac{\partial}{\partial q_1}v^{(\alpha)}_{\nu,k,\mu}(q_1),\quad \mu\le\ell-2e_j,$$

from $-\lambda_j\alpha_jq_j^2B_{j,1}v^{(\alpha)}$ and $q_j^2q_1^{2\sigma}B_{j,1}\frac{\partial}{\partial q_1}v^{(\alpha)}$.

Let $\alpha\in\mathbb{Z}^{n-1}$ be given. We shall solve (16) inductively with respect to ℓ, $|\ell|=0,1,2,\ldots$. For this purpose we consider two cases: (A) $\ell-\alpha\notin\mathbb{Z}^{n-1}_+$, (B) $\ell-\alpha\in\mathbb{Z}^{n-1}_+$. Let (α,ℓ) satisfy (A). We have $\ell-k-\alpha\ne0$ for every $k\in\mathbb{Z}^{n-1}_+$. We want to determine $v^{(\alpha)}_{\nu,k,\ell}(q_1)$. By the non-resonance condition (6) we have $\lambda\cdot(\ell-k-\alpha)\ne0$ if and only if $\ell-k-\alpha\ne0$. In the right-hand side of (17) there appear $v^{(\alpha)}_{\nu',k,\ell-\beta}$'s for which $\beta\ge0$, $\beta\ne0$. It follows that α and $\ell-\beta$ satisfy (A). Expand $v^{(\alpha)}_{\nu,k,\ell}$ into the formal power series of q_1 and insert it into (16). One easily sees that every coefficient is uniquely determined if the right-hand side F is known, i.e., $v^{(\alpha)}_{\nu',k,\ell-\beta}$'s $\beta\ne0$ are given. Next we substitute $v^{(\alpha)}_{\nu',k,\ell-\beta}$ in F with the recurrence relations for $v^{(\alpha)}_{\nu',k,\ell-\beta}$'s which can be constructed similarly as $v^{(\alpha)}_{\nu,k,\ell}(q_1)$. By repeating the same argument, we finally arrive at the relation that the right-hand side of (17) vanishes, i.e., $F=0$ because we have $\ell-\beta\notin\mathbb{Z}^{n-1}_+$ after a finite times of substitutions. Hence, by (16) we obtain $v^{(\alpha)}_{\nu,k,\ell}=0$.

Next we consider the case (B). We set $k(\alpha,\ell)=\ell-\alpha\in\mathbb{Z}^{n-1}_+$. Let $\ell=0$. Because F in (16) vanishes, we have $v^{(\alpha)}_{\nu,k,\ell}=0$ if $k\ne k(\alpha,0)=-\alpha$. If $k=k(\alpha,0)=-\alpha\in\mathbb{Z}^{n-1}_+$, then we have $(q_1\frac{\partial}{\partial q_1}-2\sigma\nu)v^{(\alpha)}_{\nu,k,0}=0$. We take

$$v^{(\alpha)}_{\nu,-\alpha,0} = c_{\alpha,\nu}q_1^{2\sigma\nu}, \tag{18}$$

where $c_{\alpha,\nu}$ is an arbitrary constant.

Let ℓ be such that $|\ell|=1$. Because $v^{(\alpha)}_{\nu,k,\ell}$ vanish unless $\ell-\alpha\ge0$ by (A), we may assume $\ell-\alpha\ge0$. By the definition of F the non-vanishing term in F is given by $v^{(\alpha)}_{\nu,k+e_j,\mu}$, $\mu\le\ell-e_j$ for some j. Hence we have $|\mu|=0$. It follows that, if $v^{(\alpha)}_{\nu,k+e_j,\mu}\ne0$, then we have $\mu=0$ and $k+e_j=-\alpha\ge0$. If $|k|\ge|\ell-\alpha|$, then we have $k+e_j\ne-\alpha$, which yields

