FUNCTION SPACES X
BANACH CENTER PUBLICATIONS, VOLUME 102
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2014

RELATIONS BETWEEN SOME CLASSES OF FUNCTIONS OF GENERALIZED BOUNDED VARIATION

AMIRAN GOGATISHVILI

Institute of Mathematics of the Academy of Sciences of the Czech Republic
Žitná 25, 115 67 Praha 1, Czech Republic
E-mail: gogatish@math.cas.cz

USHANGI GOGINAVA and GEORGE TEPHNADZE

Department of Mathematics, Faculty of Exact and Natural Sciences
Iv. Javakhishvili Tbilisi State University, Chavchavadze str. 1, Tbilisi 0128, Georgia
E-mail: zazagoginava@gmail.com, giorgitephnadze@gmail.com

Abstract. We prove inclusion relations between generalizing Waterman's and generalized Wiener's classes for functions of two variable.

The notion of function of bounded variation was introduced by C. Jordan [16]. Generalizing this notion N. Wiener [30] has considered the class BV_p of functions. L. Young [31] introduced the notion of functions of Φ-variation. In [26] D. Waterman has introduced the following concept of generalized bounded variation.

DEFINITION 1. Let $\Lambda = \{\lambda_n : n \geq 1\}$ be an increasing sequence of positive numbers such that $\sum_{n=1}^{\infty}(1/\lambda_n) = \infty$. A function f is said to be of Λ-*bounded variation* ($f \in \Lambda BV$), if for every choice of nonoverlapping intervals $\{I_n : n \geq 1\}$, we have

$$\sum_{n=1}^{\infty} \frac{|f(I_n)|}{\lambda_n} < \infty,$$

where $I_n = [a_n, b_n] \subset [0,1]$ and $f(I_n) = f(b_n) - f(a_n)$. If $f \in \Lambda BV$, then Λ-*variation of* f is defined to be the supremum of such sums, denoted by $V_\Lambda(f)$.

2010 *Mathematics Subject Classification*: 42C10.

Key words and phrases: Waterman's class, generalized Wiener's class.

The research was supported by Shota Rustaveli National Science Foundation grant no.13/06 (Geometry of function spaces, interpolation and embedding theorems).

The research of A. Gogatishvili was partly supported by the grants 201/08/0383 and 13-14743S of the Grant agency of the Czech Republic.

The paper is in final form and no version of it will be published elsewhere.

DOI: 10.4064/bc102-0-5

Properties of functions of the class ΛBV as well as the convergence and summability properties of their Fourier series have been investigated in [22]–[29].

For everywhere bounded 1-periodic functions, Z. Chanturia [6] has introduced the concept of the modulus of variation.

H. Kita and K. Yoneda [18] studied generalized Wiener classes $BV(p(n)\uparrow p)$. They introduced

Definition 2. Let f be a finite 1-periodic function defined on the interval $(-\infty,+\infty)$. $\Delta=\{t_i : i=0,\pm1,\pm2,\dots\}$ is said to be a *partition with period* 1 if

$$\dots<t_{-1}<t_0<t_1<t_2<\dots<t_m<t_{m+1}<\dots, \tag{1}$$

and $t_{k+m}=t_k+1$ when $k=0,\pm1,\pm2,\dots$, where m is a natural number. Let $p(n)$ be an increasing sequence such that $1\le p(n)\uparrow p$, $n\to\infty$, where $1\le p\le+\infty$. We say that a function f *belongs to the class* $BV(p(n)\uparrow p)$ if

$$V(f,p(n)\uparrow p)\equiv\sup_{n\ge1}\sup_{\Delta}\left\{\left(\sum_{k=1}^{m}|f(I_k)|^{p(n)}\right)^{1/p(n)} : \inf_k|I_k|\ge\frac{1}{2^n}\right\}<+\infty.$$

We note that if $p(n)=p$ for each natural number, where $1\le p<+\infty$, then the class $BV(p(n)\uparrow p)$ coincides with the Wiener class V_p.

