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GENERALIZED HAMILTONIAN DYNAMICS AFTER DIRAC AND TULCZYJEW

FIORELLA BARONE

Dipartimento Interuniversitario di Matematica, Università di Bari
Via E. Orabona 4, 70125 Bari, Italy
E-mail: barone@pascal.dm.uniba.it

RENATO GRASSINI

Dipartimento di Matematica e Applicazioni R. Caccioppoli
Università di Napoli Federico II
Via Cintia, Monte S. Angelo, 80126 Napoli, Italy
E-mail: renato.grassini@dma.unina.it

Abstract. Dirac's generalized Hamiltonian dynamics is given an accurate geometric formulation as an implicit differential equation and is compared with Tulczyjew's formulation of dynamics. From the comparison it follows that Dirac's equation—unlike Tulczyjew's—fails to give a complete picture of the real laws of classical and relativistic dynamics.

1. Introduction. (i) Generalized Hamiltonian dynamics is the name given by P. A. M. Dirac [8, 9] to his own attempt to provide a Hamiltonian formulation for the dynamics of physical systems with singular Lagrangians.

Dirac's approach starts from traditional Lagrangian dynamics (based on Hamilton's variational principle and Euler-Lagrange equations in coordinate formulation) and aims to extend the classical method of Legendre transformation from hyperregular to singular Lagrangians. The main result (Hamiltonian equations with Lagrange multipliers for constrained systems) has been geometrically interpreted by W. M. Tulczyjew [14, 18] as an implicit differential equation on T^*Q (cotangent bundle of the configuration space Q of the system).

Dirac's geometrized equation, however, contrasts—in a number of examples—with the implicit differential equation on T^*Q proposed, on the base of a more general conception of Legendre transformation, by the same author (see, e.g., Tulczyjew [16, 17] and Tulczyjew et al. [12, 13, 19]).

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Moreover, in the presentation of the above geometric equations, any explicit link with (a geometric formulation of) traditional Lagrangian dynamics seems to have been lost.

In such a situation, what we need—in our opinion—is to give the whole process of transition from Lagrangian to Hamiltonian dynamics a systematic geometric reconstruction, so as to be able to deduce (rather than only state) Dirac and Tulczyjew's equations from a coherent geometric framework and, by doing so, to get a deeper insight into the theoretical reasons for their differences.

That is the aim of the present paper.

(ii) Our line of thought is the following. We start from Lagrangian dynamics, where—for a system described by a (regular or singular) Lagrangian L defined on an open submanifold M of TQ (the tangent bundle of Q)—the possible motions are assumed to be the solution curves in Q of Hamilton's variational principle. In this connection, we focus on the problem of characterizing (in terms of differential equations) the motions of the system or, equivalently, the corresponding trajectories in TQ, obtained from (and bijectively related to) the motions in Q via tangent lifting.

In Sec. 3, we recall [1] that the trajectories of the system in TQ are the integral curves of a second-order implicit *Euler-Lagrange equation* $\mathcal{E} = \mathcal{D} \cap T^2Q$, which will be shown to arise from the intersection of the *Hamilton-Dirac equation* $\mathcal{D}$ generated by the energy of L on $M \subset TQ$ (carrying a structure of Dirac manifold [6,20]) with the well known second-order tangent bundle $T^2Q \subset TTQ$.

Now remark that a Lagrangian L determines not only the evolution law of the system in TQ through its Euler-Lagrange equation $\mathcal{E}$, but also a transition law from TQ to T^*Q—linking velocities to momenta—through its Legendre mapping (or fibre derivative) $\mathcal{L}$. So one is led to face the higher-rank problem of characterizing the trajectories of the system in T^*Q—obtained from (and bijectively related to) the trajectories in TQ via Legendre mapping.

In Sec.4, we shall assume hypotheses of almost-regularity for L, which guarantee the existence—on a 'constraint' submanifold M_1 of T^*Q—of a Hamiltonian function corresponding to L in the sense of the ordinary Legendre transformation. Then, through the operation of transforming $\mathcal{E}$ by $T\mathcal{L}$ (the tangent of $\mathcal{L}$) we shall prove that the trajectories of the system in T^*Q are the integral curves of a 'second-order' implicit differential equation $\mathcal{H} = \mathcal{D}_1 \cap T_2$ on T^*Q, which still arises from the intersection of the *Hamilton-Dirac equation* $\mathcal{D}_1$ generated by the Hamiltonian of L on $M_1 \subset T^*Q$ (carrying a structure of Dirac manifold) with a new kind of 'second-order' tangent bundle $T_2 \subset TT^*Q$ (obtained from T^2Q through $T\mathcal{L}$). We explicitly stress the fact that the above result rests on the second-order character of $\mathcal{H}$, i.e. $\mathcal{H} \subset T_2$, which is *not* generally shared by $\mathcal{D}_1$ (the above mentioned Dirac's geometrized equation).

The problem of characterizing the trajectories in T^*Q can successfully be dealt with also when the almost-regularity hypotheses are dropped.

In Sec. 5, the *Tulczyjew equation* $\mathcal{T}$ (generated by a generalized Hamiltonian) is taken into consideration. Then, through the operation of transforming $\mathcal{E}$ by $T\mathcal{L}$, we shall prove that—owing to the second-order property $\mathcal{T} \subset T_2$—the trajectories of the system in T^*Q are exactly the integral curves of $\mathcal{T}$. As a consequence, in the almost-regular case, $\mathcal{T}$ turns out to be equivalent to $\mathcal{H}$ (rather than $\mathcal{D}_1$).

In Sec. 6, the almost-regular example of a relativistic particle in a gravitational and electromagnetic field will confirm the role of $\mathcal{T}$ or, equivalently, $\mathcal{H}$ (but not $\mathcal{D}_1$) as the true law of Hamiltonian dynamics.

In Sec. 7, we conclude with some brief remarks, where the focal points of the work are underlined and looked at in perspective for further research.

2. Preliminaries. Here is a list of notations and geometric tools used in this paper.

(i) For any smooth manifold M, we shall adopt the following notations.

TM and T^*M are the tangent and cotangent bundles of M, whose bundle projections are denoted $\tau_M : TM \to M$ and $\pi_M : T^*M \to M$.

$TM \oplus T^*M := \{(x,\xi) \in TM \times T^*M \,|\, \tau_M(x) = \pi_M(\xi)\}$ is the Whitney sum of TM and T^*M.

$\chi(M)$ is the Lie algebra of vector fields on M.

$\Lambda(M)$ is the exterior graded algebra of M (in particular, $\Lambda^0(M) \subset \Lambda(M)$ is the ring of real-valued smooth functions on M).

Let $f : N \to M$ be a smooth mapping between manifolds N and M.

$Tf : TN \to TM$ is the tangent mapping of f (whose restriction to the fibre $T_yN := \tau_N^{-1}(y)$ over a point $y \in N$ is denoted $T_yf : T_yN \to T_{f(y)}M$).

$f^* : \Lambda(M) \to \Lambda(N)$ is the pull-back of the exterior algebra of M into that of N.

If $c : I \to M$ is a smooth curve in M (defined on an open interval I of the real line $\mathbb{R}$), then Tc defines a section $\dot c$ of τ_M along c, called the tangent lifting of c, given by

$$\dot c := Tc \circ \left.\frac{d}{dt}\right|_I : I \to TM$$

($\frac{d}{dt} \in \chi(\mathbb{R})$ being the vector field associated with the natural chart $t := \mathrm{id}_{\mathbb{R}}$), and $T\dot c$ similarly defines the second tangent lifting $\ddot c : I \to TTM$.

If $\psi : N \to M$ is a submersion, then $V\psi$ is the vertical vector bundle of ψ, whose fibre over any $y \in N$ is

$$V_y\psi := \ker T_y\psi$$

and $V^o\psi$ is its annihilator, with typical fibre

$$(V_y\psi)^o := \{\eta \in T_y^*N \mid \langle \eta \mid u\rangle = 0, \forall\, u \in V_y\psi\}$$

(where $\langle\,|\,\rangle$ denotes the natural pairing between forms and vectors).

As is known, there exists a unique vector bundle morphism (over ψ)

$$\varpi_\psi : V^o\psi \to T^*M$$

satisfying, for all $y \in N$ and $\eta \in V_y^o\psi$,

$$\eta = \varpi_\psi(\eta) \circ T_y\psi$$

Let $\omega \in \Lambda^2(M)$ be a 2-form on M. The vector bundle morphism

$$^\flat : TM \to T^*M : x \mapsto {}^\flat x := i_x\omega := \langle \omega(\tau_M(x)) \mid x, \cdot\rangle$$

is called the musical morphism associated with ω.