$v^{(\alpha)}_{\nu,k+e_j,\mu}=0$ and $F=0$. Hence, by (16) we have the following expression

$$\begin{aligned} v^{(\alpha)}_{\nu,k,\ell}&=0 \quad (\nu=0,1,\dots) \text{ if } \ \ell-\alpha-k\neq 0,\ |k|\geq|\ell-\alpha|,\\ v^{(\alpha)}_{\nu,k,\ell}&=c_{\alpha,\nu,\ell}q_1^{2\sigma\nu} \quad (\nu=0,1,\dots) \text{ if } \ \ell-\alpha-k=0. \end{aligned} \tag{19}$$

In the case $|k|<|\ell-\alpha|$, we have $\ell-\alpha-k\neq 0$, and we can recursively determine $v^{(\alpha)}_{\nu,k,\ell}$ as the formal power series of q_1. We note that the Gevrey order of $v^{(\alpha)}_{\nu,k,\ell}$ increases at most by $2\sigma-1$ if one solves the recurrence relation once.

Suppose that for some integer $s\geq 0$ we have determined the solutions of (16), $v^{(\alpha)}_{\nu,k,\ell}$ for $\nu=0,1,\dots$; $k\in\mathbb{Z}_+^{n-1}$; ℓ, $|\ell|=0,1,2\dots,s$, such that (19) holds with some constant $c_{\alpha,\nu,\ell}$. We will solve (16) for $|\ell|=s+1$ so that (19) holds. Let k and ℓ satisfy $|k|\geq|\ell-\alpha|$. We want to show $F=0$ in (16). In view of the definition of F, we first consider $v^{(\alpha)}_{\nu,k+e_j,\mu}$ where $\mu\leq\ell-e_j$ for some j. In order to show that this term vanishes by (19), we will show that $|k+e_j|>|\mu-\alpha|$. This relation follows from $|k+e_j|>|\ell-e_j-\alpha|\geq|\mu-\alpha|$ because the last inequality follows from the inductive assumption. Next we consider the term $v^{(\alpha)}_{\nu,k,\mu}$ where $\mu\leq\ell-2e_j$ for some j. Then it is sufficient to verify $|k|>|\ell-2e_j-\alpha|$, which holds by assumption. One can similarly verify that the other terms vanish, from which we obtain $F=0$. Therefore, by (16) we have the first relation of (19). As for the second one we can argue as in the case $|\ell|=1$. Clearly, if $|k|<|\ell-\alpha|$, then we can determine $v^{(\alpha)}_{\nu,k,\ell}$ by solving (16) recursively. This proves that we can determine $v^{(\alpha)}_{\nu,k,\ell}$ in the case $|\ell|=s+1$ so that (19) holds. This proves the assertion.

We remark that if we take arbitrary constants $c_{\alpha,\nu,\ell}$, $\ell-\alpha=k$, to be zero except for a finite number of k's or ν's, then we see that the formal solution is a polynomial of p_1 and p. We note that the sum with respect to ℓ is an infinite sum, in general. Moreover, in view of the arbitrariness of α and $c_{\alpha,\nu,\ell}$ in (19) we obtain at least $2(n-1)$ functionally independent formal first integrals. Therefore we have

Theorem 2.1. *Assume* (5) *and* (6). *Then the Hamiltonian system with the Hamiltonian* $H=H_0+H_1$ *given by* (3)–(4) *has* $2(n-1)$ *functionally independent formal first integrals of the form* (8) *which are polynomials of* p_1 *and* p.