Properties of functions of the class $BV(p(n)\uparrow p)$ as well as the uniform convergence and divergence at point of their Fourier series with respect to trigonometric and Walsh system have been investigated in [9], [12], [17].

Generalizing the class $BV(p(n)\uparrow p)$ T. Akhobadze (see [1, 2]) has considered the classes of functions $BV(p(n)\uparrow p,\varphi)$ and $B\Lambda(p(n)\uparrow p,\varphi)$.

The relation between different classes of generalized bounded variation was taken into account in the works of M. Avdispahić [4], A. Kováčik [19], A. Belov [5], Z. Chanturia [7], T. Akhobadze [3], M. Medvedeva [21] and U. Goginava [11, 13].

Let f be a real and measurable function of two variables of period 1 with respect to each variable. Given intervals $J_1=(a,b)$, $J_2=(c,d)$ and points x,y from $I:=[0,1]$, we define

$$f(J_1,y):=f(b,y)-f(a,y),\qquad f(x,J_2):=f(x,d)-f(x,c)$$

and for the rectangle $A=(a,b)\times(c,d)$, we set

$$f(A)=f(J_1,J_2):=f(a,c)-f(a,d)-f(b,c)+f(b,d).$$

Let $E=\{I_i\}$ be a collection of nonoverlapping intervals from I ordered in an arbitrary way and let Ω be the set of all such collections E.

For the sequence of positive numbers $\Lambda=\{\lambda_n\}_{n=1}^{\infty}$ we define

$$\Lambda V_1(f)=\sup_{y\in I}\sup_{\{I_i\}\in\Omega}\sum_i\frac{|f(I_i,y)|}{\lambda_i},$$

$$\Lambda V_2(f)=\sup_{x\in I}\sup_{\{J_j\}\in\Omega}\sum_j\frac{|f(x,J_j)|}{\lambda_j},$$

$$\Lambda V_{1,2}(f)=\sup_{\{I_i\},\{J_j\}\in\Omega}\sum_i\sum_j\frac{|f(I_i,J_j)|}{\lambda_i\lambda_j}.$$

DEFINITION 3. We say that the function f *has bounded* Λ*-variation on* $I^2 := [0,1]\times[0,1]$ and write $f \in \Lambda BV$, if

$$\Lambda V(f) := \Lambda V_1(f) + \Lambda V_2(f) + \Lambda V_{1,2}(f) < \infty.$$

We say that the function f *has bounded partial* Λ*-variation* and write $f \in P\Lambda BV$ if

$$P\Lambda V(f) := \Lambda V_1(f) + \Lambda V_2(f) < \infty.$$

If $\lambda_n \equiv 1$ (or if $0 < c < \lambda_n < C < \infty$, $n = 1, 2, \dots$) the classes ΛBV and $P\Lambda BV$ coincide with the Hardy class BV and PBV respectively. Hence it is reasonable to assume that $\lambda_n \to \infty$ and since the intervals in $E = \{I_i\}$ are ordered arbitrarily, we will suppose, without loss of generality, that the sequence $\{\lambda_n\}$ is increasing. Thus, in what follows we suppose that

$$1 < \lambda_1 \le \lambda_2 \le \dots, \qquad \lim_{n\to\infty} \lambda_n = \infty, \qquad \sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty. \tag{2}$$

In the case when $\lambda_n = n$, $n = 1, 2, \dots$, we say *Harmonic Variation* instead of Λ-variation and write H instead of Λ (HBV, $PHBV$, $HV(f)$, etc.).

The notion of Λ-variation was introduced by Waterman [26] in one-dimensional case and Sahakian [24] in two-dimensional case. The notion of bounded partial variation (class PBV) was introduced by Goginava [10]. These classes of functions of generalized bounded variation play an important role in the theory of Fourier series.

We have proved in [14] the following theorem.

THEOREM 4 (Goginava, Sahakian). *Let* $\Lambda = \{\lambda_n = n\gamma_n\}$ *and* $\gamma_n \ge \gamma_{n+1} > 0$, *where* $n = 1, 2, \dots$.