ω is said to be nondegenerate if $^\flat$ is an isomorphism.

If $d\omega = 0$, d being the exterior derivative of forms, ω will be called a presymplectic 2-form (prefix 'pre' is dropped when ω is nondegenerate).

Recall the canonical example of a symplectic 2-form (on a cotangent bundle T^*M)

$$\omega_M := -d\vartheta_M$$

obtained from Liouville 1-form

$$\vartheta_M : T^*M \to T^*T^*M : \xi \mapsto \vartheta_M(\xi) := \xi \circ T_\xi \pi_M.$$

Let us now recall two basic tangent derivations [18].

$i_T : \Lambda(M) \to \Lambda(TM)$ is the tangent derivation (of degree -1) which vanishes on $\Lambda^0(M)$ and acts on $\Lambda^1(M)$ by $\theta \in \Lambda^1(M) \mapsto i_T\theta \in \Lambda^0(TM)$ with

$$i_T\theta : TM \to \mathbb{R} : x \mapsto i_T\theta(x) := i_x\theta := \langle \theta(\tau_M(x)) \mid x \rangle.$$

Then i_T will act on $\Lambda^2(M)$ by $\omega \in \Lambda^2(M) \mapsto i_T\omega \in \Lambda^1(TM)$ with

$$i_T\omega : TM \to T^*TM : x \mapsto i_T\omega(x) := i_x\omega \circ T_x\tau_M.$$

The commutator $d_T : \Lambda(M) \to \Lambda(TM)$ of i_T and d is the tangent derivation (of zero degree)

$$d_T := i_T d + d i_T$$

satisfying, for any $\psi : N \to M$,

$$d_T \psi^* = (T\psi)^* d_T.$$

(ii) In the geometry of the iterated bundles associated with a smooth manifold Q, a key role is played by the following canonical morphisms.

First, we recall the diffeomorphism [18]

$$\alpha : TT^*Q \to T^*TQ$$

uniquely determined by conditions

$$\pi_{TQ} \circ \alpha = T\pi_Q, \qquad d_T\vartheta_Q = \alpha^*\vartheta_{TQ}$$

Remark that, for any $v \in T_qQ$ and $\theta_v \in T_v^*TQ$, one has

$$(\tau_{T^*Q} \circ \alpha^{-1})(\theta_v) = \theta_v \circ \nu_v$$

(where $\nu_v : T_qQ \to V_v\tau_Q$ is the canonical isomorphism of $T_qQ = \tau_Q^{-1}(q)$ onto its own tangent space $T_v(T_qQ) = V_v\tau_Q$).

Then, for any function $L \in \Lambda^0(M)$ defined on an open submanifold M of TQ, the bundle morphism

$$FL := \tau_{T^*Q} \circ \alpha^{-1} \circ dL : M \to T^*Q : v \mapsto FL(v) = dL(v) \circ \nu_v$$

is the fibre derivative of L.

Next, we recall the musical isomorphism

$$\beta : TT^*Q \to T^*T^*Q : z \mapsto {}^\beta z := i_z\omega_Q$$

associated with the canonical symplectic 2-form ω_Q of T^*Q.

Remark that, for any $p \in T_q^*Q$ and $h_p \in T_p^*T^*Q$, one has

$$(T\pi_Q \circ \beta^{-1})(h_p) = h_p \circ \nu_p$$

(where $\nu_p : T_q^*Q \to V_p\pi_Q$ is the canonical isomorphism of $T_q^*Q = \pi_Q^{-1}(q)$ onto its own tangent space $T_p(T_q^*Q) = V_p\pi_Q$).

Then, for any function $H \in \Lambda^0(W)$ defined on an open submanifold W of T^*Q, the bundle morphism

$$FH := T\pi_Q \circ \beta^{-1} \circ dH : W \to TQ : p \mapsto FH(p) = dH(p) \circ \nu_p$$

is the fibre derivative of H.

Finally, we recall the vertical vector bundle endomorphism [11]

$$S : TTQ \to TTQ$$

defined by putting, for any $v \in TQ$,

$$S_v := S|_{T_vTQ} := \nu_v \circ T_v\tau_Q.$$

Associated with S there are two derivations [11, 10].

$i_S : \Lambda(TQ) \to \Lambda(TQ)$ is the derivation (of zero degree) which vanishes on $\Lambda^0(TQ)$ and acts on $\Lambda^1(TQ)$ by $\theta \in \Lambda^1(TQ) \mapsto i_S\theta \in \Lambda^1(TQ)$ with

$$i_S\theta : TQ \to T^*TQ : v \mapsto i_S\theta(v) := \theta(v) \circ S_v.$$

The commutator $d_S : \Lambda(TQ) \to \Lambda(TQ)$ of i_S and d is the derivation (of degree 1)

$$d_S := i_S d - d i_S$$

Clearly, i_S and d_S act as derivations on the exterior algebra of any open submanifold M of TQ.

In particular, if $L \in \Lambda^0(M)$, one has

$$d_S L = (FL)^*\vartheta_Q$$

whence

$$dd_S L = -(FL)^*\omega_Q.$$

Owing to the above result, the presymplectic 2-form $dd_S L \in \Lambda^2(M)$ turns out to be symplectic iff FL is a local diffeomorphism.

3. Lagrangian dynamics. Lagrangian dynamics—for a mechanical system described in terms of a (generally) singular Lagrangian—will be framed into a simple and compact geometric scheme.

(i) Let (Q, L) be (the mathematical model of) a mechanical system, consisting of a smooth manifold Q (the *configuration space* of the system) and a smooth function $L \in \Lambda^o(M)$ defined on an open submanifold M of TQ (the *Lagrangian* function of the system).

According to classical dynamics, the *motions* of (Q, L) are the smooth curves in Q satisfying Hamilton's variational principle.

From a geometric formulation of variational calculus [1], it follows that a smooth curve γ in Q is a motion of (Q, L) iff

$$\operatorname{Im}\dot\gamma \subset M$$

and

$$^\flat\ddot{\gamma} = dE \circ \dot{\gamma}$$

(where $^\flat$ denotes the musical morphism associated with the *Poincaré-Cartan* presymplectic 2-form $\omega := -dd_S L \in \Lambda^2(M)$, and $E := \Delta L - L \in \Lambda^0(M)$ is the *energy* function defined by putting $\Delta L := i_\Delta dL$ with $\Delta : M \to TM : v \mapsto \Delta(v) := \nu_v(v)$).

(ii) The dynamics of (Q, L) can naturally be moved onto TQ (the *velocity phase space* of the system) as follows.

For any motion γ of (Q, L), its *tangent lifting* $\dot{\gamma}$, a smooth curve lying on M, will be called a *velocity phase space trajectory* (or VPS trajectory) of (Q, L).

The correspondence $\gamma \mapsto c := \dot{\gamma}$ between motions and VPS trajectories of (Q, L) is obviously invertible, the inverse being the projection $c \mapsto \gamma := \tau_Q \circ c$.

The problem of determining the motions can then be solved by determining the VPS trajectories, i.e. the smooth curves c's in TQ satisfying

$$\operatorname{Im} c \subset M, \tag{3.1}$$

$$^\flat\dot{c} = dE \circ c, \tag{3.2}$$

$$c = (\tau_Q \circ c)^{\cdot}. \tag{3.3}$$

The above trajectories will prove to be the integral curves of an implicit differential equation $\mathcal{E}$ on TQ, i.e.

$$\operatorname{Im} \dot{c} \subset \mathcal{E} \tag{3.4}$$

with

$$\mathcal{E} \subset TTQ \tag{3.5}$$

Such an equation will soon be worked out and its mathematical structure analysed.

(iii) Conditions (3.1) and (3.2) also read

$$\operatorname{Im} \dot{c} \subset \mathcal{D} \tag{3.6}$$

with

$$\mathcal{D} := \{x \in TM \mid {}^\flat x = dE(\tau_M(x))\}.$$

$\mathcal{D}$ is an implicit differential equation on M (and then on TQ), whose underlying geometric structure will now be examined.

First recall that a *Dirac manifold* [6, 20] is a couple (M, Ω), consisting of a smooth manifold M and a Dirac structure $\Omega \subset TM \oplus T^*M$.

Then recall that, on a Dirac manifold (M, Ω), to any *Hamiltonian* function $E \in \Lambda^0(M)$ there corresponds an implicit differential equation [20]

$$\mathcal{D}_E := \{x \in TM \mid (x, dE(\tau_M(x))) \in \Omega\}$$

which will be called the *Hamilton-Dirac equation* generated by E on (M, Ω).

Now turn back to the presymplectic manifold (M, ω) and the energy function E introduced in (i).

