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THE JACOBI VARIATIONAL PRINCIPLE REVISITED

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Dedicated to Włodzimierz Tulczyjew on his 70th birthday

1. Introduction. One of the key problems of the 18^{th} century dynamics was finding procedures which could facilitate the integration of systems of Newtonian equations of motion. These procedures turned out to be especially effective and universal for systems which admitted integrals of the motion. Some dynamical variables could then be eliminated from the system and, as a result of it, the number of equations left to be solved was adequately reduced. In the 19^{th} century a new problem emerged in this context. If the original Newtonian equations were Euler-Lagrange equations of a certain Lagrangian L, was it always true that the reduced equations could be brought to an Euler-Lagrange form? And if yes, how could one find the new Lagrangian in terms of the original one? In the particular case when the existence of a first integral was a consequence of the fact that one of the generalized coordinates was a cyclic variable of the original Lagrangian, the question was answered positively by Routh in 1876.

About ten years later Jacobi studied another problem of this kind, by then yet unsolved on the variational level. It was the problem of a Lagrangian that did not depend explicitly on the time t. Jacobi showed that with the help of the energy conservation law the original action could be reduced to another action which described only the spatial path of the dynamical system. His original proof was based on a very general variational principle that for the first time had been formulated by L. Euler, who called it the Maupertuis principle, and later was developed by J. L. Lagrange, about one hundred years before Jacobi's work. A result of the choice of the method of the original proof was that in many text-books the Jacobi principle was presented under the name of the Maupertuis principle, without even making any mention to the original principle of that name at all. According to Arnold [1], Jacobi complained that his principle was presented, in almost

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all text-books known to him, in a non-understandable way. I agree with Arnold's opinion that this tradition is continued up to the present. The aim of this lecture is to present some of the results of my work on this subject, which in my opinion shed new light on the Jacobi principle.

Section 2 contains a short review of the Routh procedure of eliminating a cyclic variable from the Lagrange equations of motion. This review is presented in order to fix the framework which will be employed in the next sections. The procedure described here is just a simple by-product of the general Routh formalism whose details may be found in some texts on analytical dynamics; for further references see e.g. [4].

In Section 3 the case is studied of a dynamical system whose Lagrangian L does not depend explicitly on time. It is demonstrated that the action of the system entering the usual Hamilton principle can be brought to a form which enables one to employ the Routh procedure, in order to eliminate from the action integral the piece of information about the temporal evolution of the system. As a result of such elimination, the Hamilton principle is reduced to the Jacobi principle whose Euler-Lagrange equations determine only the spatial trajectories of the system.

Since the Jacobi Lagrangian is a homogeneous function of degree one in the velocities, in Section 4 first a review of some properties of the Euler-Lagrange equations with a general homogeneous Lagrangian of such a kind is presented. In particular, the freedom of lifting solutions from the configuration space to the space of states is discussed. Next, conclusions of this discussion are used to interpret the procedure of solving the complete dynamical Lagrange equations as lifting the trajectories that are solutions to the Jacobi problem from the configuration space to world lines in the space of states.

In Section 5 the same problem of lifting geometrical trajectories to world lines is presented on the level of variational principles. Technically, it resolves itself to the inverse Jacobi problem discussed in [2] which solves the following question. Given any Lagrange function L_H homogeneous of degree one in the velocities and a function G of positions and velocities, how one can find a Lagrange function L such that G is its energy function and L_H its Jacobi Lagrangian?

In the text which follows, an abbreviated notation is used, in accordance with which expressions like e.g. $(q^i, \dot{q}^j)$ stand for sequences $(q^1, q^2, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^n)$ or, depending on ranges in which the indices vary, for some other sequences of a similar type. The summation convention is employed throughout the article.

2. Routh's procedure. Let us consider a dynamical system whose action is

$$\mathcal{W}[q^\alpha] = \int_{t_1}^{t_2} \mathcal{L}(q^i(t), \dot{q}^\beta(t), t)\, dt, \tag{2.1}$$

where t is the time, $\alpha, \beta = 0, 1, \ldots, n$ and $q^\alpha = q^\alpha(t)$ is the motion of the system in a configuration space $\mathbb{Q}^{n+1}$. It is assumed that the Lagrangian $\mathcal{L}$ is non-degenerate and the variable q^0 is cyclic, i.e., $\partial\mathcal{L}/\partial q^0 = 0$. The corresponding conservation law

$$p_0 = \frac{\partial \mathcal{L}}{\partial \dot{q}^0} = \Gamma(\dot{q}^0(t), q^i(t), \dot{q}^j(t), t) = \text{const}, \tag{2.2}$$

where $i, j = 1, 2, \dots, n$, permits us to reduce the number of independent variables to $(q^i, \dot{q}^j)$ and, moreover, to demonstrate that the new dynamical system, described by the reduced set of variables, has again a Lagrangian which can be found as a result of the Routh procedure. In accordance with this procedure, Eq. (2.2) should at first be solved with respect to the variable $\dot{q}^0$, which leaves us with a relationship $\dot{q}^0(t) = \phi(p_0, q^i(t), \dot{q}^j(t), t)$, where p_0 is an arbitrary, but fixed, value of the integration constant from (2.2). Next, the Routh function $\mathcal{R}_{p_0}$, parametrized by the values of p_0, ought to be determined as

$$\mathcal{R}_{p_0}(q^i, \dot{q}^j, t) = \mathcal{L}(q^i, \phi(p_0, q^j, \dot{q}^k), \dot{q}^l, t) - \dot{q}^0 p_0. \tag{2.3}$$

