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ASYMPTOTIC ANALYSIS AND SPECIAL VALUES OF GENERALISED MULTIPLE ZETA FUNCTIONS

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Abstract. This is an expository article, based on the talk with the same title, given at the 2011 FASDE II Conference in Będlewo, Poland. In the introduction we define Multiple Zeta Values and certain historical remarks are given. Then we present several results on Multiple Zeta Values and, in particular, we introduce certain meromorphic differential equations associated to their generating function. Finally, we make some conclusive remarks on generalisations of Multiple Zeta Values as well as the meromorphic differential equations.

1. Introduction

1.1. Riemann zeta function. Recall that Riemann zeta function ζ is defined as

$$\zeta(s) := \sum_{n>0} n^{-s}, \tag{1}$$

for $\operatorname{Re} s > 1$. It can be continued analytically to $\mathbb{C} \setminus \{1\}$ and $(1-s)\zeta(s)$ is an entire function. A lot of attention has been brought to evaluation of $\zeta(n)$ for positive integers $n > 2$. In 1735 Euler found his famous formula

$$\zeta(2n) = -\frac{(2\pi i)^{2n} B_{2n}}{2(2n)!}\,, \tag{2}$$

where B_m are Bernoulli numbers defined by the generating function

$$\frac{x}{1-e^{-x}} = \sum_{m\geq 0} B_m \frac{x^m}{m!}\,. \tag{3}$$

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In particular, $\zeta(2n) = q \times \pi^{2n}$, where $n \in \mathbb{N}$ and $q \in \mathbb{Q}$. However, such a formula for $\zeta(2n+1)$ is not known and we know a lot less about these numbers. Euler's method was based on the fact that

$$\frac{1}{\Gamma(1+x)} = e^{\gamma x} \prod_{n>0} \left(1 + \frac{x}{n}\right) e^{-x/n}, \tag{4}$$

and

$$\frac{1}{\Gamma(1+x)\Gamma(1-x)} = \prod_{n>0} \left(1 - \frac{x^2}{n^2}\right) = 1 - \sum_{n>0} \frac{x^2}{n^2} + O(x^4) = 1 - \zeta(2)x^2 + O(x^4). \tag{5}$$

On the other hand we have

$$\frac{1}{\Gamma(1+x)\Gamma(1-x)} = \frac{\sin \pi x}{\pi x} = 1 - \frac{\pi^2}{6}\, x^2 + O(x^4), \tag{6}$$

which gives the result.

Formulas for $\zeta(2n)$ for $n > 1$ can be obtained essentially in the same way.[1]

Unfortunately, a similar explicit formula for $\Gamma(1+x)\Gamma(1+\mu x)\Gamma(1+\overline{\mu} x)$, where $\mu = 2^{-1}(1+\sqrt{-3})$, which would allow for immediate evaluation of $\zeta(3)$, is not known. The same problem appears in all other positive odd numbers. In fact, we have the following

CONJECTURE 1.1. *The numbers* $\zeta(2), \zeta(3), \zeta(5), \zeta(7), \zeta(11), \ldots,$ *are algebraically independent over* $\mathbb{Q}$.

1.2. Multiple zeta function. *Multiple zeta values* (in short MZV) are defined as

$$\zeta(s_1, s_2, \ldots, s_p) := \sum_{n_1 > n_2 > \ldots > n_p} n_1^{-s_1} n_2^{-s_2} \cdots n_p^{-s_p}, \tag{7}$$

for $(s_1, s_2, \ldots, s_p) \in \mathbb{N}^p$ and whenever the series (7) converges. They appeared for the first time in Euler's [5]. He found the following formula relating multiple zeta values to 'single' ones:

$$\sum_{n>0} \frac{H_n}{(n+1)^2} = \zeta(2,1) = \zeta(3) = \sum_{n>0} \frac{1}{n^3}\,, \tag{8}$$

where H_m is the m-th harmonic number. The formula (8) is a particular example of the remarkable identity (here p is fixed integer and $s_1 + s_2 + \ldots + s_p = s$)

$$\sum_{s_1>1, s_2>0, \ldots, s_p>0} \zeta(s_1, s_2, \ldots, s_p) = \zeta(s), \tag{9}$$

which follows from Ohno's relations (see [9]).

MZV's are values at integral points of a more general multiple zeta functions, where parameters $s_1, s_2, \ldots, s_p$ in (7) are complex. In [15], Zhao Jianqiang found the analytic continuation of (7) to $\mathbb{C}^p \setminus S$ and proved that $\zeta(s_1, s_2, \ldots, s_p)$ has a simple pole along S. Here S is a certain linear subset of $\mathbb{C}^p$.