Remark that (M, ω) can as well be regarded as a Dirac manifold by putting

$$\Omega := \operatorname{Im} \operatorname{Graph}{}^\flat = \{(x, \xi) \in TM \oplus T^*M \mid {}^\flat x = \xi\}$$

and then

$$\mathcal{D}_E = \mathcal{D}.$$

So $\mathcal{D}$ is the Hamilton-Dirac equation generated by E on (M, ω).

If L is a regular Lagrangian (i.e., ω is symplectic), and only in that case, the equation $\mathcal{D}$ takes the explicit form (on M)

$$\mathcal{D} = \operatorname{Im} \Gamma_E$$

where $\Gamma_E := \flat^{-1} \circ dE \in \chi(M)$ is an ordinary Hamiltonian vector field, characterized by $i_{\Gamma_E}\omega = dE$, on the symplectic manifold (M, ω).

(iv) Condition (3.3), i.e. $\tau_{TQ} \circ \dot{c} = T\tau_Q \circ \dot{c}$, also reads

$$\operatorname{Im} \dot{c} \subset T^2 Q \tag{3.7}$$

with

$$T^2 Q := \{x \in TTQ \mid \tau_{TQ}(x) = T\tau_Q(x)\}.$$

T^2Q is an implicit differential equation on TQ, which—as well as any equation contained in it—exhibits the typical *second-order character*, consisting in the fact that the projection $c \mapsto \gamma := \tau_Q \circ c$ of its integral curves onto the corresponding base integral curves is inverted by the tangent lifting $\gamma \mapsto c := \dot{\gamma}$.

(v) Conditions (3.6) and (3.7), characterizing the VPS trajectories of (Q, L), can equivalently be expressed in the form (3.4) and (3.5) by putting

$$\mathcal{E} := \mathcal{D} \cap T^2 Q.$$

So *the* VPS *trajectories of* (Q, L) *are the integral curves of the equation* $\mathcal{E}$ (called the *Euler-Lagrange equation*).

Observe the structure of $\mathcal{E}$, *extracted* from the Hamilton-Dirac equation $\mathcal{D}$ via intersection with the second-order equation T^2Q.

Owing to such a structure, $\mathcal{E}$ is not generally equivalent to $\mathcal{D}$, for the latter may admit more integral curves than the former does (not all of the integral curves of $\mathcal{D}$ will then correspond to possible motions of the system).

The problem of integrating $\mathcal{E}$ will in principle be solved by determining the integral curves of $\mathcal{D}$, characterized by condition (3.6), and then sorting out those which satisfy the *second-order condition* (3.7).

Note that, if L is a regular Lagrangian, the second-order condition (3.7) is *hidden* by the well known circumstance [11, 7] $\mathcal{D} = \operatorname{Im} \Gamma_E \subset T^2Q$, i.e. $\mathcal{E} = \mathcal{D} = \operatorname{Im} \Gamma_E$. Owing to the above result, indeed, the VPS trajectories of (Q, L) turn out to be characterized by the only condition (3.6), which takes the normal form $\dot{c} = \Gamma_E \circ c$ (with $\operatorname{Im} c \subset M$).

4. Hamiltonian dynamics after Dirac. Dirac's approach to Hamiltonian dynamics, starting from Lagrangian dynamics, will be examined (and revised) through a systematic geometric reconstruction.

(i) The dynamics of (Q, L) can as well be moved onto T^*Q (the *momentum phase space* of the system) by means of the *Legendre morphism*

$$\mathcal{L} := FL : M \to T^*Q.$$

For any motion γ of (Q, L), its *Legendre lifting* $k := \mathcal{L} \circ \dot{\gamma}$, a smooth curve lying on $M_1 := \operatorname{Im} \mathcal{L}$, will be called a *momentum phase space trajectory* (or MPS trajectory) of (Q, L).

The correspondence $\gamma \mapsto k := \mathcal{L} \circ \dot{\gamma}$ between motions and MPS trajectories of (Q, L) is obviously invertible, the inverse being the projection $k \mapsto \gamma := \pi_Q \circ k$.

Determining the motions is now only a part of the higher-rank problem of determining the MPS trajectories, i.e. the smooth curves in T^*Q which correspond to the VPS trajectories through $\mathcal{L}$.

As the VPS trajectories are the integral curves of the implicit differential equation $\mathcal{E}$ on TQ, the MPS trajectories are expected to be the integral curves of an implicit differential equation on T^*Q obtained from $\mathcal{E}$ via $T\mathcal{L}$.

Such an equation will now be worked out in the case of an almost-regular Lagrangian, i.e. one satisfying the following hypotheses:

(a) $M_1 := \operatorname{Im} \mathcal{L}$ is an embedded submanifold of T^*Q.
(b) $\mathcal{L}_1 : M \to M_1$, defined by $\iota_1 \circ \mathcal{L}_1 = \mathcal{L}$ with $\iota_1 : M_1 \hookrightarrow T^*Q$, is a submersion.
(c) $E := \Delta L - L$ is projectable by $\mathcal{L}$, i.e. $E = \mathcal{L}^* H$, $H \in \Lambda^0(W)$ being defined on an open submanifold W of T^*Q containing M_1.

The above hypotheses generalize some of the features of a hyperregular Lagrangian (whose Legendre morphism is an injective local diffeomorphism).

Indeed, if $\mathcal{L}$ is a local diffeomorphism, conditions (a) and (b) are automatically fulfilled, since M_1 is an open submanifold of T^*Q and $\mathcal{L}_1$ is a local diffeomorphism as well. Moreover, if—and only if—$\mathcal{L}$ is injective too, condition (c) is fulfilled (with $H := E \circ \mathcal{L}_1^{-1}$ uniquely determined on $W := M_1$).

(ii) To start with, $T\mathcal{L}$ will be made to act on $\mathcal{D}$. Let $x \in TM$ and put $z := T\mathcal{L}(x) = T\mathcal{L}_1(x) \in TM_1$. Recall that $x \in \mathcal{D}$ iff ${}^\flat x = dE(v)$ with $v := \tau_M(x)$.

If M_1 is given the presymplectic 2-form

$$\omega_1 := \iota_1^* \omega_Q$$

and ${}^{\flat_1} : TM_1 \to T^*M_1$ denotes the corresponding musical morphism, from

$$\omega = \mathcal{L}^* \omega_Q = \mathcal{L}_1^* \iota_1^* \omega_Q = \mathcal{L}_1^* \omega_1$$

it follows that

$${}^\flat x = \langle \omega_1(\mathcal{L}_1(v)) \mid T_v\mathcal{L}_1(x), T_v\mathcal{L}_1(\cdot) \rangle = {}^{\flat_1} z \circ T_v\mathcal{L}_1.$$

Moreover, from $E = \mathcal{L}^* H = \mathcal{L}_1^* \iota_1^* H = \mathcal{L}_1^* H_1$ with $H_1 := \iota_1^* H$, it follows that

$$dE(v) = dH_1(\mathcal{L}_1(v)) \circ T_v\mathcal{L}_1 = dH_1(\tau_{M_1}(z)) \circ T_v\mathcal{L}_1.$$

As $\mathcal{L}_1$ is a submersion, condition ${}^\flat x = dE(v)$ turns out to be equivalent to

$${}^{\flat_1} z = dH_1(\tau_{M_1}(z)).$$

So one has

$$x \in \mathcal{D} \Leftrightarrow x \in TM,\ T\mathcal{L}(x) \in \mathcal{D}_1 \tag{4.1}$$

with

$$\mathcal{D}_1 := \{ z \in TM_1 \mid {}^{\flat_1} z = dH_1(\tau_{M_1}(z)) \}.$$

$\mathcal{D}_1$ is the Hamilton-Dirac equation generated by H_1 on (M_1, ω_1).

It can be given an alternative formulation, making direct use of the canonical symplectic 2-form of T^*Q, as follows.

Let $z \in TT^*Q$. Recall that $z \in \mathcal{D}_1$ iff $z \in TM_1$, whence $p := \tau_{T^*Q}(z) \in M_1$, and ${}^{\flat_1}z = dH_1(p)$.