In virtue of (2.2), the function $\mathcal{R}_{p_0}$ does not depend on $\dot{q}^0$. Furthermore, because of (2.2), the following equalities are valid:

$$\frac{\partial \mathcal{R}_{p_0}}{\partial q^i} = \frac{\partial \mathcal{L}}{\partial q^i}, \qquad \frac{\partial \mathcal{R}_{p_0}}{\partial \dot{q}^i} = \frac{\partial \mathcal{L}}{\partial \dot{q}^i}.$$

Thus the action of the reduced dynamical system is

$$\mathcal{W}_{p_0}[q^i] = \int_{t_1}^{t_2} \mathcal{R}_{p_0}(q^i(t), \dot{q}^j(t), t)dt, \tag{2.4}$$

where p_0 ought to be treated as a parameter. The Hamilton principle based on this action leads one to the Euler-Lagrange equations on $q^i = q^i(p_0, t)$. Knowing explicitly a solution of these equations, the corresponding function $q^0 = q^0(p_0, t)$ can then be determined by solving the differential equation

$$\dot{q}^0 = -\frac{\partial \mathcal{R}_{p_0}}{\partial p_0} = \tilde{\phi}(p_0, t), \tag{2.5}$$

where the function $\tilde{\phi}(p_0, t)$ is a solution of the equation

$$\Gamma(\tilde{\phi}(p_0, t), q^i(t), \dot{q}^j(t), t) = p_0 \tag{2.6}$$

into which the now known functions $q^i(t)$ and $\dot{q}^j(t)$ should be substituted.

3. The Jacobi variational principle. Let us consider now a class of non-degenerate Lagrangians $L(q^i, \dot{q}^j)$ which do not explicitly depend on time. In such a case, as is known, the Lagrange equations imply the energy conservation law $G(q^i, \dot{q}^j) = E$, where

$$G(q^i, \dot{q}^j) = \dot{q}^i \frac{\partial L}{\partial \dot{q}^i} - L \tag{3.1}$$

is the energy function, and E is the energy constant.

In order to be able to apply the Routh formalism to the present case, we transform the original action

$$W[q] = \int_{t_1}^{t_2} L(q^i(t), \dot{q}^j(t))dt, \tag{3.2}$$

which defines a dynamical system of n degrees of freedom, to the form

$$\mathcal{W}[\theta, x^i] = \int_{\tau_1}^{\tau_2} \Lambda(x^i(\tau), \theta'(\tau), x'^j(\tau))d\tau, \tag{3.3}$$

which in turn defines a system of $n+1$ degrees of freedom described by $n+1$ independent variables (θ, x^i) which are functions of a parameter τ. Primes are used in (3.3) to denote differentiation with respect to τ. A transformation of this kind can be achieved just by assuming that the time t is a monotonic function of another parameter τ, i.e. $t = \theta(\tau)$, with $\theta'(\tau) \neq 0$ everywhere. Introducing then the notation

$$x^i(\tau) = q^i(\theta(\tau)), \quad \text{and, as a result,} \quad x'^i(\tau) = \dot{q}^i(\theta(\tau))\,\theta'(\tau), \tag{3.4}$$

we can easily rewrite the action (3.2) in the form (3.3), where

$$\Lambda(x^i(\tau), \theta'(\tau), x'^j(\tau)) = L\left(x^i(\tau), \frac{x'^j(\tau)}{\theta'(\tau)}\right)\theta'(\tau). \tag{3.5}$$

The new Lagrangian Λ is a homogeneous function of degree one in the variables (θ', x'^i). The appropriate Hamilton variational principle will thus lead us to n independent differential equations of motion regardless of the fact that the system is described by $n+1$ dynamical variables. The Lagrangian Λ does not explicitly depend on θ. Therefore, this variable plays here the same role as q^0 does in the case of the Lagrangian $\mathcal{L}$ of the previous section. Now, due to Eqs. (3.5) and (3.1), the counterpart of Eq. (2.2) reads as

$$\begin{aligned} p_0 &= \frac{\partial \Lambda}{\partial \theta'} \\ &= L\left(x^i(\tau), \frac{x'^j(\tau)}{\theta'(\tau)}\right) - \frac{x'^k(\tau)}{\theta'(\tau)}\frac{\partial L}{\partial \dot{q}^k}\left(x^i(\tau), \frac{x'^j(\tau)}{\theta'(\tau)}\right) = -G\left(x^i(\tau), \frac{x'^j(\tau)}{\theta'(\tau)}\right). \end{aligned} \tag{3.6}$$

Comparing Eq. (3.6) with (2.2), we see that now $\Gamma = \tilde{\Gamma}(\theta', x^i, x'^j) = -G(x^i, \frac{x'^j}{\theta'})$, and as a result of the conservation law $G(q^i, \dot{q}^j) = E$, we have $p_0 = -E$. Therefore, in order to find the generalized velocity $\dot{q}^0 = \theta'$ as a function of p_0 and of the remaining dynamical variables x^i, x'^j, we have to solve the equation

$$G\left(x^i(\tau), \frac{x'^j(\tau)}{\theta'}\right) = E \tag{3.7}$$

with respect to θ'.