In the standard terminology, the number p is called the depth and the number $|s| := s_1 + \ldots + s_p$ is called the weight of multiple zeta value/function (7).

[1] The easiest way of calculating $\zeta(2n)$ is, perhaps, with use of the function

$$\frac{1 - \pi x \cot \pi x}{2} = \sum_{n>0} \zeta(2n) x^{2n}.$$

This formula can be obtained from (6).

MZV's satisfy a lot of relations. Some of them are quite simple, as

$$\zeta(r)\zeta(s) = \zeta(r,s) + \zeta(s,r) + \zeta(r+s), \tag{10}$$

coming from

$$\sum_{n>0} m^{-r} \sum_{n>0} n^{-s} = \Big(\sum_{m>n} + \sum_{n>m} + \sum_{m=n}\Big) m^{-r} n^{-s} \tag{11}$$

and some of them are not so obvious. Because of these relations, we have the following (see [11])

Conjecture 1.2 (Drinfeld, Kontsevich, Zagier). *Denote by Z_k the $\mathbb{Q}$-vector space spanned by MZV's of a given weight k. Then $Z = \bigoplus Z_k$. If $d_k := \dim Z_k$, then for $k > 2$ the formula*

$$d_k = d_{k-2} + d_{k-3} \tag{12}$$

holds. Equivalently,

$$\sum_{k\geq 0} d_k x^k = \frac{1}{1-t^2-t^3}\,. \tag{13}$$

So far only the estimates $\dim_{\mathbb{Q}} Z_k \leq d_k$ have been shown by Deligne, Goncharov and, independently, by Terasoma (see [4] and [10]).

2. Multiple polylogarithms, generating functions and differential equations. Consider the (so called, Drinfeld–Kontsevich) integral

$$\operatorname{Li}(s_1, s_2, \ldots, s_p, t) := \int_0^t \omega_0(t_1) \int_0^{t_1} \omega_0(t_2) \cdots \int_0^{t_{s_1-1}} \omega_1(t_{s_1}) \int_0^{t_{s_1}} \omega_0(t_{s_1+1}) \cdots \int_0^{t_{|s|-1}} \omega_1(t_{|s|}), \tag{14}$$

where $\omega_0 = dt/t$, $\omega_1 = dt/(1-t)$. We have

$$\operatorname{Li}(s_1, s_2, \ldots, s_p, 1) = \zeta(s_1, s_2, \ldots, s_p). \tag{15}$$

Note that if we define

$$P := t\partial t, \quad Q := (1-t)\partial t, \quad \text{and} \quad T := QP^{s_1-1}QP^{s_2-1}\cdots QP^{s_p-1}, \tag{16}$$

then

$$T \operatorname{Li}(s_1, s_2, \ldots, s_p, t) = 1 \tag{17}$$

and, more generally,

$$T^n \operatorname{Li}(\{s_1, s_2, \ldots, s_p\}^n, t) = 1, \tag{18}$$

where we employ the obvious analogue of MZV notation.[2] The operator T may be seen as a left inverse of the integral operator associated to (14). Thus, we put

$$\begin{aligned} F(s_1, s_2, \ldots, s_p, \lambda, t) &:= 1 - \lambda^{|s|} \cdot \operatorname{Li}(s_1, s_2, \ldots, s_p, t) + \lambda^{2|s|} \cdot \operatorname{Li}(\{s_1, s_2, \ldots, s_p\}^2, t) - \ldots \\ &= \big(I - \lambda^{|s|}T^{-1} + \lambda^{2|s|}T^{-2} - \ldots\big).1. \end{aligned} \tag{19}$$

From the formula for geometric series, from (19) one gets

$$F(s_1, s_2, \ldots, s_p, \lambda, t) = (I + \lambda^{|s|}T^{-1})^{-1}.1 = (T + \lambda^{|s|})^{-1}T.1 \tag{20}$$

[2] By definition, we have

$$\operatorname{Li}(\{s_1, s_2, \ldots, s_p\}^n, t) = \operatorname{Li}(s_1, s_2, \ldots, s_p, s_1, s_2, \ldots, s_p, \ldots, s_1, s_2, \ldots, s_p, t),$$

where the sequence $s_1, s_2, \ldots, s_p$ is repeated n times.

and applying to both sides $T+\lambda^{|s|}$, we obtain

$$(T+\lambda^{|s|})F(s_1,s_2,\dots,s_p;\lambda,t)=0. \tag{21}$$

On the other hand, from (15), we have

$$F(s_1,s_2,\dots,s_p;1,\lambda)=\sum_{n\geq 0}\lambda^{|s|n}\zeta(\{s_1,s_2,\dots,s_p\}^n), \tag{22}$$

i.e., we constructed generating function of the sequence $\zeta(\{s_1,s_2,\dots,s_p\}^n)$.