3. Summability of formal integrals.

We first show the $(2\sigma-1)$-summability of $v^{(\alpha)}$ for every α in (14). We define the set of singular directions

$$\begin{aligned} S_0:=\big\{&z\in\mathbb{C}:\exists\nu\geq 0,\ \exists k\geq 0,\ \exists\ell\geq 0,\ \exists\alpha\geq 0 \text{ such that}\\ &(2\sigma-1)z^{2\sigma-1}+\lambda\cdot(\ell-\alpha-k)=0,\ v^{(\alpha)}_{\nu,k,\ell}\neq 0,\ \ell-\alpha-k\geq 0\big\}\setminus 0. \end{aligned}\tag{20}$$

For a neighborhood Ω_0 of the origin and the convex cone Ω_1 with vertex at the origin, we define $\Sigma_0:=\Omega_0\cup\Omega_1$. Then we assume that there exists Σ_0 such that the closure $\overline{S_0}$ satisfies

$$\overline{S_0}\cap\Sigma_0=\emptyset. \tag{21}$$

Theorem 3.1. *Assume* (5), (6) *and* (21). *Let* v *be a formal first integral given in Theorem* 2.1 *which is a polynomial in* p *and* p_1. *Then, for each* $\alpha\geq 0$ *in* (8)*)* $v^{(\alpha)}$ *is* $(2\sigma-1)$-*summable in every direction of* Ω_1 *with respect to* q_1. *More precisely, for every* $\xi\in\Omega_1$

there exists a neighborhood V_0 of the origin $q = 0$ such that $v^{(\alpha)}$ is analytic in $q \in V_0$ and $(2\sigma - 1)$-summable with respect to q_1 in the direction ξ.

Before proving the theorem we give a corollary, in which we have the summability of v_α.

COROLLARY 3.2. *Suppose* (6). *Assume*

$$B_j = B_{j,0}(q_1, q), \quad 2 \le j \le n, \tag{22}$$

for some $B_{j,0}$ analytic at $q_1 = 0$ and polynomial in $q = (q_2, \dots, q_n)$. Let $v = \sum_{\alpha \ge 0} v^{(\alpha)} E^\alpha$ be the formal first integral as in Theorem 2.1 *which is a polynomial in p and p_1. Then the set of singular directions S_0 is finite, and for each α $v^{(\alpha)}$ is a polynomial in q and $(2\sigma - 1)$-summable with respect to q_1. More precisely, for every $\xi \notin S_0$ $v^{(\alpha)}$ is $(2\sigma - 1)$-summable with respect to q_1 in the direction ξ.*

In order to prove Theorem 3.1 we prepare a lemma. Let $\kappa > 0$, $r > 0$ and $\theta \in \mathbb{R}$, and $0 < \varepsilon < \pi$ be given. Let γ_κ denote the path from the origin along $\arg z = \theta + (\varepsilon + \pi)/(2\kappa)$ to some z_1 of modulus r, then along the circle $|z| = r$ to the ray $\arg z = \theta - (\varepsilon + \pi)/(2\kappa)$, and back to the origin along this ray. Let $\mathcal{B}_\kappa$ denote the Borel transform

$$(\mathcal{B}_\kappa f)(\zeta) = \frac{1}{2\pi i} \int_{\gamma_\kappa} t^\kappa f(t) \exp(\zeta^\kappa t^{-\kappa})\, dt^{-\kappa}. \tag{23}$$

Then, by simple computations we have

$$\mathcal{B}_\kappa \Big(t^{\kappa+1} \frac{d}{dt} f \Big)(\zeta) = \kappa \zeta^\kappa \mathcal{B}_\kappa(f)(\zeta) - \kappa \mathcal{B}_\kappa(t^\kappa f)(\zeta). \tag{24}$$

Let $c > 0$ and Ω be a domain in $\mathbb{C}$. Define $H_c(\Omega)$ as the Banach space of all f which is holomorphic and of exponential growth of order c in Ω with the norm

$$\|f\|_c := \sup_{z \in \Omega} \big| f(z) e^{-cz^\kappa} \big| < \infty. \tag{25}$$

Then we have

LEMMA 3.3. *Let $\lambda > 0$. Then there exists $K_0 > 0$ such that*

$$\|\mathcal{B}_\kappa(t^\lambda f)\|_c \le K_0 \|\mathcal{B}_\kappa(f)\|_c, \quad \mathcal{B}_\kappa(f) \in H_c(\Omega). \tag{26}$$

Moreover, K_0 can be taken arbitrarily small if we take $c > 0$ sufficiently large.