From $\omega_1 := \iota_1^*\omega_Q$ and $H_1 := \iota_1^* H$, it follows that

$$^{\flat_1}z = \langle \omega_Q(\iota_1(p)) \mid T_p\iota_1(z), T_p\iota_1(\cdot)\rangle = {}^{\beta}z \circ T_p\iota_1 = {}^{\beta}z|_{T_pM_1}.$$

and

$$dH_1(p) = dH(\iota_1(p)) \circ T_p\iota_1 = dH(p)|_{T_pM_1}.$$

Condition ${}^{\flat_1}z = dH_1(p)$ then reads ${}^{\beta}z - dH(p) \in (T_pM_1)^o$ or, equivalently, $z - X_H(p) \in {}^{\beta^{-1}}(T_pM_1)^o$ (where we have put $X_H := \beta^{-1} \circ dH \in \chi(W)$). So we obtain

$$\mathcal{D}_1 = TM_1 \cap \widehat{\mathcal{D}}_1 \tag{4.2}$$

with $\widehat{\mathcal{D}}_1$—equivalent to $\mathcal{D}_1$—given by

$$\widehat{\mathcal{D}}_1 = \{z \in TT^*Q \mid p := \tau_{T^*Q}(z) \in M_1, z - X_H(p) \in {}^{\beta^{-1}}(T_pM_1)^o\}$$

(see [14, 18]).

Now we shall focus on the case of a singular Lagrangian, by reinforcing hypothesis (a) as follows:

(a') $M_1 = \phi^{-1}(\mu)$, where $\phi = (\phi^1, \ldots, \phi^m) : W \to \mathbb{R}^m$ (with $0 < m < \dim T^*Q$ and $\operatorname{Im}\phi \ni \mu$) is a submersion at every point of M_1.

Clearly, (a') implies (a) with $\dim M_1 < \dim T^*Q$, which in turn implies the singularity of L.

From (a') it follows that, at any $p \in M_1$,

$$(T_pM_1)^o = \operatorname{Span} d\phi(p)$$

(where $d\phi(p) = (d\phi^1(p), \ldots, d\phi^m(p)) : T_pT^*Q \to \mathbb{R}^m$ is the differential of ϕ at p) and then

$$^{\beta^{-1}}(T_pM_1)^o = \operatorname{Span} X_\phi(p)$$

(where we have put $X_\phi = (X_{\phi^1}, \ldots, X_{\phi^m})$ with $X_{\phi^a} := \beta^{-1} \circ d\phi^a \in \chi(W)$ for all $a = 1, \ldots, m$; in the sequel, an index-free summation convention will be adopted, say $\lambda X_\phi(p) := \lambda_1 X_{\phi^1}(p) + \ldots + \lambda_m X_{\phi^m}(p)$ for any $\lambda = (\lambda_1, \ldots \lambda_m) \in \mathbb{R}^m$).

So, in the singular case (a'), $\widehat{\mathcal{D}}_1$ is expressed, in terms of Lagrange multipliers $\lambda \in \mathbb{R}^m$, by

$$\widehat{\mathcal{D}}_1 := \{z \in TT^*Q \mid p := \tau_{T^*Q}(z) \in \phi^{-1}(\mu), \exists\, \lambda \in \mathbb{R}^m : z = X_H(p) + \lambda X_\phi(p)\} \tag{4.3}$$

In the hyperregular case, as ${}^{\beta^{-1}}(T_pM_1)^o = {}^{\beta^{-1}}(T_pT^*Q)^o$ is the null subspace of T_pT^*Q (for all $p \in M_1$), the equation $\mathcal{D}_1 = \widehat{\mathcal{D}}_1$ takes the explicit form (on M_1)

$$\mathcal{D}_1 = \operatorname{Im} X_H.$$

(iii) Now $T\mathcal{L}$ will be made to act on T^2Q. Let $x \in TM$ and put $z := T\mathcal{L}(x) \in TT^*Q$. From $\pi_Q \circ \mathcal{L} = \tau_Q|_M$, it follows that, if $x \in T^2Q$, one has

$$T\pi_Q(z) = T\pi_Q(T\mathcal{L}(x)) = T\tau_Q(x) = \tau_{TQ}(x) = \tau_M(x) \in M$$

whence

$$\tau_{T^*Q}(z) = \tau_{T^*Q}(T\mathcal{L}(x)) = \mathcal{L}(\tau_M(x)) = \mathcal{L}(T\pi_Q(z)).$$

So we obtain

$$x \in T^2_M Q := T^2 Q \cap TM \Rightarrow T\mathcal{L}(x) \in T_2 \tag{4.4}$$

with

$$T_2 := \{z \in TT^*Q \mid T\pi_Q(z) \in M,\ \tau_{T^*Q}(z) = \mathcal{L}(T\pi_Q(z))\}.$$

T_2 is an implicit differential equation on T^*Q, which—as well as any equation contained in it—exhibits a sort of *second-order character*, consisting in the fact that the projection $k \mapsto \gamma := \pi_Q \circ k$ of its integral curves onto the corresponding base integral curves is inverted by the Legendre lifting $\gamma \mapsto k := \mathcal{L} \circ \dot\gamma$.

(iv) The operation of transforming $\mathcal{E}$ by $T\mathcal{L}$, expressed by (4.1) and (4.4), can be synthesized by

$$x \in \mathcal{E} \Leftrightarrow x \in T^2_M Q,\ T\mathcal{L}(x) \in \mathcal{H} \tag{4.5}$$

with

$$\mathcal{H} := \mathcal{D}_1 \cap T_2.$$

The above equation also reads

$$\mathcal{H} = TM_1 \cap \widehat{\mathcal{H}} \tag{4.6}$$

with $\widehat{\mathcal{H}} := \widehat{\mathcal{D}}_1 \cap T_2$—equivalent to $\mathcal{H}$—expressed by

$$\begin{aligned}\widehat{\mathcal{H}} = \{z \in TT^*Q \mid T\pi_Q(z) \in M,\ p := \tau_{T^*Q}(z) = \mathcal{L}(T\pi_Q(z)),\\ z - X_H(p) \in {}^{\beta^{-1}}(T_pM_1)^o\ \}.\end{aligned} \tag{4.7}$$

In the singular case (a'), $\widehat{\mathcal{H}}$ is expressed, in terms of Lagrange multipliers, by

$$\begin{aligned}\widehat{\mathcal{H}} = \{z \in TT^*Q \mid T\pi_Q(z) \in M,\ p := \tau_{T^*Q}(z) = \mathcal{L}(T\pi_Q(z)),\\ \exists\, \lambda \in \mathbb{R}^m : z = X_H(p) + \lambda X_\phi(p)\}.\end{aligned} \tag{4.8}$$

A special situation, leading to the elimination of the unknown Lagrange multipliers, occurs when hypothesis (a') is further reinforced by assuming that

(a") $M_1 = \phi^{-1}(\mu)$, where $\phi = (\phi^1, \dots, \phi^m) : W \to \mathbb{R}^m$ (with $0 < m < \dim T^*Q$ and $\operatorname{Im}\phi \ni \mu$) is such that $F\phi(p) = (F\phi^1(p), \dots, F\phi^m(p))$ is a linearly independent system at every point $p \in M_1$.

First remark that the vector field

$$\Gamma_H : M \to V\tau_Q : v \mapsto \Gamma_H(v) := \nu_v(FH(\mathcal{L}(v)))$$

satisfies

$$T_v\mathcal{L}(\Gamma_H(v)) = \nu_{\mathcal{L}(v)}(F\mathcal{L}^*H(v))$$

and the vector field

$$\Delta : M \to V\tau_Q : v \mapsto \Delta(v) := \nu_v(v)$$

satisfies

$$T_v\mathcal{L}(\Delta(v)) = \nu_{\mathcal{L}(v)}(FE(v)).$$

Hence, owing to (c),

$$\Delta(v) - \Gamma_H(v) \in \ker T_v\mathcal{L}. \tag{4.9}$$

Then remark that also the vector fields

$$\Gamma_{\phi^a} : M \to V\tau_Q : v \mapsto \Gamma_{\phi^a}(v) := \nu_v(F\phi^a(\mathcal{L}(v)))$$

$(a = 1, \ldots, m)$ satisfy $\Gamma_{\phi^a}(v) \in \ker T_v\mathcal{L}$ (since $T_v\mathcal{L}(\Gamma_{\phi^a}(v)) = \nu_{\mathcal{L}(v)}(F\mathcal{L}^*\phi^a(v))$ and $\mathcal{L}^*\phi^a = \mu^a$). Owing to (a"), $\Gamma_\phi(v) := (\Gamma_{\phi^1}(v), \ldots, \Gamma_{\phi^m}(v))$ is then a basis of $\ker T_v\mathcal{L}$.

As a consequence, there exists a unique m-tuple $J = (J_1, \ldots, J_m)$ of real-valued functions on M such that

$$\Delta(v) = \Gamma_H(v) + J(v)\Gamma_\phi(v), \tag{4.10}$$

that is, $v = FH(\mathcal{L}(v)) + J(v)F\phi(\mathcal{L}(v))$ for all $v \in M$.