Unlike L, the homogeneous Lagrangian Λ is degenerate. This fact, however, is of no importance in the special case when the Routh formalism is used for the reduction of a single dynamical variable. The solvability of Eq. (2.2) is just assured by the non-vanishing of the partial derivative of Γ with respect to $\dot{q}^0$. Now to check whether Eq. (3.7) is solvable with respect to θ', we have to compute at first $\partial\tilde{\Gamma}/\partial\theta'$. Using the definition (3.1) of G and that of $\tilde{\Gamma}(\theta', x^i, x'^j)$, we obtain

$$\frac{\partial \tilde{\Gamma}}{\partial \theta'} = \frac{1}{\theta'}\frac{\partial^2 L}{\partial v^i \partial v^j} v^i v^j. \tag{3.8}$$

For real motions, the non-vanishing of the expression on the right hand side above is a sufficient condition that the original action (3.2) attain its extremum. This condition was already implicitly assumed while posing the Hamilton principle for the action (3.2). Now the non-vanishing of (3.8) implies the existence of a function ϕ_E such that the value of the variable θ' given by the equation

$$\theta'(\tau) = \phi_E(x^i(\tau), x'^j(\tau)) \tag{3.9}$$

is an algebraic solution to Eq. (3.7). For some Lagrangians L which occur in physical problems the corresponding functions ϕ_E can be found explicitly.

The implicit-function theorem applied to Eq. (3.7) permits us to compute the derivatives $\frac{\partial \phi_E}{\partial x'^i}$ as

$$\frac{\partial \phi_E}{\partial x'^i} = \phi_E \frac{\partial G}{\partial v^i}\Big(\frac{\partial G}{\partial v^j}x'^j\Big)^{-1},$$

which implies

$$\frac{\partial \phi_E}{\partial x'^i}x'^i = \phi_E. \tag{3.10}$$

Thus the function ϕ_E determined by Eq. (3.7), due to the implicit-function theorem and Euler's identity, is homogeneous of degree one in the variables x'^i. This in turn implies that the relation (3.9) is covariant with respect to reparametrizations $\tau \to \tau'$.

As was already said, the Lagrangian Λ does not explicitly depend on θ. Now we are prepared to transform Λ to a corresponding Routh function, denoted here by L_E,

$$\begin{aligned} L_E(x^i, x'^j) &= \Lambda(x^i, \phi_E(x^j, x'^k), x'^l) - p_0\, \phi_E(x^i, x'^j) \\ &= \Big[L\Big(x^i, \frac{x'^j}{\phi_E(x^k, x'^l)}\Big) + E\Big]\phi_E(x^r, x'^s) \qquad (3.11)\\ &= x'^i\Big[\frac{\partial L}{\partial \dot q^i}\Big(x^k, \frac{x'^l}{\phi_E(x^r, x'^s)}\Big)\Big]. \qquad (3.12) \end{aligned}$$

In accordance with Section 1, the Lagrangian L_E describes a reduced system which resulted from eliminating the information about the time evolution from the original system with the Lagrangian L. The variables x^i that remain describe trajectories (i.e. spatial paths) of the system. The first Lagrangian of such type was, in a special case, found by Jacobi. Therefore, L_E is called here the Jacobi Lagrangian of a Lagrangian L which does not depend explicitly on time t. Due to the homogeneity of the function ϕ_E, it follows from the definition (3.11) that L_E is a homogeneous function of degree one in the variables x'^i. To my knowledge, this derivation of L_E was never published before.

4. Lagrange equations of a homogeneous Lagrangian. Let us for a moment turn over to a review of some of the peculiarities of a Hamilton-like principle formulated for a Lagrangian $L_H = L_H(x^i(\tau), x'^j(\tau))$ which is a homogeneous function of degree one in the variables x'^j.

Upon making the usual assumptions about the variations $\delta x^i(\tau)$, the action

$$\mathcal{W}_H[x^i] = \int_{\tau_1}^{\tau_2} L_H(x^i(\tau), x'^j(\tau))d\tau \tag{4.1}$$

leads us to the system of Euler-Lagrange differential equations

$$\frac{\delta L_H}{\delta x^i} = \frac{\partial L_H}{\partial x^i} - \frac{d}{d\tau}\left(\frac{\partial L_H}{\partial x'^i}\right) = 0 \tag{4.2}$$

which, however, due to the homogeneity of L_H, are not independent of one another, for their r.h. sides satisfy a strong identity $x'^i \frac{\delta L_H}{\delta x^i} \equiv 0$. In applications, one usually assumes that the rank of the Hesse matrix of the Lagrangian L_H equals $n-1$, which means that the r.h. sides of Eqs. (4.2) do not satisfy any further strong identities of a similar kind.