For the proof we refer the reader to [BY].

Proof of Theorem 3.1. We define $\Omega = \Sigma_0$. In view of the inductive definition of $v^{(\alpha)}_{\nu,k,\ell}$'s with respect to ℓ, the first non-vanishing term $v^{(\alpha)}_{\nu,k,\ell}$ is a polynomial of q_1. Hence it is $(2\sigma - 1)$-summable in q_1. Therefore it is sufficient to show, by induction, that if F in (16) is $(2\sigma - 1)$-summable, then $v^{(\alpha)}_{\nu,k,\ell}$ is $(2\sigma - 1)$-summable as well.

Set $\kappa = 2\sigma - 1$. In the following we omit the suffix (α) in $v^{(\alpha)}_{\nu,k,\ell}$ for the sake of simplicity. Suppose that there exists an integer N such that $\mathcal{B}_\kappa(v_{\nu,k,\mu}) \in H_c(\Omega)$ for all ν, k and μ, $|\mu| \le N$. We want to show $\mathcal{B}_\kappa(v_{\nu,k,\ell}) \in H_c(\Omega)$, $|\ell| = N + 1$. Let ζ be the dual variable of q_1 with respect to the Borel transform. Let $\chi_\lambda(D)$ be defined by

$$\chi_\lambda(D) \mathcal{B}_\kappa(f)(\zeta) := \mathcal{B}_\kappa(q_1^\lambda f)(\zeta), \quad \mathcal{B}_\kappa(f) \in H_c(\Omega).$$

By Lemma 3.3 $\chi_\lambda(D)$ is a linear continuous operator on $H_c(\Omega)$. Moreover, by taking $c > 0$ sufficiently large, we may assume that the norm can be made arbitrarily small.

We apply the $(2\sigma-1)$-Borel transform to both sides of (16) with respect to q_1. Then we have

$$\big((2\sigma-1)\zeta^{2\sigma-1} - (2\sigma(\nu+1)-1)\chi_{2\sigma-1}(D) + \lambda\cdot(\ell-k-\alpha)\big)\mathcal{B}_{2\sigma-1}(v^{(\alpha)}_{\nu,k,\ell}) = g(\zeta), \tag{27}$$

where $g(\zeta)$ is the partial Borel transform of F with respect to q_1.

First we note that the formal Borel transform $\tilde{\mathcal{B}}_{2\sigma-1}(v^{(\alpha)}_{\nu,k,\ell})$ satisfies a relation similar to (27). In view of the construction of formal series $v^{(\alpha)}_{\nu,k,\ell}$, $\tilde{\mathcal{B}}_{2\sigma-1}(v^{(\alpha)}_{\nu,k,0})(\zeta)$ is an entire function of ζ. In order to determine $\tilde{\mathcal{B}}_{2\sigma-1}(v^{(\alpha)}_{\nu,k,e_j})$ for $e_j=(0,\dots,1,\dots,0)$ $(j=1,2,\dots,n)$ we note that the right-hand side $g(\zeta)$ is an entire function of ζ because it contains only the formal Borel transform of $v^{(\alpha)}_{\nu,k,0}$. By inverting the operator $\big((2\sigma-1)\zeta^{2\sigma-1} - (2\sigma(\nu+1)-1)\chi_{2\sigma-1}(D) + \lambda\cdot(\ell-k-\alpha)\big)$ we see that $\tilde{\mathcal{B}}_{2\sigma-1}(v^{(\alpha)}_{\nu,k,e_j})$ is holomorphic in some neighborhood of the origin $\zeta=0$ because the right-hand side is analytic at the origin. By the inductive argument we see that $\tilde{\mathcal{B}}_{2\sigma-1}(v^{(\alpha)}_{\nu,k,\ell})$ is holomorphic at the origin $\zeta=0$.