1) *If*

$$\sum_{n=1}^{\infty} \frac{\gamma_n}{n} < \infty, \tag{3}$$

then $P\Lambda BV \subset HBV$.

2) *If for some* $\delta > 0$

$$\gamma_n = O(\gamma_{n^{[1+\delta]}}) \quad as \quad n \to \infty \tag{4}$$

and

$$\sum_{n=1}^{\infty} \frac{\gamma_n}{n} = \infty, \tag{5}$$

then $P\Lambda BV \not\subset HBV$.

Dyachenko and Waterman [8] introduced another class of functions of generalized bounded variation. Denoting by Γ the the set of finite collections of nonoverlapping rectangles $A_k := [\alpha_k, \beta_k] \times [\gamma_k, \delta_k] \subset T^2$ we define

$$\Lambda^* V(f) := \sup_{\{A_k\}\in\Gamma} \sum_k \frac{|f(A_k)|}{\lambda_k}.$$

DEFINITION 5 (Dyachenko, Waterman). Let f be a real function on I^2. We say that $f \in \Lambda^* BV$ if

$$\Lambda V(f) := \Lambda V_1(f) + \Lambda V_2(f) + \Lambda^* V(f) < \infty.$$

In [15] Goginava and Sahakian introduced a new class of functions of generalized bounded variation and investigated the convergence of Fourier series of function of this class.

For the sequence $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ we put

$$\Lambda^{\#}V_1(f) = \sup_{\{y_i\}\subset T} \sup_{\{I_i\}\in\Omega} \sum_i \frac{|f(I_i, y_i)|}{\lambda_i},$$

$$\Lambda^{\#}V_2(f) = \sup_{\{x_j\}\subset T} \sup_{\{J_j\}\in\Omega} \sum_j \frac{|f(x_j, J_j|}{\lambda_j}.$$

Definition 6 (Goginava, Sahakian). We say that the function f *belongs to the class* $\Lambda^{\#}BV$, if

$$\Lambda^{\#}V(f) := \Lambda^{\#}V_1(f) + \Lambda^{\#}V_2(f) < \infty.$$

The following theorem was proved in [15].

Theorem 7.

a) *If*

$$\overline{\lim_{n\to\infty}} \frac{\lambda_n \log(n+1)}{n} < \infty, \tag{6}$$

then

$$\Lambda^{\#}BV \subset HBV.$$

b) *If* $\frac{\lambda_n}{n} \downarrow 0$ *and*

$$\overline{\lim_{n\to\infty}} \frac{\lambda_n \log(n+1)}{n} = +\infty,$$

then

$$\Lambda^{\#}BV \not\subset HBV.$$

In this paper we introduce new classes of bounded generalized variation.

Let f be a function defined on R^2 and 1-periodic with respect to each variable. Δ_1 and Δ_2 are said to be partitions with period 1, if

$$\Delta_i: \quad \ldots < t_{-1}^{(i)} < t_0^{(i)} < t_1^{(i)} < \ldots < t_{m_i}^{(i)} < t_{m_i+1}^{(i)} < \ldots, \qquad i = 1, 2,$$

satisfies $t_{k+m_i}^{(i)} = t_k^{(i)} + 1$ for $k = 0, \pm 1, \pm 2, \ldots$, where m_i, $i = 1, 2$, are positive integers.

Definition 8. Let $p(n)$ be an increasing sequence such that $1 \le p(n) \uparrow p$, $n \to \infty$, where $1 \le p \le +\infty$. We say that a function f *belongs to the class* $BV^{\#}(p(n)\uparrow p)$ if

$$V_1^{\#}(f, p(n)\uparrow p) := \sup_{\{y_i\}\subset I} \sup_{n\ge 1} \sup_{\Delta_1} \left\{ \left(\sum_{i=1}^{m_1} |f(I_i, y_i)|^{p(n)} \right)^{1/p(n)} : \inf_i |I_i| \ge \frac{1}{2^n} \right\} < +\infty,$$

and

$$V_2^{\#}(f, p(n)\uparrow p) := \sup_{\{x_j\}\subset I} \sup_{n\ge 1} \sup_{\Delta_2} \left\{ \left(\sum_{j=1}^{m_2} |f(x_j, J_j)|^{p(n)} \right)^{1/p(n)} : \inf_j |J_j| \ge \frac{1}{2^n} \right\} < +\infty,$$

where

$$I_i := (t_{i-1}^{(1)}, t_i^{(1)}), \quad J_j := (t_{j-1}^{(2)}, t_j^{(2)}).$$

$C(I^2)$ and $B(I^2)$ are the spaces of continuous and bounded functions given on I^2, respectively.

