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EMBEDDINGS OF DOUBLING WEIGHTED BESOV SPACES

DOROTHEE D. HAROSKE and PHILIPP SKANDERA

Institute of Mathematics, Friedrich-Schiller-University Jena
D-07737 Jena, Germany
E-mail: dorothee.haroske@uni-jena.de, philipp.skandera@uni-jena.de

Abstract. We study continuous embeddings of Besov spaces of type $B^s_{p,q}(\mathbb{R}^n, w)$, where $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$, and the weight w is doubling. This approach generalises recent results about embeddings of Muckenhoupt weighted Besov spaces, cf. [11, 13, 14]. Our main argument relies on appropriate atomic decomposition techniques of such weighted spaces; here we benefit from earlier results by Bownik [2]. In addition, we discuss some other related weight classes briefly and compare corresponding results.

1. Introduction. In recent years some attention has been paid to the continuity and compactness of embeddings of weighted function spaces of Besov and Sobolev (or, more generally, Triebel–Lizorkin) type as well as to certain quantities like approximation and entropy numbers which provide a refined description of this compactness. These investigations (in the above described context) started in [9, 15, 16] (with a partial forerunner in [24, Ch. V, §3]), and were continued and extended in the series of papers [17–19, 27]. As an application one obtains spectral estimates of certain pseudo-differential operators in the spirit of the program proposed by Edmunds and Triebel [5]. In all those papers above the class of so-called 'admissible' weights was considered: These are smooth weights with no singular points. One can take $w(x) = (1 + |x|^2)^{\alpha/2}$, $\alpha \in \mathbb{R}$, $x \in \mathbb{R}^n$, as a prominent example. Later we developed parallel investigations in the context of Muckenhoupt weights $\mathcal{A}_\infty$ in [11–14]. In contrast to 'admissible' weights the $\mathcal{A}_\infty$ weights may have local singularities which can influence properties of the embeddings of function spaces. Here the weight $w(x) = |x|^\varrho$, $\varrho > -n$, may serve as a typical example. Of particular interest are always necessary and sufficient conditions on the parameters and weights

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of the Besov spaces which guarantee continuity and compactness of such embeddings. Afterwards one tries to determine the exact asymptotic behaviour of corresponding entropy and approximation numbers. Our methods in [11,16] essentially rely on appropriate wavelet decompositions in weighted spaces which admit to deal with weighted sequence spaces instead of weighted function spaces in the sequel. The approach is similar when local Muckenhoupt weights $\mathcal{A}_\infty^{\mathrm{loc}}$ are considered; this class was introduced in [25] and contains both the admissible and the Muckenhoupt weights. Results in this context have been obtained quite recently in [38].

We follow a different approach now and consider the extension of Muckenhoupt weights by doubling weights. These are weight functions which satisfy

$$\int_{B(x,2r)} w(y)\,\mathrm{d}y \le 2^{n\beta} \int_{B(x,r)} w(y)\,\mathrm{d}y$$

for all $x \in \mathbb{R}^n$ and $r > 0$, where $B(x,r)$ denotes the (open) ball centred at x with radius r. The smallest constant β with the above property is called the doubling constant (concerning balls). It is well-known that Muckenhoupt weights $\mathcal{A}_\infty$ form a proper subset of doubling weights, cf. [7, 36]. Sometimes, motivated also by some questions arising in fractal geometry, the setting of doubling (or, respectively, non-doubling) weights seems more appropriate and obvious than the class $\mathcal{A}_\infty$. This was our starting point to inquire what results known from the context of Muckenhoupt weights can be transferred to this more general situation, and to what extent. Since we also aim at applications in the sense sketched above we start with continuous embeddings in doubling weighted Besov spaces. Naturally one would continue this by observations about compactness and other scales of spaces, but this is postponed to another paper.

Our main theorem here concerns embeddings of weighted Besov spaces $B^s_{p,q}(\mathbb{R}^n, w)$, where w is doubling. More precisely, let $s_1 \ge s_2$, $0 < p_1, p_2 < \infty$, $0 < q_1, q_2 \le \infty$ and w_1, w_2 doubling weights. Then the embedding

$$\mathrm{id} : B^{s_1}_{p_1,q_1}(\mathbb{R}^n, w_1) \to B^{s_2}_{p_2,q_2}(\mathbb{R}^n, w_2)$$

is continuous, if

$$\left\{2^{-\nu(s_1-s_2)} \left\| \{w_1(Q_{\nu,m})^{-1/p_1} w_2(Q_{\nu,m})^{1/p_2}\}_m \,|\, \ell_{p^*} \right\| \right\}_\nu \in \ell_{q^*},$$

where $\frac{1}{p^*} = \max(\frac{1}{p_2} - \frac{1}{p_1}, 0)$, $\frac{1}{q^*} = \max(\frac{1}{q_2} - \frac{1}{q_1}, 0)$, $w_k(Q_{\nu,m}) = \int_{Q_{\nu,m}} w_k(y)\,\mathrm{d}y$, $k = 1, 2$, and $Q_{\nu,m}$ are dyadic cubes with side-length $2^{-\nu}$. This outcome coincides with the corresponding one in [11] for the special case of Muckenhoupt weights, see also [38] for local Muckenhoupt weights. The method to prove this result relies on appropriate atomic decompositions in such weighted Besov spaces. Here we benefit from an earlier paper by Bownik [2] and adapt it to our needs. As we shall discuss below, this result for doubling weights is not covered by its fore-runners in case of Muckenhoupt or admissible weights, but really new to the best of our knowledge. Though some of the used techniques resemble similar considerations in related cases, one has to check all arguments carefully concerning their validity and applicability. This turns out, in particular, when we deal with some consequences of our main theorem: assuming that $w_1 = w_2 = w$

and $\delta = s_1 - \frac{n}{p_1} - s_2 + \frac{n}{p_2} > n(\gamma - 1)(\frac{1}{p_1} - \frac{1}{p_2})$, where γ is the doubling constant concerning cubes, we need, in addition, some stronger assumption than in case of Muckenhoupt weights to ensure continuity of the embedding, that is, we have to assume that $\inf \int_{Q_{0,x}} w(y)\,\mathrm{d}y \geq c > 0$, where the infimum is taken over all unit cubes centred at $x \in \mathbb{Q}^n$. Such phenomena occur occasionally when dealing with the larger class of doubling weights.