We shall show that $g(\zeta)\in H_c(\Omega)$. Indeed, in view of the definition of R in (10) F is the sum of products of some $v_{\nu',k',\mu}$ and holomorphic functions of q_1. This implies that their Borel transforms are in $H_c(\Omega)$. Hence we have the assertion.

We also note that the Borel transform of the differentiation $q_1^{2\sigma}(\partial/\partial q_1)$ in R is equal to $(2\sigma-1)\zeta^{2\sigma-1}-(2\sigma-1)\chi_{2\sigma-1}(D)$. In order to show that $\mathcal{B}_\kappa(v_{\nu,k,\ell})\in H_c(\Omega)$ we may assume that $\ell-k-\alpha\neq 0$. Indeed, the number of terms satisfying $\ell-k-\alpha=0$ is finite in view of the finiteness of k, and, by definition, the corresponding $v_{\nu,k,\ell}$ is a polynomial of q_1.

Assume that there exists $K>0$ such that

$$\big\|\zeta^{2\sigma-1}\big((2\sigma-1)\zeta^{2\sigma-1} - (2\sigma(\nu+1)-1)\chi_{2\sigma-1}(D) + \lambda\cdot(\ell-k-\alpha)\big)^{-1}\big\| \le K \tag{28}$$

for $\ell\in\mathbb{Z}_+^{n-1}$ and $\zeta\in\Omega$. Then we obtain $\mathcal{B}_\kappa(v_{\nu,k,\ell})\in H_c(\Omega)$ by the recurrence relation whose norm of the right-hand side is bounded by constant times of that of $v_{\nu,k,\mu}$ for $|\mu|<|\ell|$. Hence we have proved the $(2\sigma-1)$-summability of every coefficient of our formal integral with respect to q_1 as desired.

As for the convergence with respect to ℓ, we obtain the inductive estimate of $v_{\nu,k,\ell}$ with respect to $|\ell|$. Indeed, $\mathcal{B}_\kappa(v_{\nu,k,\ell})$ is calculated from the recurrence relation from the previous ones by operating the bounded operator as the one in (28) to the right-hand side.

Hence it remains to show (28). Because the number of pairs of ν, k and α is finite we take arbitrary ν, k and α and we fix them. Let ζ_ℓ satisfy $(2\sigma-1)\zeta_\ell^{2\sigma-1}+\lambda\cdot(\ell-k-\alpha)=0$ and let ω_j $(j=1,2,\dots,2\sigma-1)$ be the $(2\sigma-1)$-th root of unity. Then we have

$$\begin{aligned}(2\sigma-1)\zeta^{2\sigma-1} &- (2\sigma(\nu+1)-1)\chi_{2\sigma-1}(D) + \lambda\cdot(\ell-k-\alpha)\\ &= (2\sigma-1)(\zeta^{2\sigma-1}-\zeta_\ell^{2\sigma-1}) - (2\sigma(\nu+1)-1)\chi_{2\sigma-1}(D).\end{aligned}$$

We have $\zeta^{2\sigma-1}-\zeta_\ell^{2\sigma-1} = \prod_{j=1}^{2\sigma-1}(\zeta-\zeta_\ell\omega_j)$. By (21) there exists $c_1>0$ such that $|\zeta-\zeta_\ell\omega_j|\ge c_1|\zeta|$ for all $\zeta\in\Omega$ and $j=1,2,\dots,2\sigma-1$. It follows that there exists $c_2>0$ such that $\zeta^{2\sigma-1}-\zeta_\ell^{2\sigma-1}\ge c_2(|\zeta|^{2\sigma-1}+1)$. Recalling that the norm of $(2\sigma(\nu+1)-1)\chi_{2\sigma-1}(D)$ can be made arbitrarily small by the preceding lemma we obtain (28). ∎