A consequence of this assumption is that the solution to an initial value problem of the equations of motion (4.2) is not unique, but can be expressed in terms of one arbitrary function. This can be more precisely stated as

LEMMA. *If a set of n functions $x^i : \mathbb{R} \supset [a,b] \to \mathbb{R}$, $i = 1,\ldots,n$, is a solution of Eqs. (4.2), in which $L_H = L_H(x^i, x'^j)$ is a homogeneous function with respect to x'^j of degree one, and the Hesse matrix $\left[\frac{\partial^2 L_H}{\partial x'^i \partial x'^j}\right]$ is of rank $n-1$, then the set of composite functions $q^i = x^i \circ \psi$, where ψ is a C_2 function $\psi\colon [a,b] \to [a',b']$ such that $\psi' \neq 0$, is also a solution of Eqs. (4.2).*

Proof. This follows by inspection. ∎

Let us discuss now the procedure of integrating the Lagrange equations (4.2). The discussion may be held in either of the following two equivalent ways:

4.1. *A complete elimination of the parameter.* If a certain solution of Eqs. (4.2) corresponding to some initial conditions is found explicitly in terms of a parameter τ as $x^i = x^i(\tau)$, then, by eliminating the parameter τ from these n equations, we obtain $n-1$ relations of the form $F_K(x^1,\ldots,x^n) = 0$, where $K = 1,\ldots, n-1$. By virtue of the lemma, any other solution of (4.2), which satisfies the same initial conditions as the previous one, is of the form $q^i(\tau) = x^i(\psi(\tau))$. Because the functions $x^i(\cdot)$ are here the same as before, by eliminating now $\psi(\tau)$, we must derive the same $n-1$ relations as in the first case, which can be also written in a more universal form

$$F_K(q^1,\ldots, q^n) = 0, \text{ where } K = 1,\ldots, n-1. \tag{4.3}$$

Here the arguments q^i of F_K should be interpreted as coordinates of points in a configuration space $\mathbb{Q}^n$. A choice of $n-1$ variables q^K, for $K = 1,\ldots,n-1$, from all the n variables q^i such that the Jacobian

$$\frac{\partial(F_1,\ldots,F_{n-1})}{\partial(q^1,\ldots,q^{n-1})} \neq 0$$

permits one to solve Eqs. (4.2) for q^K, $K = 1,\ldots,n-1$, in terms of the free variable q^n: $q^K = q^K(q^n)$. These last equations determine a curve in a region of the configuration space $\mathbb{Q}^n$ in which all the steps described above are permissible. The equations (4.3), on the other hand, render a global description of a trajectory $\mathcal{P}_1$, understood as a one-dimensional locus of points in $\mathbb{Q}^n$.

Supplementing the $n-1$ equations (4.3) by an equation of the form

$$F_n(q^1,\ldots, q^n) = \tau, \tag{4.4}$$

where $F_n\colon \mathbb{R}^n \to \mathbb{R}$ is a smooth function chosen in such a way that

$$\frac{\partial(F_1,\ldots,F_n)}{\partial(q^1,\ldots,q^n)} \neq 0,$$

and besides that chosen arbitrarily, we can solve the system of n Eqs. (4.3) and (4.4) for q^i in terms of τ, obtaining $q^i = q^i(\tau)$. Thus, any function F_n that satisfies the assumptions just made above introduces, with the aid of Eq. (4.4), a parameter description of a family of trajectories $\{\mathcal{P}_1\}$ which are labelled by sets of initial data posed on differential equations (4.2). A parameter description of a family of parametrized curves is called *integrable* iff for any two curves $q_1^i(\tau_1)$ and $q_2^i(\tau_2)$ from the family, the equality $q_1^i(\tau_1) = q_2^i(\tau_2)$ for $i = 1,\ldots,n$ implies $\tau_1 = \tau_2$. Evidently, a parameter descripton imposed by a foliation of the type (4.4) is integrable. An integrable parameter description of a family of trajectories $\{\mathcal{P}_1\}$ can be used to define a mapping $\{\mathcal{P}_1\} \to \mathbb{Q}^n \times \mathbb{R}$, defined for every parametrized curve $q^i(\tau)$ from the family as $(q^1(\tau),\ldots,q^n(\tau)) \mapsto (q^1(\tau),\ldots,q^n(\tau),\tau)$. Any composed mapping, defined as $\tau \mapsto (q^1(\tau),\ldots,q^n(\tau),\tau)$, defines a parametrized curve in $\mathbb{Q}^n \times \mathbb{R}$. One introduces, as usual, by means of reparametrizations an equivalence relation in the set of parametrized curves in $\mathbb{Q}^n \times \mathbb{R}$. After dividing the set by this equivalence relation, one obtains a set of oriented loci of points in $\mathbb{Q}^n \times \mathbb{R}$ which are called *world lines* over corresponding trajectories from the the family $\{\mathcal{P}_1\} \subset \mathbb{Q}^n$. In this context, the space $\mathbb{Q}^n \times \mathbb{R}$ is called sometimes the *space of states* [1)] over the configuration space $\mathbb{Q}^n$. The Jacobi action principle (4.1) when considered alone, or, more precisely, its Lagrange equations (4.2) together with appropriate initial conditions, allow us to determine only the set of trajectories $\{\mathcal{P}_1\}$ in $\mathbb{Q}^n$. The foliation (4.4) introduces an additional piece of information that after all steps described above lifts every trajectory from $\{\mathcal{P}_1\}$ to a corresponding world line.