Note that many of the arguments and results have their direct counterparts in the scale of Sobolev, or more general, Triebel–Lizorkin spaces. But this, as well as refined studies concerning compactness or necessary assumptions of those embeddings are out of the scope of the present paper.

The paper is organised as follows. In Section 2 we recall basic facts about the above weight classes and weighted Besov spaces. Section 3 is devoted to the continuity of the embeddings, where we first state and prove our main result and conclude the short paper by some discussion of known counterparts and collect further consequences. We always illustrate our presentation with some well-known examples.

2. Weighted function spaces. We fix some notation. By $\mathbb{N}$ we mean the set of natural numbers, by $\mathbb{N}_0$ the set $\mathbb{N} \cup \{0\}$, and by $\mathbb{Z}^n$ the set of all lattice points in $\mathbb{R}^n$ having integer components. The positive part of a real function f is denoted by $f_+(x) = \max(f(x), 0)$, the integer part of $a \in \mathbb{R}$ by $\lfloor a \rfloor = \max\{k \in \mathbb{Z} : k \leq a\}$. If $0 < u \leq \infty$, the number u' is given by $\frac{1}{u'} = (1 - \frac{1}{u})_+$. For two non-negative functions ϕ, ψ we mean by $\phi(t) \sim \psi(t)$ that there exist constants $c_1, c_2 > 0$ such that $c_1\phi(t) \leq \psi(t) \leq c_2\phi(t)$ for all admitted values of t. Given two (quasi-) Banach spaces X and Y, we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding of X in Y is continuous.

Let for $m \in \mathbb{Z}^n$ and $\nu \in \mathbb{N}_0$, $Q_{\nu,m}$ denote the n-dimensional (open) cube with sides parallel to the axes of coordinates, centred at $2^{-\nu}m$ and with side length $2^{-\nu}$. Occasionally we shall also deal with n-dimensional (open) cubes $Q = Q(x, l)$ with sides parallel to the axes of coordinates, centred at x and with side length l. Then $2Q$ stands for the cube centred at x and with doubled side-length $2l$, i.e., $2Q = Q(x, 2l)$. For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x, r)$ denote the open ball $B(x, r) = \{y \in \mathbb{R}^n : |y - x| < r\}$.

All unimportant positive constants will be denoted by c, occasionally with subscripts. For convenience, let both $\mathrm{d}x$ and $|\cdot|$ stand for the (n-dimensional) Lebesgue measure in the sequel. As we shall always deal with function spaces on $\mathbb{R}^n$, we may often omit the '$\mathbb{R}^n$' from their notation for convenience.

2.1. Weights. By a weight w we shall always mean a locally integrable function w, positive a.e. in the sequel. We are mainly interested in doubling weights, but for later use we briefly recall, in addition, the notions of Muckenhoupt weights and admissible weights and some of their characteristic features.

Muckenhoupt weights. It is well-known that this weight class is closely connected with the boundedness of the Hardy–Littlewood maximal operator M given by

$$Mf(x) = \sup_{B(x,r)\in\mathcal{B}} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\,\mathrm{d}y, \quad x \in \mathbb{R}^n, \tag{1}$$

acting in weighted Lebesgue spaces. Here $\mathcal{B}$ is the collection of all open balls $B(x,r)$ centred at $x \in \mathbb{R}^n$, $r > 0$.

DEFINITION 2.1. Let w be a weight on $\mathbb{R}^n$.

(i) Then w *belongs to the Muckenhoupt class* $\mathcal{A}_p$, $1 < p < \infty$, if there exists a constant $0 < A < \infty$ such that for all balls B the following inequality holds

$$\Big(\frac{1}{|B|}\int_B w(x)\,\mathrm{d}x\Big)^{1/p} \cdot \Big(\frac{1}{|B|}\int_B w(x)^{-p'/p}\,\mathrm{d}x\Big)^{1/p'} \le A. \tag{2}$$

(ii) Then w *belongs to the Muckenhoupt class* $\mathcal{A}_1$ if there exists a constant $0 < A < \infty$ such that the inequality

$$Mw(x) \le Aw(x) \tag{3}$$

holds for almost all $x \in \mathbb{R}^n$.

(iii) The *Muckenhoupt class* $\mathcal{A}_\infty$ is given by

$$\mathcal{A}_\infty = \bigcup_{p>1} \mathcal{A}_p. \tag{4}$$

Since the pioneering work of Muckenhoupt [21–23], these classes of weight functions have been studied in great detail, we refer, in particular, to the monographs [8, 28–30] for a complete account on the theory of Muckenhoupt weights. As usual, we use the abbreviation

$$w(\Omega) = \int_\Omega w(x)\,\mathrm{d}x, \tag{5}$$

where $\Omega \subset \mathbb{R}^n$ is some bounded, measurable set. Then $w \in \mathcal{A}_r$, $1 \le r < \infty$, implies that

$$\frac{|E|}{|B|} \le c'\Big(\frac{w(E)}{w(B)}\Big)^{1/r}, \qquad E \subset B. \tag{6}$$

Another property of Muckenhoupt weights that will be used in the sequel is that $w \in \mathcal{A}_p$, $p > 1$, implies the existence of some number $r < p$ such that $w \in \mathcal{A}_r$. This is closely connected with the so-called 'reverse Hölder inequality', see [28, Ch. V, §3, Prop. 3, Cor.]. In our case this fact will re-emerge in the number

$$r_w = \inf\{r \ge 1 : w \in \mathcal{A}_r\}, \quad w \in \mathcal{A}_\infty, \tag{7}$$

that plays some role later on. Obviously, $1 \le r_w < \infty$, and $w \in \mathcal{A}_{r_w}$ implies $r_w = 1$.