4. Normalizing transformation. In the next theorem we study the relation between our formal solution in the preceding theorem and the generating function of a normalizing symplectic transformation. Assume $A \subset \mathbb{Z}_+^{n-1}$. Let $V_m^{(\alpha)}$ $(m = 2, \ldots, n)$ be the first integrals of χ_H constructed as in Theorem 3.1. Namely, the coefficients of q^ℓ for $\ell \not\geq e_m + \alpha$ vanish, while for $\ell = e_m + \alpha$ they are equal to $p_m q_m q^\alpha$. We inductively construct the coefficients for $\ell \geq e_m + \alpha$ as in Theorem 3.1. We say that $\lambda_2, \ldots, \lambda_n$ *satisfy the Poincaré condition* if the convex hull of $\lambda_2, \ldots, \lambda_n$ in $\mathbb{C}$ does not contain the origin. Then we have

THEOREM 4.1. *Assume* (6). *Suppose that*

$$B_j(q_1, t, q) = \tilde{B}_j(t, q), \quad j = 2, \ldots, n, \tag{29}$$

where $\tilde{B}_j$ is a polynomial of t with coefficients analytic at $q = 0$. Suppose that the Poincaré condition is satisfied. Let $2 \leq m \leq n$ be an integer. Then $V_m^{(\alpha)}$ $(\alpha \in A)$ are analytic at the origin and are functionally independent.

Expand $\sum_j q_j^2 \tilde{B}_j = \sum_\mu c_\mu q^\mu$, and let W be the analytic function whose coefficient of q^ℓ is given by $c_\ell / \lambda \cdot \ell$ if $|\ell| \geq 2$, and 0 if otherwise. Then W satisfies

$$q_m \frac{\partial}{\partial q_m} W = q_m p_m - V_m^{(0)}. \tag{30}$$

If we define $\tilde{W}$ by $\tilde{W} := \sum_{j=2}^n q_j y_j - W(q)$, then the (partial) symplectic transformation $(q, p) \mapsto (y, -x)$ given by

$$q_1 = x_1, \ p_1 = y_1, \ x_j = \tilde{W}_{y_j} = q_j, \ p_j = \tilde{W}_{q_j} = y_j - W_{q_j} \ (j = 2, \ldots, n) \tag{31}$$

transforms χ_H to $\chi_{\tilde{H}_0}$, where $\tilde{H}_0 := x_1^{2\sigma} y_1 + \sum_{j=2}^n \lambda_j x_j y_j$.

REMARK. By Theorem 4.1 χ_H is C^ω-Liouville integrable and the transformation (31) is the (resonant) Birkhoff transformation. Indeed, W gives the generating function of the partial symplectic transformation (cf. [I]).

Proof of Theorem 4.1. Let m $(2 \leq m \leq n)$ be an integer and let V_m be the first integral as in Theorem 4.1. Then the functional independentness of V_m $(m = 2, \ldots, n)$ is clear in view of the above constructions.

In order to have the representation of V_m, set $V_m = E^\alpha \sum_\ell v_\ell^{(\alpha)}(q_1, q_1^{2\sigma} p_1, p) q^\ell$. We will show that $v_\ell^{(\alpha)}$ is analytic at $q_1 = 0$ for all ℓ. For $|\ell| \leq 1$ the assertion is trivial from the choice of arbitrary functions. We also note that $v_{e_m}^{(\alpha)} = p_m$. In order to determine $v_\ell^{(\alpha)}$ for $|\ell| \geq 2$, we substitute the expansion into $\chi_H v = 0$ and compare the coefficients of $(p_1 q_1^{2\sigma})^\nu p^k q^\ell$. Then we have the recurrence relation similar to (16)

$$\big(q_1^{2\sigma} \partial_{q_1} - \lambda \cdot (\ell - \alpha)\big) v_\ell^{(\alpha)} = F_\ell(v_\gamma^{(\alpha)}, \gamma < \ell), \tag{32}$$