The formulas

$$\zeta(2)=\int_{0<t_1<t_2<1}\frac{dt_1}{1-t_1}\,\frac{dt_2}{t_2}\,, \quad \text{and} \quad \mathrm{Li}_2(t)=\int_{0<t_1<t_2<t}\frac{dt_1}{1-t_1}\,\frac{dt_2}{t_2}\,, \tag{23}$$

for the classical dilogarithm were already known to Leibniz, who introduced it in his letter to Johann Bernoulli. Leibniz also introduced general classical polylogarithms

$$\mathrm{Li}_s(t):=\sum_{n>0}\frac{t^n}{n^s}\,, \tag{24}$$

where $|t|<1$.

3. Asymptotic analysis of meromorphic differential equations associated to generating functions of MZV's. In papers [12], [13] and [14], together with Henryk Żołądek, we studied asymptotics of certain linear meromorphic ODE associated to generating functions of multiple zeta values. In papers [12] and [14] we studied particular examples of equations associated to $\zeta(2)$ and $\zeta(3)$ obtaining two new proofs of the identity $\zeta(2)=\pi^2/6$. Paper [13] contains new general results on asymptotics of meromorphic ODE's in complex domain. In particular, we proved the generalisation of the WKB method to (almost) arbitrary equations.

In mathematical physics, the WKB approximation or WKB method[3] is a method for finding approximate solutions to linear partial differential equations with spatially varying coefficients. It is typically used for a semiclassical calculation in quantum mechanics in which the wavefunction is recast as an exponential function, semiclassically expanded, and then either the amplitude or the phase is taken to be slowly changing.

Take, for example, the equation

$$(T+\lambda^2)x=0, \qquad \text{where} \qquad T:=(1-t)\partial_t(t\partial_t), \tag{25}$$

It is a particular case of MZV equation, associated to the sequence $\zeta(\{2\}^n)$. One of regular

[3]This method is named after physicists Wentzel, Kramers, and Brillouin, who all developed it in 1926. In 1923, mathematician Harold Jeffreys had developed a general method of approximating solutions to linear, second-order differential equations, which includes the Schrödinger equation. But even though the Schrödinger equation was developed two years later, Wentzel, Kramers, and Brillouin were apparently unaware of this earlier work, so Jeffreys is often neglected credit. Early texts in quantum mechanics contain any number of combinations of their initials, including WBK, BWK, WKBJ, JWKB and BWKJ. Earlier references to the method are: Carlini in 1817, Liouville in 1837, Green in 1837, Rayleigh in 1912 and Gans in 1915. Liouville and Green may be called the founders of the method, in 1837, and it is also commonly referred to as the Liouville–Green or LG method. For more details one can check the WKB page on Wikipedia.

solutions of (25) is given by the hypergeometric series

$$ {}_2F_1\left(\begin{matrix}\lambda, -\lambda \\ 1\end{matrix}\,\middle|\, t\right) := \sum_{n=0}^{\infty} \frac{(\lambda)_n(-\lambda)_n}{n!} \cdot \frac{t^n}{n!}\,, $$

Another independent solution is given by

$$ G(t) + \log t \cdot {}_2F_1\left(\begin{matrix}\lambda, -\lambda \\ 1\end{matrix}\,\middle|\, t\right), \tag{26} $$

where G is holomorphic in zero.

Recall that the classical Euler–Gauss hypergeometric series is defined by

$$ {}_2F_1\left(\begin{matrix}u, v \\ w\end{matrix}\,\middle|\, t\right) := \sum_{n=0}^{\infty} \frac{(u)_n(v)_n}{(w)_n} \frac{t^n}{n!}\,, \tag{27} $$

where $(x)_n := x(x+1)\cdots(x+n-1)$ is the Pochhammer symbol. The generalised hypergeometric series is defined analogously:

$$ {}_pF_q\left(\begin{matrix}u_1, \ldots, u_p \\ w_1, \ldots, w_q\end{matrix}\,\middle|\, t\right) := \sum_{n=0}^{\infty} \frac{(u_1)_n\cdots(u_p)_n}{(w_1)_n\cdots(w_q)_n} \frac{t^n}{n!}, \tag{28} $$

whenever $p \le q+1$.