EXAMPLE 2.2. One of the most prominent examples of a Muckenhoupt weight $w \in \mathcal{A}_\infty$ is given by $w(x) = |x|^\varrho$, $\varrho > -n$. We modified this example in [11, 14] by

$$w_{a,b}(x) = \begin{cases} |x|^a, & |x| < 1, \\ |x|^b, & |x| \ge 1, \end{cases} \tag{8}$$

where $a, b > -n$. Straightforward calculation shows that for $1 < r < \infty$,

$$w_{a,b} \in \mathcal{A}_r \quad \text{if and only if} \quad -n < a, b < n(r-1),$$

such that $r_{w_{a,b}} = 1 + \frac{\max(a,b,0)}{n}$. Moreover, $w_{a,b} \in \mathcal{A}_1$ when $\max(a,b) \le 0$. For further examples we refer to [6, 10, 11, 13].

REMARK 2.3. Rychkov introduced in [25] the class of local Muckenhoupt weights $\mathcal{A}_p^{\mathrm{loc}}$, $1 < p < \infty$, by an essential modification of (2) in Definition 2.1: one only requires the corresponding inequality to hold for small balls, that is, when $|B| \le 1$. The class $\mathcal{A}_\infty^{\mathrm{loc}} = \bigcup_{p>1} \mathcal{A}_p^{\mathrm{loc}}$ obviously extends $\mathcal{A}_\infty$. A typical example which is contained in $\mathcal{A}_\infty^{\mathrm{loc}}$, but not in $\mathcal{A}_\infty$, is given by

$$w_{a,\exp}(x) = \begin{cases} |x|^a, & \text{if} \quad |x| \le 1, \\ \exp(|x|-1), & \text{if} \quad |x| > 1, \end{cases}$$

where $a > -n$, see [38].

Doubling weights. We come to the most important weight class in this paper which naturally extends Muckenhoupt weights.

DEFINITION 2.4. We say that a nonnegative Borel measure μ on $\mathbb{R}^n$ is *doubling* (*concerning balls*) if there exists a constant $\beta > 0$ such that

$$\mu(B(x,2r)) \le 2^{n\beta} \mu(B(x,r)), \qquad \text{for all} \quad x \in \mathbb{R}^n, \ r > 0. \tag{9}$$

The smallest such β is called *doubling constant* of μ.

REMARK 2.5. Note that the doubling measure μ need not be absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$, cf. [4]. On the other hand, any weight $w \in \mathcal{A}_\infty$ defines a doubling measure μ by $\mathrm{d}\mu = w(x)\,\mathrm{d}x$ in view of (6), see also Example 2.8 below.

In the following we are only interested in doubling measures, which are absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$. So we introduce the so-called doubling weights.

DEFINITION 2.6. Let w be a locally integrable, positive a.e. function on $\mathbb{R}^n$. w is called *doubling* (*concerning balls*) if there exists a constant $\beta > 0$ such that

$$w(B(x,2r)) \le 2^{n\beta} w(B(x,r)), \qquad \text{for all} \quad x \in \mathbb{R}^n, \ r > 0. \tag{10}$$

The smallest such β is called *doubling constant* of w (concerning balls).

EXAMPLE 2.7. Plainly $w \equiv 1$ is doubling, because $|B(x,2r)| = 2^n |B(x,r)|$ for arbitrary balls $B(x,r)$, i.e., $\beta = 1$.

EXAMPLE 2.8. All Muckenhoupt weights $w \in \mathcal{A}_\infty$ are doubling with $\beta = cr_w$ given by (7). On the contrary, there exist doubling weights which do not belong to $\mathcal{A}_\infty$, see [7, 36]. Hence $\mathcal{A}_\infty$ is a proper subset of all doubling weights which are absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$.

Now we introduce another definition of doubling weights with respect to cubes. This is an equivalent definition. The constants depend on the dimension.

DEFINITION 2.9. Let w be a locally integrable, positive a.e. function on $\mathbb{R}^n$. w is called *doubling* (*concerning cubes*) if there exists a constant $\gamma > 0$ such that for all cubes Q

$$w(2Q) \le 2^{n\gamma} w(Q). \tag{11}$$

The smallest such γ is called *doubling constant* of w (concerning cubes).

PROPOSITION 2.10. *Let w be a locally integrable, positive a.e. function on $\mathbb{R}^n$.*

(i) *The conditions* (10) *and* (11) *are equivalent.*

(ii) *For the doubling constants*

$$\frac{1}{c}\,\beta \leq \gamma \leq c\beta \tag{12}$$

where $c = \lfloor \log_2(\sqrt{n}) \rfloor + 1$.

(iii) *The doubling constants satisfy* $\beta \geq 1$ *and* $\gamma \geq 1$.