where $\ell - \alpha \neq 0$ and we regard $t := q_1^{2\sigma} p_1$ as an independent variable. Indeed, by (29) the term $-2\sigma q_1^{2\sigma-1} \nu$ in (16) vanishes because p_1 appears in V_m in the form $q_1^{2\sigma} p_1$. In view of (10) we see that the term $\partial_{p_1} B_j \frac{\partial}{\partial q_1} - \partial_{q_1} B_j \frac{\partial}{\partial p_1}$ vanishes because, by induction, the inhomogeneous term depends on $q_1^{2\sigma} p_1$. Hence (10) decreases the power of p. It follows that $v_\ell^{(\alpha)}$ $(|\ell| \geq 2)$ is a function of q, $t = q_1^{2\sigma} p_1$ and q_1.

In view of (10) and (12) $F_\ell(v_\gamma^{(\alpha)}, \gamma < \ell)$ is equal to the coefficient of q^ℓ in

$$\sum_j \nabla_q (q_j^2 \tilde{B}_j) \cdot \nabla_p \Big(\sum_\gamma v_\gamma^{(\alpha)} q^\gamma \Big) - \langle \lambda, \alpha \rangle \sum_{j=2}^n q_j^2 B_{j,1} \Big(\sum_\gamma v_\gamma^{(\alpha)} q^\gamma \Big). \tag{33}$$

Recalling that $\partial_{p_1} B_j = B_{j,1} q_1^{2\sigma}$, $B_j = \tilde{B}_j$ we obtain $\sum q_j^2 B_{j,1} = \partial_t \sum q_j^2 \tilde{B}_j(t,q)$.

Let $|\ell| = 2$. Because $\nabla_p v_\gamma^{(\alpha)}$ does not vanish only for $\gamma = e_m$ and $v_{e_m}^{(\alpha)} = p_m$ it follows that the first term of (33) is equal to $\sum_j q_m \frac{\partial}{\partial q_m}(q_j^2 \tilde{B}_j)$. If we expand $\sum_j q_j^2 \tilde{B}_j = \sum_\mu c_\mu(t) q^\mu$, then we have

$$F_\ell(v_\gamma^{(\alpha)}) = \ell_m c_\ell - \langle \lambda, \alpha \rangle \sum_{\gamma+\mu=\ell, |\mu|\geq 2} (\partial_t c_\mu) v_\gamma^{(\alpha)}. \tag{34}$$

By the inductive assumption on $v_\ell^{(\alpha)}$, F_ℓ in (32) is independent of q_1. Hence the unique formal solution is given by $v_\ell^{(\alpha)} = -F_\ell/\lambda \cdot (\ell - \alpha)$, which is independent of q_1. Therefore, by induction on $|\ell|$, we can determine $v_\ell^{(\alpha)}$ from (32) being independent of q_1. Hence we obtain a formal integral. As for the convergence of the formal series, the Poincaré condition implies the convergence of the formal solution.

Let $\alpha = 0$. Then we have $F_\ell(v_\gamma^{(0)}) = \ell_m c_\ell$, which implies $v_\ell^{(0)} = -\ell_m c_\ell/\lambda \cdot \ell$. Therefore we have

$$V_m^{(0)} = p_m q_m - \sum_{|\ell|\geq 2} \frac{\ell_m c_\ell}{\lambda \cdot \ell} q^\ell \tag{35}$$

and W satisfies (30). Moreover, the Hamiltonian $\tilde{H}_0$ is transformed to

$$\begin{aligned} q_1^{2\sigma} p_1 + \sum \lambda_j p_j q_j + \sum \lambda_m q_m W_{q_m} &= H_0 + \sum_m \lambda_m (q_m p_m - V_m^{(0)}) \\ &= H_0 + \sum_m \lambda_m \Big(\sum_{|\ell|\geq 2} \frac{\ell_m c_\ell}{\lambda \cdot \ell} q^\ell \Big) = H_0 + \sum q_j^2 \tilde{B}_j = H. \end{aligned}$$

Hence we see that (31) transforms χ_H to $\chi_{\tilde{H}_0}$. This ends the proof. ∎

5. C^ω-non-integrability and C^∞-integrability. As we stated in the introduction, our Hamiltonian system is not C^ω-integrable in general. Although this fact is not used in the proofs of the preceding theorems, we will briefly state the C^ω-non-integrability for the readers' convenience.