Now let $z \in \widehat{\mathcal{H}}$. By applying the above result to $T\pi_Q(z) \in M$ and recalling that $p := \tau_{T^*Q}(z) = \mathcal{L}(T\pi_Q(z)) \in M_1$, one obtains

$$T\pi_Q(z) = FH(p) + J(T\pi_Q(z))F\phi(p).$$

Moreover, from $z = X_H(p) + \lambda X_\phi(p)$ for some $\lambda \in \mathbb{R}^m$, it follows that $T\pi_Q(z) = FH(p) + \lambda F\phi(p)$. Hence, owing to (a"),

$$\lambda = J(T\pi_Q(z)). \tag{4.11}$$

So, in the singular case (a"), one has

$$\widehat{\mathcal{H}} = \{z \in TT^*Q \mid v := T\pi_Q(z) \in M,\ p := \tau_{T^*Q}(z) = \mathcal{L}(v),\ z = X_H(p) + J(v)X_\phi(p)\,\}. \tag{4.12}$$

which shows the announced elimination of the unknown Lagrange multipliers λ (replaced by the values (4.11) of the known functions J on M).

Finally note that, in the hyperregular case, (4.9) reads $\Delta(v) - \Gamma_H(v) = 0$, i.e. $v = FH(\mathcal{L}(v))$, for all $v \in M$. As a consequence, for any $z := X_H(p) \in \mathcal{D}_1$, one has

$$T\pi_Q(z) = FH(p) = FH(\mathcal{L}(v)) = v \in M$$

with $v := \mathcal{L}_1^{-1}(p)$, and then

$$\tau_{T^*Q}(z) = p = \mathcal{L}_1(v) = \mathcal{L}(T\pi_Q(z)).$$

That means $\mathcal{D}_1 = \operatorname{Im} X_H \subset T_2$, i.e.

$$\mathcal{H} = \mathcal{D}_1 = \operatorname{Im} X_H. \tag{4.13}$$

(v) Let us now turn back to the link between $\mathcal{E}$ and $\mathcal{H}$ given by (4.5), that is,

$$\mathcal{E} = (T\mathcal{L})^{-1}(\mathcal{H}) \cap T^2Q. \tag{4.14}$$

Let us also remark that the definition of $\mathcal{H}$ obviously exhibits its second-order character, i.e.

$$\mathcal{H} \subset T_2. \tag{4.15}$$

From (4.14) and (4.15), we shall deduce the following characterization of the integral curves of $\mathcal{H}$. Let k be a smooth curve in T^*Q. Firstly, assume that k is an MPS trajectory, i.e. $k = \mathcal{L} \circ c$ with $\operatorname{Im} \dot{c} \subset \mathcal{E}$. In such a case, from (4.14) one immediately obtains

$$\mathrm{Im}\,\dot{k} = \mathrm{Im}\,(T\mathcal{L} \circ \dot{c}) = T\mathcal{L}(\mathrm{Im}\,\dot{c}) \subset T\mathcal{L}(\mathcal{E}) \subset \mathcal{H},$$

i.e. k is an integral curve of $\mathcal{H}$.

Conversely, assume that k is an integral curve of $\mathcal{H}$, i.e. $\mathrm{Im}\,\dot{k} \subset \mathcal{H}$. Owing to (4.15), one has $\mathrm{Im}\,c \subset M$ and $k = \mathcal{L} \circ c$ with $c := (\pi_Q \circ k)^{\cdot} = T\pi_Q \circ \dot{k}$. Then, from

$$T\mathcal{L}(\mathrm{Im}\,\dot{c}) = \mathrm{Im}\,(T\mathcal{L} \circ \dot{c}) = \mathrm{Im}\,\dot{k} \subset \mathcal{H}$$

and

$$\mathrm{Im}\,\dot{c} = \mathrm{Im}\,(\pi_Q \circ k)^{\cdot\cdot} \subset T^2Q$$

it follows that $\mathrm{Im}\,\dot{c} \subset (T\mathcal{L})^{-1}(\mathcal{H}) \cap T^2Q$. Therefore we obtain $k = \mathcal{L} \circ c$ with $\mathrm{Im}\,\dot{c} \subset \mathcal{E}$, i.e. k is an MPS trajectory.

So *the integral curves of* $\mathcal{H}$ *are precisely the* MPS *trajectories of* (Q, L).

The above result does not extend to the Hamilton-Dirac equation $\mathcal{D}_1$, for the latter—though fulfilling the condition $\mathcal{E} = (T\mathcal{L})^{-1}(\mathcal{D}_1) \cap T^2Q$ because of (4.1) and then admitting the MPS trajectories among its integral curves—need *not* exhibit the second-order character $\mathcal{D}_1 \subset T_2$ which would restrict its integral curves to the above trajectories.

The situation here is exactly the same as in Lagrangian dynamics.

Observe the structure of $\mathcal{H}$, *extracted* from the Hamilton-Dirac equation $\mathcal{D}_1$ via intersection with the 'second-order' equation T_2.

Owing to such a structure, the problem of integrating $\mathcal{H}$ will in principle be solved by determining the integral curves k's of $\mathcal{D}_1$, characterized by the condition $\mathrm{Im}\,\dot{k} \subset \mathcal{D}_1$, and then sorting out those which satisfy the *second-order condition* $\mathrm{Im}\,\dot{k} \subset T_2$.

For instance, in the singular case (a'), a smooth curve k in T^*Q is an integral curve of $\mathcal{D}_1$—or $\widehat{\mathcal{D}}_1$, given by (4.3)—iff it satisfies the constraint

$$\phi \circ k = \mu \tag{4.16}$$

and there exists an m-tuple $\Lambda = (\Lambda_1, \ldots, \Lambda_m)$ of time-dependent Lagrange multipliers such that

$$\dot{k} = X_H \circ k + \Lambda(X_\phi \circ k) \tag{4.17}$$

(conditions (4.16) and (4.17) exactly correspond, in coordinate formalism, to Dirac's equations [8, 9] of generalized Hamiltonian dynamics).

However, such a k will be an MPS trajectory of the system, i.e. an integral curve of $\mathcal{H}$—or $\widehat{\mathcal{H}}$, given by (4.8)—iff it satisfies the stronger Legendre condition

$$\mathrm{Im}\,\dot{\gamma} \subset M, \quad k = \mathcal{L} \circ \dot{\gamma} \tag{4.18}$$

with

$$\gamma := \pi_Q \circ k$$

coupled with Dirac's condition (4.17).

If (a') is replaced by (a"), the MPS trajectories will be characterized by Legendre's condition (4.18) coupled with a version of (4.17) where the unknown multipliers Λ no longer appear, namely

$$\dot{k} = X_H \circ k + (J \circ \dot{\gamma})(X_\phi \circ k) \tag{4.19}$$

(see [5] for a deduction of (4.19) in coordinate formalism).

Finally note that, in the hyperregular case, the second order condition $\operatorname{Im}\dot{k} \subset T_2$ is *hidden* by the circumstance $\mathcal{D}_1 = \operatorname{Im} X_H \subset T_2$.

As a consequence, the MPS trajectories turn out to be characterized by the only condition $\operatorname{Im}\dot{k} \subset \mathcal{D}_1$, which takes the normal form $\dot{k} = X_H \circ k$ (with $\operatorname{Im} k \subset M_1$).

5. Hamiltonian dynamics after Tulczyjew. Tulczyjew's approach to Hamiltonian dynamics—based on a more general idea of Legendre transformation—will be related to (the revised version of) Dirac's.

(i) According to Tulczyjew (see, e.g., Tulczyjew and Urbański [19]), Legendre transformation is the mapping $L \mapsto \widetilde{H}$ which takes any Lagrangian function L, defined on an open manifold $M \subset TQ$, onto the Hamiltonian Morse family $\widetilde{H}$ defined on $Y := \pi_1^{-1}(M) \subset TQ \oplus T^*Q$ (π_1 being the natural projection of $TQ \oplus T^*Q$ onto TQ) by putting

$$\widetilde{H} : Y \to \mathbb{R} : y = (v,p) \mapsto \widetilde{H}(y) := \langle p \mid v \rangle - L(v).$$

Let $\Sigma := \{y \in Y \mid d\widetilde{H}(y) \in V^o\rho\}$ be the critical set of $\widetilde{H}$ with respect to $\rho := \pi_2|_Y : Y \to T^*Q$ (π_2 being the natural projection of $TQ \oplus T^*Q$ onto T^*Q).