Proof. Assume that w satisfies (10). Let $Q = Q(x,l) = l \cdot \left[-\frac{1}{2}, \frac{1}{2}\right]^n + x$, $x \in \mathbb{R}^n$, $l > 0$, be an arbitrary cube. Then there exist balls $B_1 = B(x, \frac{l}{2})$, $B_2 = B(x, \sqrt{n}\, l)$, such that the outer ball touches the corners of the cube and the inner ball touches the inner sidewalls. Thus we have

$$\begin{aligned} w(2Q) = \int_{Q(x,2l)} w(y)\,\mathrm{d}y &\leq \int_{B(x,\sqrt{n}\,l)} w(y)\,\mathrm{d}y = w(B(x, \sqrt{n}\,l)) \\ &\leq 2^{n\beta} w\Big(B\Big(x, \frac{\sqrt{n}}{2}\, l\Big)\Big) \leq 2^{n\beta(k+1)} w\Big(B\Big(x, \frac{\sqrt{n}}{2^k}\, \frac{l}{2}\Big)\Big) \end{aligned}$$

where we applied (10) and $k \in \mathbb{N}$ is chosen such that $k \geq \log_2 \sqrt{n}$, say, $k = \lfloor \log_2(\sqrt{n}) \rfloor + 1$. Thus we can continue our estimate by

$$w(2Q) \leq 2^{n\beta(k+1)} w\Big(B\Big(x, \frac{l}{2}\Big)\Big) \leq 2^{n\beta(k+1)} w(Q)$$

and obtain for the doubling constants $\gamma \leq \beta\big(\lfloor \log_2(\sqrt{n}) \rfloor + 1\big)$.

Conversely, assume that (11) holds. Let $B = B(x,r)$, $x \in \mathbb{R}^n$, $r > 0$ be an arbitrary ball. Then we obtain in the same way 2 cubes $Q(x, \frac{2r}{\sqrt{n}})$, $Q(x, 4r)$ with

$$w(B(x,2r)) \leq w(Q(x,2r)) = w\Big(Q\Big(x, \frac{2r}{\sqrt{n}}\sqrt{n}\Big)\Big) \leq 2^{n\gamma k} w\Big(Q\Big(x, \frac{2r}{\sqrt{n}}\, \frac{\sqrt{n}}{2^k}\Big)\Big)$$

in view of (11), where we have to choose again $k \in \mathbb{N}$ such that $k \geq \log_2 \sqrt{n}$, say, $k = \lfloor \log_2 \sqrt{n} \rfloor + 1$. Hence,

$$w(B(x,2r)) \leq 2^{n\gamma(k+1)} w\Big(Q\Big(x, \frac{2r}{\sqrt{n}}\Big)\Big) \leq 2^{n\gamma(k+1)} w(B(x,r))$$

and we get $\beta \leq \gamma\big(\lfloor \log_2(\sqrt{n}) \rfloor + 1\big)$. This concludes the proof of (i) and (ii). It remains to verify (iii).

Let w be doubling (concerning cubes). Let Q be a cube with side-length 1. Moreover, let l be an arbitrary natural number. Q contains 2^{nl} disjoint open cubes Q_i with side-length 2^{-l} and

$$\bigcup_{i=1}^{2^{nl}} Q_i \subset Q, \quad |Q_i| = 2^{-nl}, \quad Q_i \cap Q_j = \emptyset.$$

Let Q_i be such an arbitrary small cube in Q. Then Q is covered by αQ_i, where $\alpha 2^{-l} \geq 2$, i.e., $\alpha \geq 2^{l+1}$. Hence we get for all $i \in \mathbb{N}$ and all $l \in \mathbb{N}$,

$$\begin{aligned} 2^{nl} \min_{j \in \mathbb{N}} w(Q_j) \leq \sum_{j=1}^{2^{nl}} w(Q_j) \leq w\Big(\bigcup_{j=1}^{2^{nl}} Q_j\Big) &\leq w(Q) \\ &\leq w(2^{l+1} Q_i) \leq 2^{n\gamma(l+1)} w(Q_i). \end{aligned}$$

For fixed $l \in \mathbb{N}$ we have only finitely many $i = 1, \dots, 2^{nl}$ and can thus choose i such that $w(Q_i)$ is minimal. Hence we obtain $\frac{l}{l+1} \leq \gamma$ for arbitrary $l \in \mathbb{N}$, that is, $\gamma \geq 1$. The proof for β is similar. ∎

REMARK 2.11. In the following we will not distinguish between doubling weights concerning balls or concerning cubes as long as their doubling constants do not play any role. Otherwise we stick to our convention to label the doubling constant concerning balls with β, and the one concerning cubes with γ.

We prove another feature of doubling weights which will be used in Section 3.3 below.

PROPOSITION 2.12. *Let w be a doubling weight. Then*

$$\int_{\mathbb{R}^n} w(y)\,\mathrm{d}y = \infty.$$

Proof. Let w be doubling with $w(B(x,2r)) \leq c w(B(x,r))$ for arbitrary $x \in \mathbb{R}^n$, $r > 0$, and $c = 2^{n\beta}$ for convenience. Let $R_0 > 0$ be arbitrary and $x_0 = \big(\frac{R_0}{2}, 0, \dots, 0\big)$, $x_1 = \big(2R_0, 0, \dots, 0\big)$. Then

$$B\Big(x_0, \frac{R_0}{2}\Big) \subset \{y \in \mathbb{R}^n : R_0 \leq |y - x_1| \leq 2R_0\}. \tag{13}$$