4.2. *The lift of trajectories on the level of differential equations.* The starting point of the procedure of lifting the trajectories, which was described at the end of the previous subsection, was based on the knowledge of solutions (4.3) of differential equations. An analogous procedure can also be performed by starting from the differential equations (4.2) directly. To this end, one should reduce the number of Eqs. (4.2) by one, and take into account only $n-1$ equations

$$\frac{\delta L_H}{\delta x^K} = 0, \text{ where } K = 1,\ldots,n-1. \tag{4.5}$$

The equations above ought then be supplemented by an n-th equation of the form

$$\phi(x^i(\tau), x'^j(\tau)) = 1, \tag{4.6}$$

where ϕ is an arbitrary smooth function that satisfies the condition

$$\det\begin{pmatrix} \frac{\partial\phi(x^k,x'^l)}{\partial x'^1} & \cdots & \frac{\partial\phi(x^k,x'^l)}{\partial x'^n} \\ \cdots & \cdots & \cdots \\ \frac{\partial^2 L_H}{\partial x'^K \partial x'^1} & \cdots & \frac{\partial^2 L_H}{\partial x'^K \partial x'^n} \\ \cdots & \cdots & \cdots \\ \frac{\partial^2 L_H}{\partial x'^{n-1} \partial x'^1} & \cdots & \frac{\partial^2 L_H}{\partial x'^{n-1} \partial x'^n} \end{pmatrix} \neq 0. \tag{4.7}$$

[1)] According to Synge's terminology, see [6].

The system of equations (4.5) and (4.6), together with appropriate initial conditions, determines then a unique solution $x^i = x^i(\tau)$. A freedom of choice of the function ϕ in (4.6) corresponds to the freedom of reparametrization of the solution. There exists a canonical choice of parametrization, which does not depend on any external element to the dynamics defined by Eqs. (4.1). It consists in taking for the function ϕ in (4.6) the square of the homogeneous Lagrangian L_H which defines the action (4.1). The corresponding parametrization of the solution of (4.2) is then its Finslerian arc length, see e.g. [5]. In general, the parametrizations introduced in the way described in the present subsection are non-integrable.

4.3. *The Jacobi Lagrange equations.* Let us return to the case when L_H is equal to the Jacobi Lagrangian L_E. The action (4.1) takes then the form

$$\mathcal{W}_E[x^i] = \int_{\tau_1}^{\tau_2} L_E(x^i(\tau), x'^j(\tau))d\tau. \tag{4.8}$$

Now the fact that the rank of the Hesse matrix of the Jacobi Lagrangian is equal to $n-1$ need not be assumed any more, because it is a consequence of the original dynamical Lagrangian L being a non-degenerate one (a proof can be found in the Appendix of [2]). If all the information we have or would like to make use of is only the Lagrangian L_E, then one should take into account that the lemma formulated at the beginning of Section 4 applies also to solutions of the Jacobi Lagrange equations

$$\frac{\delta L_E}{\delta x^i} = \frac{\partial L_E}{\partial x^i} - \frac{d}{d\tau}\left(\frac{\partial L_E}{\partial x'^i}\right) = 0. \tag{4.9}$$

These equations, together with appropriate initial conditions, can thus, in virtue of the lemma, determine only trajectories $\mathcal{P}_1$ described either by Eqs. (4.3) or in the form $q^K = q^K(q^n)$ for $K = 1, \ldots, n-1$.

In the literature, mainly for the so-called natural Lagrangians, also the world lines over the trajectories were considered, comp. e.g. [3] or [7], which were parametrized by their Jacobi arc length. The description of these world lines suffers certain anomalies. In Section 5, Example 5.2, physical reasons for such an anomalous behaviour are exhibited.

Usually, besides knowing L_E, we have also at our disposal the knowledge of the final form of the energy function $G(q^i, \dot{q}^j)$ and/or of the function $\phi_E(x^i, x'^j)$, defined in an implicit form by Eq. (3.7). In such a case, in addition to the trajectory, one can find the complete motion of the system determined by the original Lagrangian L, corresponding to a chosen value E of the energy constant. To that end, one has to complete the Jacobi Lagrange equations (4.9)—or, to be more precise, the $n-1$ equations arbitrarily picked out from the system (4.9)—by an equation of the type (3.9). In this connection, there are two equivalent procedures that permit us to find the motion $q^i(t)$ of the dynamical system under consideration.

The first of them consists in taking any particular solution $x^i(\tau)$ of (4.9) which satisfies the required initial conditions at $\tau = \tau_0$. One substitutes then this solution, together with its first derivatives, for the arguments of the function ϕ_E, obtaining a function $\tilde{\phi}_E(\tau)$. The integral

$$\theta(\tau) = \int_{\tau_0}^{\tau} \tilde{\phi}_E(\tau)\, d\tau$$

determines then a function θ such that in accordance with the Jacobi reduction procedure $t = \theta(\tau)$, and $q^i(t) = x^i(\theta^{-1}(t))$ is the motion of the original dynamical system with the action (3.2). At $t_0 = \tau_0$ this motion satisfies initial conditions which can be computed from those fulfilled by the solution $x^i(\tau)$.