In this paper we prove inclusion relations between $\Lambda^{\#}BV$ and $BV^{\#}(p(n)\uparrow\infty)$ classes.

THEOREM 9. *$\Lambda^{\#}BV \subset BV^{\#}(p(n)\uparrow\infty)$ if and only if*

$$\varlimsup_{n\to\infty} \sup_{1\le m\le 2^n} \frac{m^{1/p(n)}}{\sum_{j=1}^{m}(1/\lambda_j)} < \infty. \tag{7}$$

THEOREM 10. *Suppose that $\sum_{n=1}^{\infty}(1/\lambda_n) = +\infty$. Then there exists a function $f \in BV^{\#}(p(n)\uparrow\infty)\cap C(I^2)$ such that $f \notin \Lambda BV^{\#}$.*

COROLLARY 11. *$BV^{\#}(p(n)\uparrow\infty) \subset \Lambda^{\#}BV$ if and only if $\Lambda^{\#}BV = B(I^2)$.*

Proof of Theorem 9. Let us take an arbitrary $f \in \Lambda^{\#}BV$. Following the method of Kuprikov [20], we can prove that

$$\left(\sum_{k=1}^{m_1} |f(I_k, y_k)|^{p(n)}\right)^{1/p(n)} \le \Lambda^{\#}V_1(f) \sup_{1\le m\le 2^n} \frac{m^{1/p(n)}}{\sum_{i=1}^{m}(1/\lambda_i)} < \infty$$

and

$$\left(\sum_{k=1}^{m_2} |f(x_k, J_k)|^{p(n)}\right)^{1/p(n)} \le \Lambda^{\#}V_2(f) \sup_{1\le m\le 2^n} \frac{m^{1/p(n)}}{\sum_{i=1}^{m}(1/\lambda_i)} < \infty.$$

Therefore, $f \in \Lambda^{\#}BV(p(n)\uparrow\infty)$.

Next, we suppose that the condition (7) does not hold. As an example we construct a function from $\Lambda^{\#}BV$ which is not in $BV^{\#}(p(n)\uparrow\infty)$.

Since

$$\varlimsup_{n\to\infty} \sup_{1\le m\le 2^n} \frac{m^{1/p(n)}}{\sum_{j=1}^{m}(1/\lambda_j)} = +\infty,$$

there exists a sequence of integers $\{n'_k : k \ge 1\}$ such that

$$\lim_{k\to\infty} \frac{m(n'_k)^{1/p(n'_k)}}{\sum_{j=1}^{m(n'_k)}(1/\lambda_j)} = +\infty, \tag{8}$$

where

$$\sup_{1\le m\le 2^n} \frac{m^{1/p(n)}}{\sum_{j=1}^{m}(1/\lambda_j)} = \frac{m(n)^{1/p(n)}}{\sum_{j=1}^{m(n)}(1/\lambda_j)}.$$

We choose an increasing sequence of positive integers $\{n_k : k \ge 1\} \subset \{n'_k : k \ge 1\}$ such that

$$\frac{m(n_k)^{1/p(n_k)}}{\sum_{j=1}^{m(n_k)}(1/\lambda_j)} \ge 4^k, \tag{9}$$

$$p(n_k) \ge n_{k-1}, \tag{10}$$

$$n_k > 3n_{k-1} + 1 \qquad \text{for all } k \ge 2. \tag{11}$$

If $m(n_k) \le 2^{2n_{k-1}}$ then by (10) condition (8) does not hold. Hence without lost of generality we can suppose that $2^{2n_{k-1}} < m(n_k) \le 2^{n_k}$ for every k.