Thus, by the Gauss formula[4], we get

$$ f(\lambda) ={}_2 F_1\left(\begin{matrix}\lambda, -\lambda \\ 1\end{matrix}\,\middle|\, 1\right) := \frac{1}{\Gamma(1+\lambda)\Gamma(1+\lambda)} = \frac{\sin \pi\lambda}{\pi\lambda}\,. \tag{29} $$

From this we can calculate $\zeta(2)$.

There is another way of obtaining this result, with use of the WKB approximation. If the parameter λ in (25) is large (i.e. $\lambda \sim \infty$), then we make the assumption that the solutions have the form

$$ e^{\lambda S(t)}\lambda^{\alpha} \sum_{n<0} \phi_n(t)\lambda^n. \tag{30} $$

Substituting the above series into the equation (25) and comparing the coefficients with powers of λ, we obtain differential equation for ‘action’ S, where p is the depth of the associated multiple zeta value:

$$ t^{|s|-p}(1-t)^p(S')^{|s|} + 1 = 0. \tag{31} $$

It is, so called, ‘Hamilton-Jacobi’ equation. By solving (31), we have

$$ dS = \frac{dt}{t^{1-p/|s|}(1-t)^{p/|s|}}\,. \tag{32} $$

After determining S we can also successively find coefficients ϕ_n which satisfy series of ‘transport equations’.

In this place we find the following difficulty: WKB approximation does not tell us how to determine the exponent α. Moreover, $t=1$ is singular for the corresponding transport equation,

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[4]The Gauss formula for $t=1$,

$$ {}_2F_1\left(\begin{matrix}u, v \\ w\end{matrix}\,\middle|\, 1\right) = \frac{\Gamma(w)\Gamma(w-u-v)}{\Gamma(w-u)\Gamma(w-v)}\,, $$

is proved with use of integral representation of Euler’s integral representation for ${}_2F_1$.

so the asymptotic expansion does not represent the required function[5] in its neighbourhood. To obtain the correct answer one has to examine asymptotics of solutions of (25) for $s = 1 - t$. In this coordinate chart, operator (25) takes the form

$$T(s) = s\partial_s\,(1 - s)\partial_s + \lambda^2. \tag{33}$$

Solutions of $T(s)\phi = 0$ are of the form

$$\xi_1(\lambda, s) = \lambda^2 s + a_2 s^2 + \dots$$
$$\xi_2(\lambda, s) = \xi_1(\lambda, s)\log(\lambda^2 s) + \eta(s),$$

where η is holomorphic in zero and $\eta(0) \neq 0$. Function $\phi_2(\lambda, t)$ can be represented in the neighbourhood of zero as

$$\phi_2(\lambda, t) = c_1(\lambda)\xi_1(\lambda, 1 - t) + c_2(\lambda)\xi_2(\lambda, 1 - t). \tag{34}$$

Thus $\phi_2(\lambda, 1) = c_2(\lambda)\eta(0)$, so calculation of c_2 allows us to obtain the required result.

Calculating the exponent α, in the series for η and coefficient c_2, requires use of stationary phase approximation.

3.1. Stationary phase approximation. To calculate coefficients in asymptotic expansion, we will use stationary phase approximation, i.e. analysis of asymptotics of the oscillating integral

$$I(\lambda) := \int a(x)e^{i\lambda\phi(x)}\,d\mu(x). \tag{35}$$

If ϕ, so called *phase*, has finite number of isolated and nondegenerate critical points (for simplicity we assume that there is only one such point x), then for $\lambda \sim \infty$, we have (k denotes the dimension of a given manifold)

$$I(\lambda) = \Big(\frac{2\pi}{\lambda}\Big)^{n/2} \sum_{x\in\mathrm{crit}\phi} e^{i\lambda\phi(x)}a(x)\,\frac{e^{i\pi\,\mathrm{sgn}\,\phi''(x)/4}}{|\det\phi''(x)|^{1/2}} + O(\lambda^{-n/2-1}), \tag{36}$$

where critϕ is the set of critical points of ϕ, ϕ'' is the Hessian of the second derivative of the phase and $\mathrm{sgn}\,\phi''(x)$ is the signature of the quadratic form associated to $\phi''(x)$. Analysis of solutions of the equation (25) requires appropriate integral formulas for solutions. For $\lambda \sim \infty$, for ϕ such that $(1 - t)\partial_t\,t\partial_t\phi = -\lambda^2\phi$, we have