THEOREM 5.1. *Assume that* (6) *and the following condition* (M) *are satisfied.*

(M) *For $k = 2, 3, \ldots, n$ the equation*

$$q_1^{2\sigma} \frac{dv}{dq_1} + 2\lambda_k v = B_k(q_1, 0, 0) \tag{36}$$

has no analytic solution v at the origin.

Then the Hamiltonian system (1) *with the Hamiltonian $H = H_0 + H_1$ given by* (3) *and* (4) *is not C^ω-Liouville integrable.*

Condition (M) corresponds to the non-Abelian property of the fundamental group introduced in [BT]. We can also prove that (M) holds if and only if the monodromy of an analytic continuation of every solution of (36) along a path encircling the origin does not vanish (cf. Lemma 6 of [Y]).

Let $v = \sum_{\alpha \geq 0} v^{(\alpha)} E^\alpha$ be the first integral given by (8). By Theorem 3.1 every $v^{(\alpha)}$ is $(2\sigma - 1)$-summable in every direction of $\Omega_1 \equiv \Omega_1(v^{(\alpha)})$. Hence we write the summed one with the same letter for the sake of simplicity. We define

$$\Sigma_v = \Big\{ z \in \mathbb{C} : |\arg z - \arg \xi| < \frac{\pi}{2(2\sigma - 1)}\,,\ \xi \in \Omega_1 \Big\}. \tag{37}$$

Then we have

Theorem 5.2. *Assume* (5), (6) *and* (21). *Then*

(i) *Let* $\alpha \geq 0$ *and suppose* $\Omega_1(v^{(\alpha)}) \neq \emptyset$. *Then there exists an* $\varepsilon_0 > 0$ *and a sector* $S_1 \subset \Sigma_v$ *such that the summed* $v = v^{(\alpha)}$ *in Theorem* 3.1 *is holomorphic and is the first integral of* χ_H *in the domain*

$$q_1 \in \Sigma_v,\ |q_1| < \varepsilon_0,\ p_1 \in \mathbb{C},\ p_j \in \mathbb{C},\ |q_j| < \varepsilon_0,\ j = 2, \ldots, n. \tag{38}$$

Moreover, it is C^∞ *at* $q_1 = 0$ *when* $q_1 \in S_1$, $q_1 \to 0$.

(ii) *Assume either the Poincaré condition is satisfied or* $v^{(e_j)}$ *and* $v^{(2e_j)}$ *exist for which* S_0 *is a finite set. Set* $v = v^{(e_j)}$ *or* $v = v^{(2e_j)}$ *and let* Σ_v *and* $S_1 \subset \Sigma_v$ *be given by* (i) *and choose* $\theta \in S_1$. *Then we have* $\Omega_1(v) \neq \emptyset$, *and* v *is extended as a* C^∞ *first integral with respect to* q_1 *on* $R_\theta \cup -R_\theta \cup \{0\}$ *being analytic in* $q \in \mathbb{R}^{n-1}$ *at* $q = 0$. *Moreover, there exists a neighborhood of the origin* U *in* $\mathbb{R}$ *such that* χ_H *is* C^∞*-integrable when* $q_1 \in (R_\theta \cup \{0\}) \cap U$, $p_1, p_j, q_j \in \mathbb{R}$, $|q_j| < \varepsilon_0$ $(j \geq 2)$.

The proofs of these theorems will be published elsewhere.

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