If $\operatorname{Graph}\mathcal{L} : M \to Y : v \mapsto (v, \mathcal{L}(v))$ is the graph of the Legendre morphism $\mathcal{L} := FL$, Σ turns out to be given by

$$\Sigma = \operatorname{Im}\operatorname{Graph}\mathcal{L}.$$

By composing $d\widetilde{H}|_\Sigma : \Sigma \to V^o\rho$ with $\varpi_\rho : V^o\rho \to T^*T^*Q$, we obtain

$$h := \varpi_\rho \circ d\widetilde{H}|_\Sigma : \Sigma \to T^*T^*Q$$

(a section of π_{T^*Q} along $\rho|_\Sigma$), which is transformed by $\beta^{-1} : T^*T^*Q \to TT^*Q$ into a 'Hamiltonian field'

$$X_h := \beta^{-1} \circ h : \Sigma \to TT^*Q$$

(a section of τ_{T^*Q} along $\rho|_\Sigma$), satisfying

$$X_h \circ \operatorname{Graph}\mathcal{L} = \alpha^{-1} \circ dL.$$

With reference to a mechanical system (Q, L), the equation of dynamics (in Hamiltonian form) proposed in [19] is

$$\mathcal{T} := \operatorname{Im} X_h.$$

$\mathcal{T}$ will be called the *Tulczyjew equation*.

(ii) The Tulczyjew equation can be given a number of expressions.

Start off with the definition itsef, i.e.

$$\mathcal{T} = \{z \in TT^*Q \mid \exists\, v \in M : z = X_h(v, \mathcal{L}(v))\}.$$

Then remark that, for any $z \in \mathcal{T}$, one has

$$T\pi_Q(z) = (T\pi_Q \circ X_h \circ \operatorname{Graph}\mathcal{L})(v) = (T\pi_Q \circ \alpha^{-1} \circ dL)(v) = (\pi_{TQ} \circ dL)(v) = v \in M$$

whence

$$\mathcal{T} = \{z \in TT^*Q \mid T\pi_Q(z) \in M,\ z = X_h(T\pi_Q(z), \mathcal{L}(T\pi_Q(z)))\}. \tag{5.1}$$

Moreover, any $z \in \mathcal{T}$ satisfies

$$\tau_{T^*Q}(z) = (\tau_{T^*Q} \circ X_h \circ \operatorname{Graph} \mathcal{L})(T\pi_Q(z)) = (\rho \circ \operatorname{Graph} \mathcal{L})(T\pi_Q(z)) = \mathcal{L}(T\pi_Q(z))$$

whence

$$\mathcal{T} = \{z \in TT^*Q \,|\, T\pi_Q(z) \in M,\ \tau_{T^*Q}(z) = \mathcal{L}(T\pi_Q(z)),\ z = X_h(T\pi_Q(z), \tau_{T^*Q}(z))\}. \tag{5.2}$$

(iii) In order to find out the link between the Euler-Lagrange equation $\mathcal{E}$ and the Tulczyjew equation $\mathcal{T}$, we shall try again the operation of transforming $\mathcal{E}$ by $T\mathcal{L}$ (without making use, this time, of any additional hypothesis).

Let $x \in T^2_M Q$. As is known, $x \in \mathcal{E}$ iff $dE(\tau(x)) - {}^\flat x = 0$, where $\tau := \tau_M \circ j : T^2_M Q \hookrightarrow TM \to M$ is the bundle projection of $T^2_M Q$ onto M. We shall reexpress the above condition in terms of $z := T\mathcal{L}(x)$.

To that end, it will prove to be useful to focus on the 'pull-back' of $dE(\tau(x)) - {}^\flat x$ by τ, i.e.

$$\begin{aligned}
(dE(\tau(x)) - {}^\flat x) \circ T_x\tau &= dE(\tau(x)) \circ T_x\tau - {}^\flat x \circ T_x\tau \\
&= d\Delta L(\tau(x)) \circ T_x\tau - dL(\tau(x)) \circ T_x\tau - i_x\omega \circ T_x\tau_M \circ T_x j \\
&= \tau^* d\Delta L(x) - dL(\tau(x)) \circ T_x\tau - j^* i_T\omega(x) \\
&= (d\tau^*\Delta L - j^* i_T\omega)(x) - dL(\tau(x)) \circ T_x\tau.
\end{aligned}$$

As to $(d\tau^*\Delta L - j^* i_T\omega)(x)$, we first remark that

$$\begin{aligned}
\tau^*\Delta L = \Delta L \circ \tau = \langle dL \circ \tau \mid \Delta \circ \tau\rangle &= \langle dL \circ \tau_M \circ j \mid S \circ j\rangle \\
&= \langle d_S L \circ \tau_M \circ j \mid j\rangle = (i_T d_S L) \circ j = j^* i_T d_S L
\end{aligned}$$

whence

$$\begin{aligned}
d\tau^*\Delta L - j^* i_T\omega = j^* d i_T d_S L + j^* i_T d d_S L &= j^* d_T d_S L = j^* d_T \mathcal{L}^* \vartheta_Q \\
&= j^*(T\mathcal{L})^* d_T \vartheta_Q = j^*(T\mathcal{L})^* \alpha^* \vartheta_{TQ} = (\alpha \circ T\mathcal{L} \circ j)^* \vartheta_{TQ}
\end{aligned}$$

and then

$$\begin{aligned}
(d\tau^*\Delta L - j^* i_T\omega)(x) &= \vartheta_{TQ}(\alpha(z)) \circ T_x(\alpha \circ T\mathcal{L} \circ j) = \alpha(z) \circ T_{\alpha(z)}\pi_{TQ} \circ T_x(\alpha \circ T\mathcal{L} \circ j) \\
&= \alpha(z) \circ T_x(\pi_{TQ} \circ \alpha \circ T\mathcal{L} \circ j) \\
&= \alpha(z) \circ T_x(T\pi_Q \circ T\mathcal{L} \circ j) = \alpha(z) \circ T_x(T\tau_Q \circ j) = \alpha(z) \circ T_x\tau.
\end{aligned}$$

As to $dL(\tau(x)) \circ T_x\tau$, we just recall from Sec. 4(iii) that $T\pi_Q(z) = \tau(x) \in M$. So we obtain

$$(dE(\tau(x)) - {}^\flat x) \circ T_x\tau = (\alpha(z) - dL(T\pi_Q(z))) \circ T_x\tau,$$

that is, τ being a submersion,

$$dE(\tau(x)) - {}^\flat x = \alpha(z) - dL(T\pi_Q(z)).$$

Therefore, condition $dE(\tau(x)) - {}^\flat x = 0$ reads

$$\alpha(z) = dL(T\pi_Q(z)), z = (\alpha^{-1} \circ dL)(T\pi_Q(z)), z = X_h(T\pi_Q(z), \mathcal{L}(T\pi_Q(z))).$$

In view of (5.1), we have proved that

$$x \in \mathcal{E} \Leftrightarrow x \in T^2_M Q,\ T\mathcal{L}(x) \in \mathcal{T}.$$

(iv) The above link between $\mathcal{E}$ and $\mathcal{T}$ reads

$$\mathcal{E} = (T\mathcal{L})^{-1}(\mathcal{T}) \cap T^2Q. \tag{5.3}$$

Moreover, expression (5.3) naturally exhibits the second-order character of $\mathcal{T}$, i.e.

$$\mathcal{T} \subset T_2. \tag{5.4}$$

Observe that properties (5.3) and (5.4) are exactly the same as those encountered in Sec. 4(v). Therefore, by proceeding in the same way as we did there, from (5.3) we infer that the MPS trajectories are integral curves of $\mathcal{T}$, and then from (5.4) we infer the converse.

So *the integral curvers of* $\mathcal{T}$ *are precisely the* MPS *trajectories of* (Q, L).

From the expressions of $\mathcal{T}$, it then follows that a smooth curve k in T^*Q is an MPS trajectory iff it satisfies

$$\dot{k} = X_h \circ (c,\, \mathcal{L} \circ c)$$

for some curve c in M, or, putting $\gamma := \pi_Q \circ k$,

$$\operatorname{Im} \dot{\gamma} \subset M,\ \dot{k} = X_h \circ (\dot{\gamma}, \mathcal{L} \circ \dot{\gamma})$$

or

$$\operatorname{Im} \dot{\gamma} \subset M,\ k = \mathcal{L} \circ \dot{\gamma},\ \dot{k} = X_h \circ (\dot{\gamma}, k).$$

(v) Clearly, under the hypotheses (a), (b) and (c) of Sec. 4, the equation $\mathcal{H}$ (or $\widehat{\mathcal{H}}$)—but not generally the Hamilton-Dirac equation $\mathcal{D}_1$—is an equivalent reformulation of $\mathcal{T}$, for they share the integral curves.