$$\begin{aligned}\phi(\lambda, t) &= 1 - \frac{\lambda^2}{1!}\,\frac{t}{1!} + \frac{\lambda^2(\lambda^2 - 1)^2}{2!}\,\frac{t^2}{2!} + \dots\\ &\sim 1 - \frac{\lambda^2 t}{1!}\,\frac{t}{1!} + \frac{\lambda^4}{2!}\,\frac{t^2}{2!} + \dots = {}_0F_1(1; \lambda^2 t) = J_0(2\lambda\sqrt{t}),\end{aligned} \tag{37}$$

where

$$J_\alpha(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\,\frac{1}{\Gamma(\alpha + k + 1)}\Big(\frac{x}{2}\Big)^{2k+\alpha}, \tag{38}$$

is the Bessel function of the first kind, i.e. solution of the Bessel equation

$$t^2\ddot{\phi} + t\dot{\phi} + (t^2 - \alpha^2)\phi = 0. \tag{39}$$

[5]Unfortunately there are additional problems with determination of coefficients ϕ_n. In interesting point $t = 1$ they are singular or zero, thus it is impossible to simply calculate ϕ_n and take $t = 1$. This difficulty has been so far overcome in case of f_2. More details can be found in [12].

From the well known formulas for the oscillating integral representations of Bessel functions, one gets

$$\begin{aligned} J(0; 2x\sqrt{t}) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{2ix\sqrt{t}\sin v}\, dv \\ &= \frac{1}{2\sqrt{\pi x}\, t^{1/4}}\big(e^{-i\pi/4}e^{2ix\sqrt{t}} + e^{i\pi/4}e^{-2ix\sqrt{t}}\big) + O(x^{-3/2}t^{-3/4}). \end{aligned} \tag{40}$$

The last equality is obtained by the study of the integral

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{2ix\sqrt{t}\sin v}\, dv, \tag{41}$$

in the neighbourhood of the critical points of $\sin v$ in $(-\pi, \pi)$. We have $v_1 = -\pi/2$ and $v_2 = \pi/2$.

Analogously we get asymptotic expansions of ξ_1 and ξ_2:

$$\xi_1(x, s) \sim x\sqrt{s}\, J_1(2x\sqrt{s}) \tag{42}$$

$$\xi_2(x, s) \sim -x\sqrt{s}\lim_{\alpha\to 0}\frac{1}{\alpha}\big(J_{1-\alpha}(2x\sqrt{s}) + J_{\alpha-1}(2x\sqrt{s})\big) =: \Xi_2(z), \tag{43}$$

where $z = x^2 s$. Now one can get the formula for η around zero and, in particular, we get $\eta(0) = -1$. So $\phi_2(x, 1) = -c_2(x)$.

For more details see [12].

3.2. Back to the WKB series. For the Bessel function, and thus, also for $F(\lambda, t)$, the WKB series is of the form

$$\exp\Big\{\lambda B\Big(t, \frac{1}{2}, \frac{1}{2}\Big)\Big\}\, \lambda^{-1/2}\psi(t). \tag{44}$$

Putting it to (25), we get the equation

$$2(tB)\dot{\psi} + (t\dot{B})\psi = 0. \tag{45}$$

The solution to (45) is given by $c(t\dot{B})^{1/2} = c(1-t)^{1/4}t^{-1/4}$. The constant c can be calculated from the asymptotic form

$$c = \frac{e^{-i\pi/4}}{2\sqrt{\pi}}\,. \tag{46}$$

Around $0 \sim s = 1 - t$, we have

$$\begin{aligned} B\Big(1-s, \frac{1}{2}, \frac{1}{2}\Big) &= B\Big(\frac{1}{2}, \frac{1}{2}\Big) - \int_0^s \frac{du}{\sqrt{u(1-u)}} \\ &= \pi - \int_0^s u^{-1/2}(1 + \ldots)\, du \sim \pi - 2\sqrt{s}\,, \end{aligned} \tag{47}$$

and thus

$$\begin{aligned} F(\lambda, t) = \tilde{F}(\lambda, 1-s) &\sim \frac{s^{1/4}}{2\sqrt{\pi\lambda}}\cos(\pi\lambda - 2\lambda\sqrt{s} - \pi/4) \\ &= -\frac{s^{1/4}}{2\sqrt{\pi\lambda}}\sin(\pi\lambda - 2\lambda\sqrt{s} + \pi/4) \\ &= \frac{s^{1/4}}{2\sqrt{\pi\lambda}}\sin(\pi\lambda)\cos(2\lambda\sqrt{s} - \pi/4) - \frac{s^{1/4}}{2\sqrt{\pi\lambda}}\cos(\pi\lambda)\sin(2\lambda\sqrt{s} - \pi/4) \\ &\sim \frac{\cos\lambda}{\pi\lambda}\,\xi_1(s) - \frac{\sin\lambda}{\pi\lambda}\,\xi_2(s). \end{aligned} \tag{48}$$

In this way we obtained the result $\zeta(2) = \pi^2/6$ using a new method. For more details see [12] and [13].