Since w is doubling and non-trivial, we have $w(B(x_0, \frac{R_0}{2})) \geq a_0 > 0$. Now

$$w(B(x_1, 2R_0)) = \int_{B(x_1,R_0)} w(y)\,\mathrm{d}y + \int_{R_0 \leq |y-x_1| \leq 2R_0} w(y)\,\mathrm{d}y \geq \frac{1}{c} w(B(x_1, 2R_0)) + a_0$$

in view of (13). This leads to $w(B(x_1, 2R_0)) \geq a_0 \frac{c}{c-1}$. Set $R_1 = 4R_0$, such that $x_1 = \big(\frac{R_1}{2}, 0, \dots, 0\big)$. Inductively we define $R_{k+1} = 4R_k$, $x_k = \big(\frac{R_k}{2}, 0, \dots, 0\big)$ for $k \in \mathbb{N}_0$ and repeat the argument above. Then we get

$$w\Big(B\Big(x_k, \frac{R_k}{2}\Big)\Big) \geq a_0 \Big(\frac{c}{c-1}\Big)^k$$

which finally leads to

$$\int_{\mathbb{R}^n} w(y)\,\mathrm{d}y \geq \lim_{k\to\infty} \int_{B(x_k, R_k/2)} w(y)\,\mathrm{d}y \geq a_0 \lim_{k\to\infty} \Big(\frac{c}{c-1}\Big)^k = \infty$$

as desired. ∎

Admissible weights. We use the abbreviation $\langle x \rangle = (1 + |x|^2)^{1/2}$, $x \in \mathbb{R}^n$.

DEFINITION 2.13. The class of admissible weight functions is the collection of all positive C^∞ functions w on $\mathbb{R}^n$ with the following properties:

(i) for all $\eta \in \mathbb{N}_0^n$ there exists a positive constant c_η with

$$|\mathrm{D}^\eta w(x)| \leq c_\eta\, w(x) \quad \text{for all} \quad x \in \mathbb{R}^n;$$

(ii) there exist two constants $c > 0$ and $\alpha \geq 0$ such that

$$0 < w(x) \leq c\, w(y)\, \langle x - y \rangle^\alpha \quad \text{for all} \quad x, y \in \mathbb{R}^n.$$

REMARK 2.14. These are the weights we dealt with in [15, 16], see also [5, 17–19]. Note that for admissible weights w and v, also $1/w$ and vw are admissible weights.

Example 2.15. Obviously, $v_\alpha(x) = \langle x \rangle^\alpha$, $\alpha \in \mathbb{R}$, is an admissible weight. Note that $v_\alpha \in \mathcal{A}_\infty$ for $\alpha > -n$ unlike in case of $\alpha \le -n$. Conversely, $w_{a,b}$ given by (8) with $-n < a < 0$, $b > -n$, is not admissible in the above sense, but belongs to $\mathcal{A}_\infty$.

Remark 2.16. In view of the above examples the classes of Muckenhoupt weights $\mathcal{A}_\infty$ and admissible weights are incomparable. However, replacing the class $\mathcal{A}_\infty$ by their generalisation, the local Muckenhoupt weights $\mathcal{A}_\infty^{\mathrm{loc}}$, recall Remark 2.3, the situation changes: According to [16, 25, 26] admissible weights are special local Muckenhoupt weights, too.

2.2. Weighted Besov spaces. First we introduce the weighted Lebesgue space $L_p(w)$ with a doubling weight w as usual via the weighted L_p norm,

$$\big\| f \,|\, L_p(\mathbb{R}^n, w)\big\| = \Big(\int_{\mathbb{R}^n} |f(x)|^p w(x)\,\mathrm{d}x \Big)^{1/p}, \quad 0 < p < \infty. \tag{14}$$

It is clear that $L_p(\mathbb{R}^n, w) = L_p(\mathbb{R}^n)$ for $w \equiv 1$. For $p = \infty$ one obtains the classical (unweighted) Lebesgue space, $L_\infty(\mathbb{R}^n, w) = L_\infty(\mathbb{R}^n)$; we thus mainly restrict ourselves to $p < \infty$ in what follows. The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ and its dual $\mathcal{S}'(\mathbb{R}^n)$ of all complex-valued tempered distributions have their usual meaning here. Let $\varphi_0 = \varphi \in \mathcal{S}(\mathbb{R}^n)$ be such that

$$\operatorname{supp} \varphi \subset \{ y \in \mathbb{R}^n : |y| < 2\} \quad \text{and} \quad \varphi(x) = 1 \quad \text{if} \quad |x| \le 1,$$

and for each $j \in \mathbb{N}$ let $\varphi_j(x) = \varphi(2^{-j}x) - \varphi(2^{-j+1}x)$. Then $\{\varphi_j\}_{j=0}^\infty$ forms a smooth dyadic decomposition of unity. Given any $f \in \mathcal{S}'(\mathbb{R}^n)$, we denote by $\mathcal{F}f$ and $\mathcal{F}^{-1}f$ its Fourier transform and its inverse Fourier transform, respectively.

Definition 2.17. Let $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$, $\{\varphi_j\}_{j=0}^\infty$ a smooth dyadic decomposition of unity and let w be a doubling weight. *The weighted Besov space* $B^s_{p,q}(w) = B^s_{p,q}(\mathbb{R}^n, w)$ is the set of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\big\| f \,|\, B^s_{p,q}(w)\big\| = \Big(\sum_{j=0}^\infty 2^{jsq} \big\| \mathcal{F}^{-1}(\varphi_j \mathcal{F} f) \,|\, L_p(w)\big\|^q \Big)^{1/q}$$

is finite (with the usual modification in the limiting case $q = \infty$).

Remark 2.18. The spaces $B^s_{p,q}(w)$ are independent of the choice of the smooth dyadic decomposition of unity $\{\varphi_j\}_{j=0}^\infty$ appearing in their definitions, cf. [2]. They are quasi-Banach spaces (Banach spaces for $p, q \ge 1$). Moreover, for $w \equiv 1$ we re-obtain the usual (unweighted) Besov spaces; we refer, in particular, to the series of monographs by Triebel, [31–35], for comprehensive treatment of the unweighted spaces.