More precisely, $\widehat{\mathcal{H}}$ is just an 'enlarged' version of $\mathcal{T}$, as will now be shown. The starting point is the obvious equality

$$E = (\operatorname{Graph} \mathcal{L})^* \widetilde{H}$$

whence, for any $v \in M$ and putting $p := \mathcal{L}(v)$,

$$\begin{aligned} dE(v) &= d\widetilde{H}(v,p) \circ T_v \operatorname{Graph} \mathcal{L} = h(v,p) \circ T_{(v,p)}\rho \circ T_v \operatorname{Graph} \mathcal{L} \\ &= h(v,p) \circ T_v(\rho \circ \operatorname{Graph} \mathcal{L}) = h(v,p) \circ T_v\mathcal{L}. \end{aligned}$$

On the other hand, from the hypothesis $E = \mathcal{L}^* H$ it follows that $dE(v) = dH(p) \circ T_v\mathcal{L}$. Hence, $T_v\mathcal{L}$ having been assumed to be surjective onto T_pM_1 ,

$$\begin{aligned} &h(v,p)|_{T_pM_1} = dH(p)|_{T_pM_1} \\ &h(v,p) - dH(p) \in (T_pM_1)^o \\ &X_h(v,p) - X_H(p) \in {}^{\beta^{-1}}(T_pM_1)^o. \end{aligned} \tag{5.5}$$

In view of (5.5), a comparison between (4.7) and (5.2) immediately yields

$$\mathcal{T} \subset \widehat{\mathcal{H}}. \tag{5.6}$$

Focus in particular on the singular case (a"). Owing to (4.12), for any $z \in \widehat{\mathcal{H}}$ one has

$$z = X_H(p) + J(v)X_\phi(p)$$

(with $v := T\pi_Q(z)$ and $p := \tau_{T^*Q}(z) = \mathcal{L}(v)$), whence, by applying $T\pi_Q$,

$$v = FH(p) + J(v)F\phi(p).$$

Owing to (5.5), one also has

$$X_h(v,p) = X_H(p) + \lambda X_\phi(p).$$

(with $\lambda \in \mathbb{R}^m$), whence, by applying $T\pi_Q$,

$$v = FH(p) + \lambda F\phi(p).$$

Owing to (a”), we then obtain $J(v) = \lambda$ and then $z = X_h(v,p)$ that is, $z \in \mathcal{T}$. In view of (5.6), the above result means

$$\mathcal{T} = \widehat{\mathcal{H}}. \tag{5.7}$$

Of course, in the hyperregular case (when $(T_pM_1)^o = \{0\}$ for all $p \in M_1$), property (5.5) reads $X_h \circ \mathrm{Graph}\,\mathcal{L} = X_H \circ \mathcal{L}$ and then—as well as (4.13)—we have $\mathcal{T} = \mathrm{Im}\,X_H$.

6. Relativistic dynamics. Relativistic particle dynamics will exhibit a Hamiltonian setting where one can effectively contrast Hamilton-Dirac with Tulczyjew.

(i) The space-time of General Relativity is a 4-dimensional smooth manifold Q equipped with a Lorentz metric tensor

$$g : TQ \to T^*Q$$

(symmetric vector bundle isomorphism of signature $+,-,-,-$).

The causal character of g allows one to distinguish the time-like vectors (i.e. the elements $v \in TQ$ satisfying $\langle g(v) \mid v\rangle > 0$) and particularly, under the hypothesis of time-orientability (i.e. existence of a time-like vector field ζ on Q), the future-pointing vectors, sweeping the open submanifold (of TQ)

$$M := \{v \in TQ \mid \langle g(v) \mid v\rangle > 0,\, \langle g(v) \mid \zeta(\tau_Q(v))\rangle < 0\}.$$

A future-pointing, time-like, smooth curve γ in Q (i.e. one with $\mathrm{Im}\,\dot\gamma \subset M$), together with all of its orientation-preserving reparametrizations, determines an oriented orbit $\mathrm{Im}\,\gamma$, which is meant to be the world line of a material particle.

As is known, the curvature tensor of a Lorentz metric g on Q represents a gravitational field, whereas the exterior derivative of a 1-form A on Q represents an electromagnetic field.

We shall be concerned with the problem of determining the possible world lines of a particle (m,e) of proper mass $m > 0$ and electric charge $e \in \mathbb{R}$, living in the gravitational and electromagnetic fields (g,A).

Such a problem will be framed into the Hamiltonian dynamics of a system (Q,L), whose Lagrangian L, defined on the above open submanifold M of TQ, is given by the relativistic Lagrangian of a mass m minus the generalized potential of the electromagnetic force field acting on a charge e (see [3]), i.e.

$$L := m\sqrt{2K} + e\, i_T|_M A$$

with

$$K : M \to \mathbb{R} : v \mapsto K(v) := \frac{1}{2}\,|v|^2 := \frac{1}{2}\,\langle g(v) \mid v\rangle$$

and

$$i_T|_M(\cdot) := (i_T(\cdot))|_M.$$

(ii) We shall prove that L fulfils the almost-regularity conditions (a"), (b) and (c).

To that end, we focus on the Legendre morphism $\mathcal{L} := FL$. As

$$\mathcal{L}(v) = dL(v) \circ \nu_v = \frac{m}{|v|}\, dK(v) \circ \nu_v + e\, d\, i_T A(v) \circ \nu_v = \frac{m}{|v|}\, g(v) + e\, A(\tau_Q(v))$$

for all $v \in M$, we obtain

$$\mathcal{L} = \frac{m}{\sqrt{2K}}\, g|_M + e\, A \circ \tau_Q|_M.$$

The geometric structure of $M_1 := \operatorname{Im} \mathcal{L}$ will emerge from the following considerations. Put

$$\psi := \mathrm{id}_{T^*Q} - e\, A \circ \pi_Q : T^*Q \to T^*Q$$

and, on the open manifold $W := \psi^{-1}(g(M))$ of T^*Q, define

$$\phi := 2K \circ g^{-1} \circ \psi|_W : W \to \mathbb{R}.$$

A direct calculation would show that

$$F\phi = 2g^{-1} \circ \psi|_W.$$

Remark that, as $F\phi$ takes values in M, one has

$$F\phi(p) \neq 0,\ \forall\, p \in W. \tag{6.1}$$

Also remark that, if $p \in M_1$ (i.e. $p = \mathcal{L}(v) = \frac{m}{|v|}\, g(v) + e\, A(\tau_Q(v))$ for some $v \in M$), one has

$$\psi(p) = g\left(\frac{m}{|v|}\, v\right) \in g(M),$$

i.e. $p \in W$, and

$$\phi(p) = 2K(g^{-1}(\psi(p))) = 2K\left(\frac{m}{|v|}\, v\right) = m^2.$$

Conversely, if $p \in W$ and $\phi(p) = m^2$, one has

$$v := \frac{1}{m}\, g^{-1}(\psi(p)) \in M,\ |v| = 1$$

and

$$\mathcal{L}(v) = m\, g(v) + e\, A(\tau_Q(v)) = \psi(p) + e\, A(\pi_Q(p)) = p,$$

i.e. $p \in M_1$. So

$$M_1 = \phi^{-1}(m^2) \subset W. \tag{6.2}$$

From (6.1) and (6.2), it follows that condition (a") is fulfilled.

Now some information about $T\mathcal{L}$ will be obtained by taking a look at the Euler-Lagrange equation $\mathcal{E} = (T\mathcal{L})^{-1}(\mathcal{T}) \cap T^2Q$, i.e.

$$\mathcal{E} = \{x \in TM \mid T_v\mathcal{L}(x) = \alpha^{-1}(dL(v)),\ v = \tau_M(x)\}.$$

In the present case, $\mathcal{E}$ takes the form [2, 3]

$$\mathcal{E} = \{x \in TM \mid \exists\, \lambda \in \mathbb{R} : x = \Gamma(v) + \lambda\Delta(v),\ v := \tau_M(x)\}$$

(where Γ is the vector field on M characterized by $i_\Gamma\omega_K = dK + \frac{e}{m}\sqrt{2K}\, i_T|_M dA$, ω_K being the symplectic Poincaré-Cartan 2-form of K).

From the above expressions of $\mathcal{E}$, one infers that, for any $v \in M$, $\mathcal{E}_v := \mathcal{E} \cap T_vM$ (containing $\Gamma(v)$) is an affine space modelled on

$$\ker T_v\mathcal{L} = \operatorname{Span}\Delta(v). \tag{6.3}$$

From (6.1)-(6.3), it immediately follows that condition (b) is fulfilled as well.

As to the energy E of L, just remark that $E = 0$ since

$$\begin{aligned} E(v) &= \langle \mathcal{L}(v) \mid v\rangle - L(v) \\ &= \frac{m}{|v|}\langle g(v) \mid v\rangle + e\,\langle A(\tau_Q((v)) \mid v\rangle - m\,|v| - e\, i_T A(v) = 0 \end{aligned}$$

for all $v \in M$. As a consequence, condition (c) is obviously satisfied by taking $H = 0$.