Two cases are possible:

a) There exists a monotone sequence of positive integers $\{s_k : k \geq 1\} \subset \{n_k : k \geq 1\}$ such that

$$2^{2s_{k-1}} < m(s_k) \leq 2^{s_k - s_{k-1} - 1}. \tag{12}$$

Consider the function f_k defined by

$$f_k(x) = \begin{cases} h_k(2^{s_k}x - 2j + 1), & x \in [(2j-1)/2^{s_k}, 2j/2^{s_k}) \\ -h_k(2^{s_k}x - 2j - 1), & x \in [2j/2^{s_k}, (2j+1)/2^{s_k}) \\ & \text{for } j = m(s_{k-1}), \dots, m(s_k) - 1 \\ 0, & \text{otherwise} \end{cases}$$

where

$$h_k = \Big(2^k \sum_{j=1}^{m(s_k)} (1/\lambda_j)\Big)^{-1/2}.$$

Let

$$f(x, y) = \sum_{k=2}^{\infty} f_k(x) f_k(y),$$

where

$$f(x + l, y + s) = f(x, y), \quad l, s = 0, \pm 1, \pm 2, \dots.$$

First we prove that $f \in \Lambda^{\#}BV$. For every choice of nonoverlapping intervals $\{I_n : n \geq 1\}$, we get

$$\Lambda^{\#}V_1(f; p(n) \uparrow \infty) \leq \sum_{j=1}^{\infty} \frac{|f(I_j, y_j)|}{\lambda_j} \leq 4 \sum_{i=1}^{\infty} h_i^2 \sum_{j=1}^{m(s_i)} \frac{1}{\lambda_j} = 4 \sum_{i=1}^{\infty} \frac{1}{2^i} = 4.$$

Analogously, we can prove that

$$\Lambda^{\#}V_2(f; p(n) \uparrow \infty) \leq 4.$$

Next, we shall prove that $f \notin BV^{\#}(p(n) \uparrow \infty)$. By (11), (12) and from the construction of the function we get

$$\begin{aligned} V_1(f; p(n) \uparrow \infty) &\geq \left\{ \sum_{j=m(s_{k-1})}^{m(s_k)-1} \left| f\Big(\frac{2j-1}{2^{s_k}}, \frac{2j}{2^{s_k}}\Big) - f\Big(\frac{2j}{2^{s_k}}, \frac{2j}{2^{s_k}}\Big) \right|^{p(s_k)} \right\}^{1/p(s_k)} \\ &= \left\{ \sum_{j=m(s_{k-1})}^{m(s_k)-1} \left| \Big(f_k\Big(\frac{2j-1}{2^{s_k}}\Big) - f_k\Big(\frac{2j}{2^{s_k}}\Big)\Big) f_k\Big(\frac{2j}{2^{s_k}}\Big) \right|^{p(s_k)} \right\}^{1/p(s_k)} \\ &= h_k^2 \big(m(s_k) - m(s_{k-1})\big)^{1/p(s_k)} \\ &\geq c\, \frac{m(s_k)^{1/p(s_k)}}{2^k \sum_{j=1}^{m(s_k)} (1/\lambda_j)} \geq c 2^k \to \infty \qquad \text{as } k \to \infty. \end{aligned}$$

Therefore, we get $f \notin BV^{\#}(p(n) \uparrow \infty)$.

b) Without lost of generality we can suppose that

$$2^{n_k - n_{k-1} - 1} < m(n_k) \leq 2^{n_k} \quad \text{for all } k > k_0.$$

Consider the function g_k defined by

$$g_k(x)=\begin{cases} d_k(2^{n_k}x-2j+1), & x\in[(2j-1)/2^{n_k},2j/2^{n_k}) \\ -d_k(2^{n_k}x-2j-1), & x\in[2j/2^{n_k},(2j+1)/2^{n_k}) \\ & \text{for } j=2^{n_{k-1}-n_{k-2}},\dots,2^{n_k-n_{k-1}-1}-1 \\ 0, & \text{otherwise} \end{cases}$$