We have also the usual basic embeddings for these weighted spaces, that is,

$$B^{s_0}_{p,q}(w) \hookrightarrow B^{s_1}_{p,q}(w) \quad \text{and} \quad B^s_{p,q_0}(w) \hookrightarrow B^s_{p,q_1}(w), \tag{15}$$

where $0 < p < \infty$, w is doubling, $-\infty < s_1 \le s_0 < \infty$ and $0 < q_0 \le q_1 \le \infty$.

Atomic decomposition. In the proof of our main theorem we need another representation of $f \in B^s_{p,q}(w)$ than defined above; we shall use an appropriate atomic decomposition result. Let us first introduce corresponding sequence spaces and the concept of atoms.

Recall our definition of $Q_{\nu,m}$, where $m \in \mathbb{Z}^n$, $\nu \in \mathbb{N}_0$, in the beginning. For $0 < p < \infty$, $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$ we denote by $\chi^{(p)}_{\nu,m}$ the p-normalised characteristic function of the cube $Q_{\nu,m}$ defined by

$$\chi^{(p)}_{\nu,m}(x) = 2^{\nu n/p}\chi_{\nu,m}(x) = \begin{cases} 2^{\nu n/p}, & \text{if} \quad x \in Q_{\nu,m}, \\ 0, & \text{if} \quad x \notin Q_{\nu,m}. \end{cases} \tag{16}$$

It is easy to see that $\big\|\chi^{(p)}_{\nu,m} \,|\, L_p(\mathbb{R}^n)\big\| = 1$. For $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$, w doubling, we introduce suitable sequence spaces $b^s_{p,q}(w)$ by

$$b^s_{p,q}(w) = \Big\{ \lambda = \{\lambda_{\nu,m}\}_{\nu,m} : \lambda_{\nu,m} \in \mathbb{C},$$
$$\big\|\lambda \,|\, b^s_{p,q}(w)\big\| = \Big\| \Big\{ 2^{\nu(s-n/p)} \Big\| \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}|\, \chi^{(p)}_{\nu,m} \;\Big|\; L_p(w) \Big\| \Big\}_{\nu\in\mathbb{N}_0} \,|\, \ell_q \Big\| < \infty \Big\}.$$

We briefly recall the definition of atoms.

Definition 2.19. Let $K \in \mathbb{N}_0$ and $d > 1$.

(i) The complex-valued function $a \in C^K(\mathbb{R}^n)$ is said to be an 1_K*-atom* if $\operatorname{supp} a \subset dQ_{0,m}$ for some $m \in \mathbb{Z}^n$, and $|\mathrm{D}^\eta a(x)| \le 1$ for $|\eta| \le K$, $x \in \mathbb{R}^n$.

(ii) Let $L+1 \in \mathbb{N}_0$. The complex-valued function $a \in C^K(\mathbb{R}^n)$ is said to be an (K,L)-*atom* if for some $\nu \in \mathbb{N}_0$,

$$\begin{aligned} \operatorname{supp} a &\subset dQ_{\nu,m} && \text{for some } m \in \mathbb{Z}^n, \\ |\mathrm{D}^\eta a(x)| &\le 2^{|\eta|\nu} && \text{for all } x \in \mathbb{R}^n \text{ and } \eta \in \mathbb{N}_0^n \text{ with } |\eta| \le K, \\ \int_{\mathbb{R}^n} x^\theta a(x)\,\mathrm{d}x &= 0 && \text{for all } \theta \in \mathbb{N}_0^n \text{ with } |\theta| \le L. \end{aligned}$$

We shall denote an atom $a(x)$ supported in some $dQ_{\nu,m}$ by $a_{\nu,m}$ in the sequel. Choosing $L = -1$ in (ii) means that no moment conditions are required.

Remark 2.20. The next result that we want to apply is from Bownik, [2], see also [3] for parallel observations. Note that Bownik dealt with anisotropic Besov spaces with expansive dilation matrices and more general doubling measures. The difference is that there are used quasi-norms ϱ_A associated with an expansive matrix A. In the standard dyadic case $A = 2I$ a quasi-norm ϱ_A satisfies $\varrho_A(2x) = 2^n \varrho_A(x)$ instead of the usual scalar homogeneity. In particular, $\varrho_A(x) = |x|^n$ is an example for a quasi-norm for $A = 2I$. Instead of this quasi-norm $|\cdot|^n$ we will use the usual Euclidean norm $|\cdot|$ in $\mathbb{R}^n$. For more details we refer to [1, 20]. We recall that all quasi-norms associated to a fixed dilation matrix A are equivalent. Moreover, there always exists a quasi-norm ϱ_A, which is C^∞ on $\mathbb{R}^n$ except the origin. Note also that Bownik dealt with a different decomposition of unity, but we get equivalent quasi-norms. In the main part of [2] Bownik works with homogeneous spaces, later he showed that these results also hold for inhomogeneous spaces. Furthermore the atoms and the sequence spaces are L_2-normalised. In our case we have an L_∞-normalisation.

For convenience we adopt the usual notation

$$\sigma_p = n\Big(\frac{1}{p} - 1\Big)_+, \quad 0 < p \le \infty. \tag{17}$$

Then the atomic decomposition result used below reads as follows, see [2, Thm. 5.10] with the above-described modifications.