(iii) We can now turn to the Hamiltonian dynamics of (Q, L) and examine the equations therein appearing.

First remark that, in the present case, one obviously has

$$X_H = 0, \quad X_\phi(M_1) \subset TM_1.$$

Therefore, owing to (4.2) and (4.3), the Hamilton-Dirac equation $\mathcal{D}_1 = \{\, z \in TM_1 : i_z\omega_1 = 0\,\}$, the characteristic distribution of ω_1, takes the form

$$\mathcal{D}_1 = \widehat{\mathcal{D}}_1 = \{\, z \in TT^*Q \mid p := \tau_{T^*Q}(z) \in \phi^{-1}(m^2/2),\ \exists\,\lambda \in \mathbb{R} : z = \lambda X_\phi(p)\}.$$

So $\mathcal{D}_1$ is the 1-dimensional distribution spanned by $X_\phi|_{M_1} \in \chi(M_1)$.

Now observe the identities

$$\Gamma_H = 0, \quad \Gamma_\phi = \frac{2m}{\sqrt{2K}}\,\Delta$$

(the last one being due to $\Gamma_\phi(v) := \nu_v(F\phi(\mathcal{L}(v))) = \nu_v(2g^{-1} \circ \psi \circ \mathcal{L}(v)) = \nu_v(\frac{2m}{|v|}\,v) = \frac{2m}{|v|}\,\Delta(v)$ for all $v \in M$), owing to which equality (4.10) is satisfied by

$$J = \frac{\sqrt{2K}}{2m}.$$

Also note that, for any $z \in TT^*Q$ satisfying

$$p := \tau_{T^*Q}(z) \in \phi^{-1}(m^2), \quad z = \lambda\, X_\phi(p)$$

with $\lambda > 0$ one has $v := T\pi_Q(z) = \lambda\, F\phi(p) \in M$ (whence $\sqrt{2K(v)} = \lambda\sqrt{2K \circ F\phi(p)} = 2\lambda\sqrt{\phi(p)} = 2\lambda\, m$) and then

$$\begin{aligned} \mathcal{L}(v) &= \frac{m}{\sqrt{2K(v)}}\, g(v) + e\, A \circ \tau_Q(v) = \psi(p) + e\, A \circ \pi_Q(p) = p, \\ J(v) &= \frac{1}{2m}\,\sqrt{2K(v)} = \lambda. \end{aligned}$$

As a consequence, from (4.6) and (4.12), we obtain

$$\mathcal{H} = \widehat{\mathcal{H}} = \{\, z \in TT^*Q \mid p := \tau_{T^*Q}(z) \in \phi^{-1}(m^2/2),\ \exists\,\lambda > 0 : z = \lambda\, X_\phi(p)\,\}.$$

which, owing to (5.7), is the expression of the Tulczyjew equation $\mathcal{T}$ as well.

So $\mathcal{T} = \mathcal{H}$ is the 'future-pointing' component of $\mathcal{D}_1 - \mathcal{D}_1^o$ (where $\mathcal{D}_1^o$ denotes the null section of $\mathcal{D}_1$), i.e. the one containing $\operatorname{Im} X_\phi|_{M_1}$.

(iv) Some comments on the integral curves of $\mathcal{D}_1$ and $\mathcal{T}$ are now in order.

Firstly, it is clear that not all of the integral curves of $\mathcal{D}_1$ are MPS trajectories of (Q, L), for the integral curves of both $\mathcal{D}_1^o$ and the 'past-pointing' component of $\mathcal{D}_1 - \mathcal{D}_1^o$ are *not* integral curves of $\mathcal{T}$.

Then focus on $X_1 := \frac{1}{2m} X_\phi|_{M_1} \in \chi(M_1)$. An integral curve k of $\mathcal{T}_1 := \operatorname{Im} X_1 \subset \mathcal{T}$ satisfies

$$\phi \circ k = m^2, \quad \dot{k} = X_1 \circ k \tag{6.4}$$

and then the corresponding base integral curve $\gamma = \pi_Q \circ k$ is parametrized in such a way that its tangent lifting $\dot\gamma := T\pi_Q \circ \dot{k}$ fulfils the causal condition $\operatorname{Im} \dot\gamma \subset M$ with

$$|\dot\gamma| = \sqrt{2K \circ T\pi_Q \circ X_1 \circ k} = \frac{1}{2m} \sqrt{2K \circ F\phi \circ k} = \frac{1}{m} \sqrt{\phi \circ k} = 1$$

(such a parametrization is called proper time).

The base integral curves of $\operatorname{Im} X_1$—which are the same as those of $\operatorname{Im} \Gamma|_C$ (with $C := \{v \in M : |v| = 1\}$), since $T\mathcal{L} \circ \Gamma|_C = X_1 \circ \mathcal{L}$—have been shown [3] to be the (possible) life histories of the particle (i.e. the smooth curves of Q satisfying the standard laws of relativistic dynamics [15] for a particle (m, e) living in (g, A)).

As to the whole family of integral curves of $\mathcal{T}$, it is set up by precisely the orientation-preserving reparametrizations of the integral curves of $\mathcal{T}_1$.

Indeed, let k and $\chi := k \circ s$ be smooth curves in T^*Q related to each other by a reparametrization s with derivative $s' > 0$ (their tangent liftings are related to each other by $\dot\chi = s'(\dot{k} \circ s)$). Then remark that $\operatorname{Im} \dot{k} \subset \mathcal{T}_1$, i.e. (6.4), is equivalent to $\operatorname{Im} \dot\chi \subset \mathcal{T}$, i.e.

$$\phi \circ \chi = m^2, \quad \dot\chi = \Lambda(X_\phi \circ \chi)$$

with $\Lambda > 0$, if $s' = 2m\Lambda$.

The above result shows that the integral curves of $\mathcal{T}$ just determine a family of oriented orbits in T^*Q—carrying no distinguished parametrization—which project down by π_Q onto the possible world-lines (i.e. the oriented orbits of the possible life histories) of the particle.

7. Concluding remarks. In conclusion, we have been drawing a methodological line—based on the use of Legendre morphism—for deducing, in geometric terms, the Hamiltonian side of dynamics from the Lagrangian side.

A focal result is to have shown, by following such a line, that the geometric structure of Lagrangian dynamics is *shared*—in the almost-regular case—by Hamiltonian dynamics, both being governed by the *Hamilton-Dirac equations* $\mathcal{D}$ and $\mathcal{D}_1$ (on suitable Dirac manifolds) *restricted* to second-order equations via intersection with T^2Q and T_2, respectively.

For regular systems, we have seen that the second-order character of the equations $\mathcal{D} \cap T^2Q$ and $\mathcal{D}_1 \cap T_2$ is *hidden* by their reducing to $\mathcal{D}$ and $\mathcal{D}_1$, which automatically satisfy $\mathcal{D} \subset T^2Q$ and $\mathcal{D}_1 \subset T_2$.

For singular systems, the Hamilton-Dirac equations $\mathcal{D}$ and $\mathcal{D}_1$ may *not* fulfil the above second-order conditions (as in fact occurs in Relativity) and then fail to express—on their own—the laws of dynamics.

A consequence is that, on the one hand, the Hamilton-Dirac equations are the common area where problems of Lagrangian and Hamiltonian dynamics—such as integrability, symmmetries and conserved momentum mappings, reductions and reconstructions—can firstly be treated, but, on the other hand, all of the possible results should then be adapted to real dynamics by taking the restriction to second-order into due consideration.

The second-order character of the equation of dynamics—well known, in a form or another, on the Lagrangian side—had never been highlighted before (as far as we know) on the Hamiltonian side.

A further confirmation of such a character comes from the analysis of the *Tulczyjew's equation* $\mathcal{T} := \mathrm{Im}\,(\alpha^{-1} \circ dL) = \mathrm{Im}\,(\beta^{-1} \circ h)$, which has proved to be the law of dynamics for every (regular or singular) Lagrangian. As $\mathcal{T} \subset T_2$, the Tulczyjew equation is indeed second-order, and that is why—in the almost-regular case—it turns out to be equivalent to $\mathcal{D}_1 \cap T_2$ rather than $\mathcal{D}_1$.

Owing to its generality and to the fact of being independently generated by the Lagrangian and the corresponding Hamiltonian (through α and β, respectively), the Tulczyjew equation is the ideal candidate for being assumed—in some extended version—as the basic principle of both Lagrangian and Hamiltonian dynamics for more general types of constrained systems (described, e.g., by singular Lagrangians, nonpotential force fields, nonholonomic constraints and constraint reactions), as will be shown in a forthcoming paper [4].

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