4. General iterated integrals. A good motivation for considering iterated integrals comes from Picard's method for solving a system of linear ODE's. Such integrals play a crucial role in quantum field theory (see e.g. [2] or [3]).

4.1. Picard's method. Consider the system

$$\dot{x}(t) = A(t)x(t), \tag{49}$$

where $x : \mathbb{R} \supset U \to \mathbb{C}^n$, $A \in \mathrm{end}(\mathbb{C}^n) \otimes \mathcal{O}$, with initial condition $x(t_0) = x_0$. The problem (49) is equivalent to the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} A(t)x(t)\,dt, \tag{50}$$

The Picard method of solving (49)—and hence also (50)—depends on successive approximation. Let $x_0(t) \equiv x_0$ and then define

$$x_{n+1}(t) := x_0 + \int_{t_0}^{t} A(t_{n+1})x_n(t_{n+1})\,dt_{n+1}, \tag{51}$$

for $n \geq 0$. We have, for example,

$$x_1(t) = x_0 + \int_{t_0}^{t} A(t_1)x_0\,dt_1 \tag{52}$$

$$x_2(t) = x_0 + \int_{t_0}^{t} A(t_2)x_1(t_2)\,dt_2 = x_0 + \int_{t_0}^{t} A(t_2)x_0\,dt_2 + \int_{t_0<t_1<t_2<t} A(t_2)A(t_1)x_0\,dt_1\,dt_2. \tag{53}$$

Now, we can, at least formally, write $x_n(t) \to x(t)$ when $n \to \infty$. We have $x(t) = B(t)x_0$, where

$$B(t) := I + \sum_{n\geq 0} \int_{t_0<t_1<\ldots<t_n<t} A(t_n)A(t_{n-1})\cdots A(t_1)\,dt_1\cdots\,dt_n \tag{54}$$

is the sum of iterated integrals. The sum (54) converges on compact sets C, because

$$\Big|\int_{t_0<t_1<\ldots<t_n<t} A(t_n)A(t_{n-1})\cdots A(t_1)\,dt_1\cdots\,dt_n\Big| \leq \sup_{u\in C} \|A(u)\|^n\,\frac{t-t_0}{n!}\,. \tag{55}$$

Thus $x(t) = B(t)x_0$ is the solution to (49).

If $A(t)$ and $A(t')$ commute for all t, t', then one can rearrange terms in (54) to get

$$x(t) = \exp\Big\{\int_{t_0}^{t} A(u)\,du\Big\}x_0. \tag{56}$$

This formula is known in physics by the name of Dyson series, where it establishes connection between iterated integrals and quantum field theory.

A more general situation is when a product $A(t_1)\cdots A(t_k)$ vanishes for all k large enough, e.g. when A is nilpotent. Then the sum expressing $x(t)$ is finite. Thus, one should expect simpler ('algebraic') behaviour of differential systems with unipotent monodromy, as e.g. in the case of multiple polylogarithms.

4.2. Iterated integrals. Let M be a smooth manifold, $\gamma : (0,1) \longrightarrow M$—a piecewise smooth path and $\omega_1, \ldots, \omega_n \in \Omega^1(M)$. For the pullback map $\gamma^* : \Omega^1(M) \to \Omega^1([0,1])$, write

$$\gamma^*(\omega_i) =: f_i(t)\,dt. \tag{57}$$

Since ω_i is a geometric object, the integral

$$\int_\gamma \omega_i = \int_0^1 f_i(t)\,dt \tag{58}$$

does not depend on parametrisation.

DEFINITION 4.1. The iterated integral is defined by

$$I(\omega_1,\ldots,\omega_n;\gamma) := \int_\gamma \omega_1\cdots\omega_n = \int_{t_0<t_1<\ldots<t} f_1(t_1)\,dt_1\ \ldots\ f_n(t_n)\,dt_n. \tag{59}$$

We also admit linear combinations of expressions as (59). Furthermore, if the collection of forms is empty, then assume that the integral is equal to one.