Proposition 2.21. *Let* $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$ *and* w *be a doubling weight with doubling constant* β. *Let* $K, L+1 \in \mathbb{N}_0$ *with*

$$K \ge (1+\lfloor s \rfloor)_+ \quad \text{and} \quad L \ge \max\Big(-1, \Big\lfloor \frac{n(\beta-1)}{p} + \sigma_p - s \Big\rfloor\Big). \tag{18}$$

A tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ *belongs to* $B^s_{p,q}(\mathbb{R}^n, w)$ *if and only if it can be written as a series*

$$f = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m} a_{\nu,m}(x), \quad \text{converging in } \mathcal{S}'(\mathbb{R}^n), \tag{19}$$

where $a_{\nu,m}(x)$ *are* 1_K*-atoms* ($\nu = 0$) *or* (K,L)*-atoms* ($\nu \in \mathbb{N}$) *and* $\lambda = \{\lambda_{\nu,m}\}_{\nu,m} \in b^s_{p,q}(w)$. *Furthermore*

$$\inf \big\| \lambda \,|\, b^s_{p,q}(w) \big\| \tag{20}$$

is an equivalent quasi-norm in $B^s_{p,q}(\mathbb{R}^n, w)$, *where the infimum ranges over all admissible representations* (19).

We exemplify the above result in two cases and compare it with known results.

Example 2.22. Let $w \equiv 1$. Then we have by Example 2.7 $\beta = 1$, such that (18) reads as $K \ge (1+\lfloor s \rfloor)_+$ and $L \ge \max(-1, \lfloor \sigma_p - s \rfloor)$. This result coincides with [34, Thm. 13.8].

Example 2.23. Let $w \in \mathcal{A}_\infty$. Then by Example 2.8 we have $\beta = cr_w$ such that (18) can be compared with [10, Thm. 3.10]. This result has better quantitative characteristics than the ones obtained here as long as we stay in the realm of $\mathcal{A}_\infty$ weights. This is the price to pay by studying Besov spaces with doubling weights instead of $\mathcal{A}_\infty$ weights.

Remark 2.24. Weighted Besov spaces and their atomic (and wavelet) decompositions in case of admissible weights have been studied in some detail in [15–19]. As far as local Muckenhoupt weights $\mathcal{A}_p^{\text{loc}}$ are concerned, we refer to [25, 37–39].

3. Continuous embeddings

3.1. Embeddings of sequence spaces. Before we come to state our main result, we return to our description of the sequence spaces $b^s_{p,q}(w)$ and adapt it to our needs.

Let $\lambda = (\lambda_{\nu,m})_{\nu,m} \subset \mathbb{C}$, $s \in \mathbb{R}$, $0 < p < \infty$, and assume $0 < q < \infty$ for convenience. Then by the support property of the atoms,

$$\begin{aligned}
\big\| \lambda \,|\, b^s_{p,q}(w) \big\| &= \Big(\sum_{\nu=0}^{\infty} 2^{\nu(s-n/p)q} \Big\| \sum_{m \in \mathbb{Z}^n} |\lambda_{\nu,m}| \chi^{(p)}_{\nu,m} \,|\, L_p(w) \Big\|^q \Big)^{1/q} \\
&\sim \Big(\sum_{\nu=0}^{\infty} 2^{\nu(s-n/p)q} \Big(\int_{\mathbb{R}^n} \sum_{m \in \mathbb{Z}^n} |\lambda_{\nu,m}|^p 2^{\nu n} \chi_{\nu,m}(x) w(x) \,\mathrm{d}x \Big)^{q/p} \Big)^{1/q} \\
&= \Big(\sum_{\nu=0}^{\infty} 2^{\nu s q} \Big(\sum_{m \in \mathbb{Z}^n} |\lambda_{\nu,m}|^p w(Q_{\nu,m}) \Big)^{q/p} \Big)^{1/q} \\
&= \Big(\sum_{\nu=0}^{\infty} \xi_\nu^q \Big(\sum_{m \in \mathbb{Z}^n} |\lambda_{\nu,m}|^p |w_{\nu,m}|^p \Big)^{q/p} \Big)^{1/q} =: \big\| \lambda \,|\, \ell_q(\xi_\nu \ell_p(\tilde{w})) \big\|
\end{aligned} \tag{21}$$

with $\xi = (\xi_\nu)_\nu = (2^{\nu s})_\nu$, and $\widetilde{w} = (w_{\nu,m})_{\nu,m}$, $w_{\nu,m} = w(Q_{\nu,m})^{1/p}$, $\nu \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$. Note that, if w is doubling, then $0 < w(B) < \infty$ for all balls B. Therefore $\xi = (\xi_\nu)_\nu$ and $\widetilde{w} = (w_{\nu,m})_{\nu,m}$ are sequences of positive numbers.

Corollary 3.1. *Let $-\infty < s_2 \le s_1 < \infty$, $0 < p_1, p_2 < \infty$, $0 < q_1, q_2 \le \infty$ and let w_1, w_2 be doubling weights. Then $b^{s_1}_{p_1,q_1}(w_1) \hookrightarrow b^{s_2}_{p_2,q_2}(w_2)$ if and only if*

$$\Big\{2^{-\nu(s_1-s_2)} \big\| \{w_1(Q_{\nu,m})^{-1/p_1} w_2(Q_{\nu,m})^{1/p_2}\}_m \,|\, \ell_{p^*} \big\| \Big\}_\nu \in \ell_{q^*}, \tag{22}$$

where

$$\frac{1}{p^*} := \Big(\frac{1}{p_2} - \frac{1}{p_1}\Big)_+, \qquad \frac{1}{q^*} := \Big(\frac{1}{q_2} - \frac{1}{q_1}\Big)_+. \tag{23}$$