where

$$d_k=\Big(2^k\sum_{j=1}^{m(n_k)}(1/\lambda_j)\Big)^{-1/2}.$$

Let

$$g(x,y)=\sum_{k=k_0+2}^{\infty}g_k(x)g_k(y),$$

where

$$g(x+l,y+s)=g(x,y),\qquad l,s=0,\pm1,\pm2,\dots.$$

For every choice of nonoverlapping intervals $\{I_n:n\ge1\}$ we get

$$\sum_{j=1}^{\infty}\frac{|f(I_j,y_j)|}{\lambda_j}\le 4\sum_{i=k_0+1}^{\infty}d_i^2\sum_{j=1}^{2^{n_i-n_{i-1}-1}}\frac{1}{\lambda_j}$$
$$\le 4\sum_{i=k_0+1}^{\infty}d_i^2\sum_{j=1}^{m(n_i)}\frac{1}{\lambda_j}<\infty.$$

Analogously, we can prove that

$$\sum_{j=1}^{\infty}\frac{|f(x_j,J_j)|}{\lambda_j}<\infty.$$

Hence $g\in\Lambda^{\#}BV$.

Next we shall prove that $g\notin BV^{\#}(p(n)\uparrow\infty)$. By (8), (10), (11) and from the construction of the function we get

$$V_1^{\#}(g;p(n)\uparrow\infty)\ge\Big\{\sum_{j=2^{n_{k-1}-n_{k-2}}}^{2^{n_k-n_{k-1}-1}-1}\Big|g\Big(\frac{2j-1}{2^{n_k}},\frac{2j}{2^{n_k}}\Big)-g\Big(\frac{2j}{2^{n_k}},\frac{2j}{2^{n_k}}\Big)\Big|^{p(n_k)}\Big\}^{1/p(n_k)}$$
$$=\Big\{\sum_{j=2^{n_{k-1}-n_{k-2}}}^{2^{n_k-n_{k-1}-1}-1}\Big|\Big(g_k\Big(\frac{2j-1}{2^{n_k}}\Big)-g_k\Big(\frac{2j}{2^{n_k}}\Big)\Big)g_k\Big(\frac{2j}{2^{n_k}}\Big)\Big|^{p(n_k)}\Big\}^{1/p(n_k)}$$
$$=d_k^2(2^{n_k-n_{k-1}-1}-2^{n_{k-1}-n_{k-2}})^{1/p(n_k)}\ge\frac14\,d_k^2 2^{(n_k-n_{k-1})/p(n_k)}$$
$$\ge\frac{c2^{n_k/p(n_k)}}{2^{k+2}\sum_{j=1}^{m(n_k)}(1/\lambda_j)}\ge c\,\frac{m(n_k)^{1/p(n_k)}}{2^k\sum_{j=1}^{m(n_k)}(1/\lambda_j)}\ge c2^k\ \to\infty\quad\text{as } k\to\infty.$$

Therefore, we get $g\notin BV^{\#}(p(n)\uparrow\infty)$ and the proof of Theorem 1 is complete. ∎

Proof of Theorem 10. We choose an increasing sequence of positive integers $\{l_k : k \geq 1\}$ such that $l_1 = 1$ and

$$p(l_{k-1}) \geq \ln k \qquad \text{for all}\, k \geq 2. \tag{13}$$

Set for $k = 1, 2, \ldots$

$$r_k(x) = \begin{cases} 2^{l_k+1} c_k (x - 1/2^{l_k}), & \text{if } 1/2^{l_k} \leq x \leq 3/2^{l_k+1} \\ -2^{l_k+1} c_k (x - 1/2^{l_k-1}), & \text{if } 3/2^{l_k+1} \leq x \leq 1/2^{l_k-1} \\ 0, & \text{otherwise} \end{cases}$$

where

$$c_k = \Big(\sum_{j=1}^{k} \frac{1}{\lambda_j}\Big)^{-1/4}$$

and

$$r(x,y) = \sum_{k=1}^{\infty} r_k(x) r_k(y),$$

where

$$r(x+l, y+s) = r(x,y), \quad l, s = 0, \pm1, \pm2, \ldots.$$

It is easy to show that $r \in C(I^2)$.