The second procedure starts by observing that for $\tau = t$, $\theta'(\tau) = 1$, which implies that $x^i = q^i$, and thus

$$\phi_E(q^i(t), \dot{q}^j(t)) = 1. \tag{4.10}$$

The equations (4.9) are valid for any choice of parametrization. So we can write them for $\tau = t$, replace [2] the arguments in L_E by q^i and $\dot{q}^j$, and select from the so prepared equations (4.9) any [3] $n-1$ equations

$$\frac{\partial L_E}{\partial q^K} - \frac{d}{dt}\left(\frac{\partial L_E}{\partial \dot{q}^K}\right) = 0, \text{ where } K = 1, \dots, n-1. \tag{4.11}$$

By virtue of the Jacobi reduction procedure, one can easily prove that

$$\det \begin{pmatrix} \frac{\partial G}{\partial \dot{q}^1} & \cdots & \frac{\partial G}{\partial \dot{q}^n} \\ \cdots & \cdots & \cdots \\ \frac{\partial^2 L_E}{\partial \dot{q}^K \partial \dot{q}^1} & \cdots & \frac{\partial^2 L_E}{\partial \dot{q}^K \partial \dot{q}^n} \\ \cdots & \cdots & \cdots \\ \frac{\partial^2 L_E}{\partial \dot{q}^{n-1} \partial \dot{q}^1} & \cdots & \frac{\partial^2 L_E}{\partial \dot{q}^{n-1} \partial \dot{q}^n} \end{pmatrix} \neq 0. \tag{4.12}$$

Since $\frac{\partial \phi_E}{\partial \dot{q}^j}$ is proportional to $\frac{\partial G}{\partial \dot{q}^j}$, it is evident that the condition (4.7) remains valid after we replace in it ϕ by ϕ_E and L_H by L_E. Thus the initial value problem posed on the system of n equations (4.10) and (4.11) has a unique solution $q^k = q^k(t)$ which determines the motion of the dynamical system with the Lagrangian $L(q^i(t), \dot{q}^j(t))$ not depending explicitly on the Newtonian time t. Of course, in the procedure just described, the replacement of the constraint condition (4.10) by the condition

$$G(q^i(t), \dot{q}^j(t)) = E \tag{4.13}$$

would not have any influence upon the final solution. From a physical point of view, the pair $(q^i(t), t)$ geometrically represents a world line in the space of states $\mathbb{Q}^n \times \mathbb{R}$ in which the unit taken along the real axis $\mathbb{R}$ is equal to the unit of the Newtonian time t. In the space of states the parameter description of world lines with the aid of t is evidently integrable.

The Jacobi action principle (4.8), which implies the Jacobi Lagrange equations (4.9), permits us to find only a geometric trajectory being a projection of the world line $(q^i(t), t)$ on the configuration space $\mathbb{Q}^n$. The described above procedure of integrating the equations

[2] Of course, while making such a replacement, we are not allowed to employ condition (4.10) before performing all the differentiations in the Lagrange equations (4.2). The substitution, for instance, of Eq. (4.10) into (3.11) would lead to a nonsense.

[3] It ought to be remembered that any equations so selected are independent of one another due to the fact that the Hesse matrix of L_E is of rank $n-1$.

(4.9) with the help of the constraint condition (4.10), or (4.13), can be viewed upon as lifting the trajectory in $\mathbb{Q}^n$ to a complete motion $q^i = q^i(t)$. Such a lift was described here on the level of solving the differential equations.

Although, in principle, one could solve the Lagrange equations with the primary Lagrangian $L(q^i, \dot{q}^j)$ directly, in practice very often the procedure described above is the only workable one. For instance, while solving the Kepler problem, one finds first trajectories of the motion which one lifts then to complete motions by solving the Kepler equation.

5. The inverse Jacobi problem. The problem of lifting trajectories to world lines, discussed in the previous section on the level of differential equations, has its counterpart on the variational level. In a coordinate dependent way, which is very convenient for the derivation of its solution, one can formulate the problem in the following way.

Suppose a Lagrange function $L_H(x^i(\tau), x'^j(\tau))$ homogeneous of degree one in the variables x'^i is given. The arguments of L_H are trajectories in a configuration space $\mathbb{Q}^n$ described in an arbitrary parametrization as $x^i = x^i(\tau)$ together with the vectors $x'^j(\tau)$ which are tangent to these trajectories. The following questions may then be asked.

i. What data should be added to the knowledge of L_H, in order to be able to lift trajectories $x^i = x^i(\tau)$ to motions $q^i = q^i(t)$ determined by a Lagrangian $L(q^i(t), \dot{q}^j(t))$ such that the given homogeneous Lagrangian L_H is its Jacobi Lagrangian L_E?
ii. What is the algorithm that enables us to determine L in terms of an arbitrarily given L_H and the necessary additional data that make the solution to the problem unique?

A problem of this kind was formulated and solved in [2] under the name of *inverse Jacobi problem.* Now I would like to review the solution and add some comments on it.