PROPOSITION 4.1. *Integral* (59) *has the following properties:*

- $I(\omega_1,\ldots,\omega_n;\gamma)$ *does not depend on the choice of parametrisation of the path* γ.
- *If* $\gamma^{-1}(t) := \gamma(1-t)$, *then* $I(\omega_1,\ldots,\omega_n;\gamma^{-1}) = (-1)^n I(\omega_n,\ldots,\omega_1;\gamma)$.
- *If* α,β *are two paths such that* $\alpha(1)=\beta(0)$, *then*

$$I(\omega_1,\ldots,\omega_n;\alpha\circ\beta) = \sum_{i=1}^n I(\omega_1,\ldots,\omega_i;\alpha)I(\omega_{i+1},\ldots,\omega_n;\beta), \tag{60}$$

  where $\alpha\circ\beta$ *is the usual composition of paths.*
- *There is a shuffle product formula*:

$$I(\omega_1,\ldots,\omega_r;\gamma)\dot{I}(\omega_1,\ldots,\omega_r;\gamma) = \sum_{\sigma\in S(r,s)} I(\omega_{\sigma(1)},\ldots,\omega_{\sigma(r+s)};\gamma), \tag{61}$$

  where $S(r,s) := \{\sigma\in S_n : \sigma(1)<\ldots<\sigma(r)\wedge\sigma(r+1)<\ldots<\sigma(r+s)\}$ *is the set of* (r,s)*-shuffles.*

A proof of Proposition 4.1 can be found in [2]. The ‘time reversal’ property (i.e., the second property in Proposition 4.1) implies for example the Drinfeld duality for multiple zeta values (see [11]). The shuffle product (last property) plays an important role in the theory of MZV’s, but we will not exploit it in this paper. Instead, we refer the reader to [1].

As we have observed above, the iterated integrals have a lot of algebraic structure and properties noted while studying multiple zeta values.

5. Generalisations of multiple zeta functions. One motivation for introducing MZV is that they appear naturally in further terms of expansions of the generating functions of ‘ordinary’ single zeta values. For example

$$\frac{\sin\pi x}{\pi x} = \prod_{n>0}\Bigl(1-\frac{x^2}{n^2}\Bigr) = 1-x^2\sum_{n>0}\frac{1}{n^2}+x^4\sum_{n>m>0}\frac{1}{n^2m^2}+\ldots = 1-x^2\zeta(2)+x^4\zeta(2,2)-\ldots. \tag{62}$$

But the zeta function is only a particular member of a whole family of the, so called, L-functions. The most basic, nontrivial L-functions are Dirichlet L-series defined as

$$L(\chi;s) := \sum_{n>0}\chi(n)n^{-s}, \tag{63}$$

where $\chi : \mathbb{Z}_N^* \to \mathbb{C}$ is a Dirichlet character[6] and as before, we have[7] $\operatorname{Re} s > 1$.

If we now take

$$\prod_{n>0}\Big(1 - \frac{\chi(n)^3 x^3}{n^3}\Big) = 1 - x^3 \sum_{n>0} \frac{\chi(n^3)}{n^3} + x^6 \sum_{n>m>0} \frac{\chi(n^3 m^3)}{n^3 m^3} + \dots$$
$$= -x^3 L_4(\chi_1; 3) + O(x^6) =: f(x), \tag{64}$$

we can use similar methods, like those used to obtain (6), to get the formula[8]

$$f(\lambda) = \Big\{\cos\Big(\frac{\pi\lambda}{4}\Big) + \sin\Big(\frac{\pi\lambda}{4}\Big)\Big\} \cdot \Big\{\cosh\Big(\frac{\sqrt{3}\,\pi\lambda}{4}\Big) - \sin\Big(\frac{\pi\lambda}{4}\Big)\Big\}. \tag{65}$$

One then gets

$$\beta(3) = \frac{\pi^3}{32}\,. \tag{66}$$

In a similar way as in the case of MZV's, one can associate with multiple Dirichlet L-function a meromorphic differential equation. To do it, one has to observe that the Drinfeld–Kontsevich integral (14) contains the 'shift forms'

$$\omega_1 = \frac{dt}{1-t} = \frac{dt}{t}\,\frac{t}{1-t} = \frac{dt}{t} \sum_{n>0} t^n, \tag{67}$$

which should be replaced by

$$\omega_1(\chi) = \frac{dt}{t} \sum_{n>0} \chi t^n =: \frac{dt}{t}\, F(t), \tag{68}$$

where $F \in \dot{\mathbb{Q}}(t)$.