First we show that $r \in BV^{\#}(p(n) \uparrow \infty)$. Let $\{I_i\}$ be an arbitrary partition of the interval I such that $\inf_i |I_i| \geq 1/2^l$. For this fixed l, we can choose integers l_{k-1} and l_k for which $l_{k-1} \leq l < l_k$ holds. Then it follows that $p(l_{k-1}) \leq p(l) \leq p(l_k)$ and $1/2^{l_k} < 1/2^l \leq 1/2^{l_{k-1}}$.

By (13) and from the construction of the function r we obtain

$$\begin{aligned}
\Big\{\sum_{j=1}^{m} |r(I_i, y_i)|^{p(l)}\Big\}^{1/p(l)} &= \Big\{\sum_{j=1}^{k} \sum_{\{i: 2^{-l_j} \leq y_i < 2^{-l_j+1}\}} |r(I_i, y_i)|^{p(l)}\Big\}^{1/p(l)} \\
&\leq \Big\{\sum_{j=1}^{k} \Big(\sum_{\substack{I_i \cap (2^{-l_j}, 2^{-l_j+1}) \neq \varnothing \\ \{i: 2^{-l_j} \leq y_i < 2^{-l_j+1}\}}} |r(I_i, y_i)|\Big)^{p(l)}\Big\}^{1/p(l)} \\
&\leq \Big\{\sum_{j=1}^{k} \Big(\sum_{\{i: I_i \cap (2^{-l_j}, 2^{-l_j+1}) \neq \varnothing\}} \Big| r\Big(I_i, \frac{3}{2^{l_j+1}}\Big)\Big|\Big)^{p(l)}\Big\}^{1/p(l)} \\
&\leq \Big\{\sum_{j=1}^{k} \Big(\Big| r\Big(\Big(\frac{1}{2^{l_j}}, \frac{3}{2^{l_j+1}}\Big), \frac{3}{2^{l_j+1}}\Big)\Big| + \Big| r\Big(\Big(\frac{3}{2^{l_j+1}}, \frac{1}{2^{l_j-1}}\Big), \frac{3}{2^{l_j+1}}\Big)\Big|\Big)^{p(l)}\Big\}^{1/p(l)} \\
&\leq \Big\{\sum_{j=1}^{k} (2c_j^2)^{p(l)}\Big\}^{1/p(l)} \leq 2k^{1/p(l_{k-1})} \leq 4k^{1/\ln k} = 4e.
\end{aligned}$$

Therefore $r \in BV^{\#}(p(n) \uparrow \infty)$.

Finally, we prove that $r \notin \Lambda BV^{\#}$. Since $c_n \downarrow 0$, we get

$$\sum_{j=1}^{k} \frac{\left|r(1/2^{l_j}, 3/2^{l_j+1}) - r(3/2^{l_j+1}, 3/2^{l_j+1})\right|}{\lambda_j}$$

$$= \sum_{j=1}^{k} \frac{\left|(r_j(1/2^{l_j}) - r_j(3/2^{l_j+1}))r_j(3/2^{l_j+1})\right|}{\lambda_j}$$

$$= \sum_{j=1}^{k} \frac{c_j^2}{\lambda_j} \geq c_k^2 \sum_{j=1}^{k} \frac{1}{\lambda_j} = \Big(\sum_{j=1}^{k} \frac{1}{\lambda_j}\Big)^{1/2} \to \infty \qquad \text{as } k \to \infty.$$

Therefore, we get $r \notin \Lambda BV^{\#}$ and the proof of Theorem 10 is complete. ∎

Since $\Lambda BV^{\#} = B(I^2)$ if and only if $\sum_{j=1}^{\infty}(1/\lambda_j) < \infty$ the validity of Corollary 11 follows from Theorem 10.

Acknowledgements. We thank the anonymous referee for his/her remarks which have improved the final version of this paper.

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