A suggestion following from Section 4.3 is that a good candidate for the above-mentioned additional data would be another arbitrarily assigned function $G(q^i, \dot{q}^j)$ which is the hoped-for energy function of the yet unknown Lagrangian L. It is here simply called *the energy function* even if it does not yet have anything in common with any energy conservation law. After introducing the velocity variable $v^i = \dot{q}^i(t)$, we see that relation (3.1) turns into a partial differential equation

$$v^1 \frac{\partial L}{\partial v^1} + \ldots + v^n \frac{\partial L}{\partial v^n} - L = G \tag{5.1}$$

for an unknown function $L(v^i)$. In Eq. (5.1) $G = G(v^i)$ is treated as a given function, and the dependence of L and G on q^i is here suppressed since from the point of view of the differential equation (5.1) the variables q^i ought to be treated as parameters. Equation (5.1) determines the class of Lagrangians L such that every one of them has the same energy function G. Applying the standard method of integration of partial linear differential equations, each Lagrangian L from that class is found in [2] in terms of a quadrature as a functional of the energy function G. This result is achieved in the following way. Equation (5.1) is transformed to a linear homogeneous equation

$$v^1 \frac{\partial V}{\partial v^1} + \ldots + v^n \frac{\partial V}{\partial v^n} + (L+G)\frac{\partial V}{\partial L} = 0, \tag{5.2}$$

where L plays the role of an $n+1$-th independent argument of a function V which implicitly defines L as a function of v^i:

$$V(L, v^1, \dots, v^n) = 0. \tag{5.3}$$

As is shown in [2], the characteristic equations to Eq. (5.2) admit a system of independent first integrals

$$\begin{aligned} \psi_0(v^i, L) &:= \frac{1}{\sqrt{v^s v^s}}\left[L - \sqrt{v^j v^j}\, I\left(\frac{v^i}{\sqrt{v^k v^k}}, \sqrt{v^l v^l}\right)\right] = c_0, \\ \psi_i(v^j) &:= \frac{v^i}{\sqrt{v^k v^k}} = c_i, \quad \text{where} \quad I(c^i, \rho) = \int \frac{G(c^i \rho)}{\rho^2}\, d\rho. \end{aligned} \tag{5.4}$$

In accordance with the general method of solving partial differential equations of the first order, the general solution of Eq. (5.2) is of the form

$$V = V\Big(\psi_0(v^i, L), \psi_1(v^i), \dots, \psi_n(v^i)\Big), \tag{5.5}$$

where V is an arbitrary function of $n+1$ variables. Therefore, from Eq. (5.3) it follows that the Lagrangian $L(q^i, v^j)$ determined by a given energy function $G(q^i, v^j)$ is

$$L(q^i, v^j) = \sqrt{v^s v^s}\, I\left(q^i, \frac{v^j}{\sqrt{v^k v^k}}, \sqrt{v^l v^l}\right) + \Lambda(q^i, v^j), \tag{5.6}$$

where $\Lambda(q^i, v^j)$ is an arbitrary function homogeneous of degree one in the variables v^j. This is a general formula that determines a class of Lagrangians L describing a conservative dynamical system in terms of an *a priori* assigned energy function G of the system and an arbitrary homogeneous Lagrangian Λ. Formula (5.6) could also be helpful when one would like to decide whether a given conservative dynamical system is a Lagrangian one.

To solve the inverse Jacobi problem, we have to remove the arbitrariness of Λ by making use of the requirement that a given homogeneous Lagrangian $L_H(x^i, x'^j)$ be the Jacobi Lagrangian corresponding to the Lagrangian L determined by Eq. (5.6).

In order to be able to use the definition (3.11) of L_E, we have to find first the function ϕ_E by solving the equation

$$G\left(x^i, \frac{x'^j}{\phi_E}\right) = E, \tag{5.7}$$

where G is now the freely given energy function that was used in Eqs. (5.1) or (5.2) and E is an arbitrary constant. Then, after taking into account that the function Λ is homogeneous of degree one in the second set of its arguments, from (5.6) we obtain

$$\begin{aligned} &L\left(x^i, \frac{x'^j}{\phi_E(x^k, x'^l)}\right) \\ &\quad = \frac{1}{\phi_E(x^r, x'^s)}\left[\sqrt{x'^q x'^q} I\left(x^i, \frac{x'^j}{\sqrt{x'^k x'^k}}, \frac{\sqrt{x'^l x'^l}}{\phi_E(x^m, x'^n)}\right) + \Lambda(x^i, x'^j)\right]. \end{aligned} \tag{5.8}$$

We substitute now the expression (5.8) just computed into formula (3.11) on the r.h. side of which we replace L_E by L_H. All of this gives us an equation

$$L_H(x^i, x'^j) = \sqrt{x'^s x'^s}\, I\left(x^i, \frac{x'^j}{\sqrt{x'^k x'^k}}, \frac{\sqrt{x'^l x'^l}}{\phi_E(x^m, x'^n)}\right) + \Lambda(x^i, x'^j) + E\,\phi_E(x^i, x'^j) \tag{5.9}$$

which we solve for the unknown function Λ

$$\Lambda(x^i, x'^j) = L_H(x^i, x'^j) - \sqrt{x'^s x'^s}\, I\left(x^i, \frac{x'^j}{\sqrt{x'^k x'^k}}, \frac{\sqrt{x'^l x'^l}}{\phi_E(x^m, x'^n)}\right) - E\,\phi_E(x^i, x'^j). \tag{5.10}$$