In this way we get e.g. the equation associated to $\beta(p)$:

$$\big\{(1+t^2)\partial_t\, t\partial_t \,\cdots\, t\partial_t + \lambda^p\big\} f = 0. \tag{69}$$

There are two ways to study WKB asymptotics of the equation (69): one is straightforward and the other depends on plugging in the WKB series after appropriate change of variables $\mathbb{C}P^1 \setminus \{0, i, -i, \infty\} \to \mathbb{C}P^1 \setminus \{0, 1, \infty\}$.

Another generalisation of Riemann's ζ has been introduced by Hurwitz:

$$\zeta(s, \alpha) := \sum_{n>0} (n+\alpha)^{-s}, \tag{70}$$

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[6] A Dirichlet character is a (group) homomorphism from multiplicative group $\mathbb{Z}_N^*$ of the ring $\mathbb{Z}_N$ to complex numbers of unital modulus, continued (via pullback) to $\mathbb{Z}$. We set $\chi(n) = 0$, whenever $(n, N) > 1$. One can define χ axiomatically as follows. The function $\chi : \mathbb{Z} \to \mathbb{C}$ satisfies the following properties: $\chi(N+n) = \chi(n)$; if $(N, n) > 1$, then $\chi(n) = 0$; $\chi(nm) = \chi(m)\chi(n)$ for every m, n.

[7] Actually, since partial sums of χ are bounded, by Dirichlet test (which was by the way discovered by Dirichlet in connection of studying L-functions), $L(\chi; s)$ is holomorphic for $\operatorname{Re} s > 0$.

[8] The function L_4 is the simplest nontrivial Dirichlet L-series modulo 4, also known as Dirichlet Beta function. We have

$$L(\chi_1, s) = \beta(s) = \sum_{n\ge 0} (-1)^n (2n+1)^{-s},$$

for $\operatorname{Re} s > 0$. The function $L(\chi_1, s)$ is entire on $\mathbb{C}$.

for $\operatorname{Re} s > 1$. Since χ is a periodic sequence with period q, Dirichlet L-function can be represented with use of Hurwitz's zeta with $\alpha = m/q \in \mathbb{Q}$:

$$L(\chi; s) = \sum_{m=1}^{q} \chi(m) \sum_{n \geq 0} (qn + m)^{-s} = q^{-s} \sum_{m=1}^{q} \chi(m) \zeta(s, m/q). \tag{71}$$

Thus, the Dirichlet L-function can be expressed as a linear combination of Hurwitz zeta functions.

DEFINITION 5.1. *Multiple Hurwitz zeta functions* are defined by the multiple series

$$\zeta \begin{pmatrix} s_1, s_2, \ldots, s_p \\ a_1, a_2, \ldots, a_p \end{pmatrix} := \sum_{n_1 > n_2 > \ldots > n_p} (n_1 + a_1)^{-s_1} (n_2 + a_2)^{-s_2} \cdots (n_p + a_p)^{-s_p}. \tag{72}$$

For multiple Hurwitz zeta values, the Lerch[9] transcendent plays the same role as polylogarithm for Riemann's zeta.

In a similar way, we can introduce integral representations of multiple zeta values, one can also deal with multiple Hurwitz zeta values. For example, for Φ, we have

$$\Phi(t, s, \alpha) = \int_0^t \omega_0(t_1) \int_0^{t_1} \omega_0(t_2) \cdots \int_0^{t_p} \omega_{1,\alpha}(t_p), \tag{73}$$

where $\omega_{1,\alpha}(t) := t^{\alpha-1} dt/(1-t)$ and the integral is s-fold. More generally, we have

$$\Phi \left(t \left| \begin{array}{c} s_1, s_2, \ldots, s_p \\ \alpha_1, \alpha_2, \ldots, \alpha_p \end{array} \right. \right) = \int_0^t \omega_{\epsilon_1}(t_1) \int_0^{t_1} \omega_{\epsilon_2}(t_2) \cdots \int_0^{t_p} \omega_{\epsilon_p}(t_p), \tag{74}$$

where this time $\epsilon_i \in \{0, (1, \beta_i)\}$ and $\alpha_i = \beta_1 + \ldots + \beta_i$.

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[9] Lerch function is defined as

$$\Phi(t, s, \alpha) := \sum_{n \geq 0} \frac{t^n}{(n + \alpha)^s},$$

whenever the above series converges.

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