After changing the names of the variables (x^i, x'^j) to (q^i, v^j), we substitute the expression (5.10) for Λ into (5.6), obtaining

$$\begin{aligned} L(q^i, v^j) = \sqrt{v^s v^s}\left[I\left(q^i, \frac{v^j}{\sqrt{v^k v^k}}, \sqrt{v^l v^l}\right) - I\left(q^i, \frac{v^j}{\sqrt{v^k v^k}}, \frac{\sqrt{v^l v^l}}{\phi_E(q^m, v^n)}\right)\right] \\ + L_H(q^i, v^j) - E\,\phi_E(q^i, v^j). \end{aligned} \tag{5.11}$$

The formula above renders us a unique solution [4)] of the inverse Jacobi problem.

REMARK. Because both the energy conservation law $G((q^i(t), \dot q^j(t)) = E$ and the Eq. (4.10) equivalent to it are weak identities, i.e. satisfied only by the solutions of the Lagrange equations, one must not make use of them in the Lagrangian (5.11), and in its Lagrange equations one may apply them only after carrying out all the differentiations which occur there.

EXAMPLE 5.1. *Natural dynamical systems.* These are systems characterized by a Lagrangian of the form

$$L(q^i, v^j) = \frac{1}{2}\, g_{ij}(q^l) v^i v^j - V(q^l). \tag{5.12}$$

The energy function (3.1) in this case is

$$G(q^i, v^j) = \frac{1}{2}\, g_{ij}(q^l) v^i v^j + V(q^l), \tag{5.13}$$

and Eq. (5.7) leads us to the following solution:

$$\phi_E(x^i, x'^j) = \sqrt{\frac{g_{kl}(x^i) x'^k x'^l}{2(E - V(x^j))}}. \tag{5.14}$$

After substituting the function ϕ_E written above into Eq. (3.11), we obtain the Jacobi Lagrangian

$$L_E(x^i, x'^j) = \sqrt{2(E - V) g_{ij} x'^i x'^j} \tag{5.15}$$

which is quoted in nearly every text-book on theoretical mechanics.

EXAMPLE 5.2. *The inverse problem to 5.1.* Let us consider the case in which the Jacobi Lagrangian (5.15) is known and is given in the form

$$L_H(x^i, x'^j) = \sqrt{\gamma_{ij} x'^i x'^j}, \tag{5.16}$$

i.e. only the final form of the functions $\gamma_{ij}(x^l)$ is known, and information about the conformal factor is lost. What chances do we have to reconstruct an original Lagrangian

[4)] Note that Eq. (53) in [2] which corresponds to the present Eq. (5.11) contains a misprint a consequence of which is a totally false Eq. (54).

L such that its Jacobi Lagrangian is given by Eq. (5.16)? One can try to choose the energy function in the form (5.13). But even then one is left with a set of unknown functions: $g_{ij}(x^l)$, $V(x^k)$, and with an unknown value of the energy constant. While keeping all these quantities unknown, one still can substitute the energy function (5.13), the function ϕ_E in the form (5.14), and the energy constant E into formula (5.11), and look for the outcome. After some algebra, one obtains

$$L(q^i, v^j) = \frac{1}{2} g_{ij}(q^l)v^i v^j - V(q^l) + \sqrt{\gamma_{ij}v^i v^j} - \sqrt{2(E-V)g_{ij}v^i v^j}. \tag{5.17}$$

Thus, taking into account the occurrence of unknown quantities in Eq. (5.17), there is a large variety of Lagrangians L that determine the same Jacobi Lagrangian (5.16). The requirement that L be natural decreases this variety to

$$L_a(q^i, v^j) = \frac{1}{2}\, a\, \gamma_{ij}(q^l)v^i v^j - \frac{V(q^l)}{a}, \tag{5.18}$$

where each Lagrangian L_a is labelled by a non-vanishing arbitrary constant a. All the Lagrangians L_a given by Eq. (5.18), each L_a for the energy constant being equal to E/a, lead one to the same Jacobi Lagrangian (5.16). First by fixing the value of the energy constant, i.e. by fixing a and E, one can assure the uniqueness of L.

The richness of the class of Lagrangians L to which a single Jacobi Lagrangian can be lifted, when there is no other information left except for that given by Eq. (5.16), accounts for singular behaviour of Jacobi Lagrangians of natural systems studied e.g. in [3] or [7].

The last example indicates that the knowledge of the Jacobi Lagrangian L_E, supplemented even by some requirements of a general nature, is not sufficient for a unique reconstruction of the original Lagrangian L. In accordance with Eq. (5.11) it is information contained in the triple $(L_E(x^i, x'^j), G(q^i, \dot{q}^j), E)$ that is equivalent to the original dynamics described by the Lagrangian $L(q^i, \dot{q}^j)$.

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