Proof. Let $b^{s_k}_{p_k,q_k}(w_k) = \ell_{q_k}(\xi^{(k)}_\nu \ell_{p_k}(w^{(k)}))$ with $\xi^{(k)}_\nu = 2^{\nu s_k}$ and $w^{(k)} = (w^{(k)}_{\nu,m})_{\nu,m}$, $w^{(k)}_{\nu,m} = w_k(Q_{\nu,m})^{1/p_k}$, $k = 1, 2$, be given. We apply [18, Thm. 3.1] and deduce that

$$\ell_{q_1}(\xi^{(1)}_\nu \ell_{p_1}(w^{(1)})) = b^{s_1}_{p_1,q_1}(w_1) \hookrightarrow b^{s_2}_{p_2,q_2}(w_2) = \ell_{q_2}(\xi^{(2)}_\nu \ell_{p_2}(w^{(2)}))$$

holds if and only if

$$\Big\{ \frac{\xi^{(2)}_\nu}{\xi^{(1)}_\nu} \Big\| \Big\{ \frac{w^{(2)}_{\nu,m}}{w^{(1)}_{\nu,m}} \Big\}_m \,|\, \ell_{p^*} \Big\| \Big\}_\nu \in \ell_{q^*}$$

which coincides with (22). ∎

3.2. The main result

Theorem 3.2. *Let $-\infty < s_2 \le s_1 < \infty$, $0 < p_1, p_2 < \infty$, $0 < q_1, q_2 \le \infty$ and let w_1, w_2 be doubling weights with the corresponding doubling constants β_1, β_2. The embedding $B^{s_1}_{p_1,q_1}(w_1) \hookrightarrow B^{s_2}_{p_2,q_2}(w_2)$ is continuous, if*

$$\Big\{2^{-\nu(s_1-s_2)} \big\| \{w_1(Q_{\nu,m})^{-1/p_1} w_2(Q_{\nu,m})^{1/p_2}\}_m \,|\, \ell_{p^*} \big\| \Big\}_\nu \in \ell_{q^*}, \tag{24}$$

where p^* and q^* are given by (23).

Proof. Let $f \in B^{s_1}_{p_1,q_1}(w_1)$ and $\varepsilon > 0$. Let $K_1, L_1 + 1 \in \mathbb{N}_0$ be such that $K_1 \ge (1 + \lfloor s_1 \rfloor)_+$ and $L_1 \ge \max\big(-1, \lfloor \frac{n(\beta_1-1)}{p_1} + \sigma_{p_1} - s_1 \rfloor\big)$. Then by Proposition 2.21 we can find (K_1, L_1)-atoms $a_{\nu,m}(\cdot)$ and numbers $\lambda = (\lambda_{\nu,m})_{\nu,m} \in b^{s_1}_{p_1,q_1}(w_1)$ such that f can be represented as in (19) and

$$c_1 \big\| f \,|\, B^{s_1}_{p_1,q_1}(w_1) \big\| \ge \big\| \lambda \,|\, b^{s_1}_{p_1,q_1}(w_1) \big\| - \varepsilon.$$

Assumption (24) together with Corollary 3.1 implies that $b^{s_1}_{p_1,q_1}(w_1) \hookrightarrow b^{s_2}_{p_2,q_2}(w_2)$, i.e.,

$$\big\| \lambda \,|\, b^{s_2}_{p_2,q_2}(w_2) \big\| \le c_2 \big\| \lambda \,|\, b^{s_1}_{p_1,q_1}(w_1) \big\| \le c_3 \big\| f \,|\, B^{s_1}_{p_1,q_1}(w_1) \big\| + c_2 \varepsilon.$$

Hence we have

$$f = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m} a_{\nu,m}(x) \in B^{s_2}_{p_2,q_2}(w_2)$$

with $\big\| f \,|\, B^{s_2}_{p_2,q_2}(w_2) \big\| \le c_4 \big\| \lambda \,|\, b^{s_2}_{p_2,q_2}(w_2) \big\|$, granted that $a_{\nu,m}(\cdot)$ are also (K_2, L_2)-atoms, where $K_2 \ge (1 + \lfloor s_2 \rfloor)_+$ and $L_2 \ge \max\big(-1, \lfloor \frac{n(\beta_2-1)}{p_2} + \sigma_{p_2} - s_2 \rfloor\big)$. So we choose $K \ge \max(K_1, K_2)$, $L \ge \max(L_1, L_2)$ sufficiently large and start from the very beginning with associated (K, L)-atoms $a_{\nu,m}(\cdot)$. Thus we get for all $\varepsilon > 0$

$$\big\| f \,|\, B^{s_2}_{p_2,q_2}(w_2) \big\| \le c_4 \big\| \lambda \,|\, b^{s_2}_{p_2,q_2}(w_2) \big\| \le c_5 \big\| f \,|\, B^{s_1}_{p_1,q_1}(w_1) \big\| + c_6 \varepsilon,$$

and letting $\varepsilon \to 0$ completes the argument. ∎

Remark 3.3. The above result extends its counterpart for Muckenhoupt weights naturally, see [11]. Note that we obtained there even necessary and sufficient conditions, but dealing with wavelet decompositions instead of atomic ones. This has not yet been studied in case of doubling weights. In case of admissible weights parallel results can be found in [15, 16, 18].

3.3. Some consequences. We collect a few examples and immediate implications of our main result.

Example 3.4. For $w_1 \equiv w_2 \equiv 1$ we have $w_1(Q_{\nu,m}) = w_2(Q_{\nu,m}) = 2^{-\nu n}$. Thus we get in (24)

$$2^{-\nu(s_1-s_2)}\big\|\{2^{\nu n/p_1-\nu n/p_2}\}_m\,|\,\ell_{p^*}\big\| = 2^{-\nu(s_1-n/p_1)+\nu(s_2-n/p_2)}\big\|\{1\}_m\,|\,\ell_{p^*}\big\|.$$

Then $\big\|1\,|\,\ell_{p^*}\big\| < \infty$ immediately implies $p^* = \infty$, that is, $p_1 \le p_2$. We set

$$\delta := s_1 - \frac{n}{p_1} - s_2 + \frac{n}{p_2} \tag{25}$$

as the difference of the differential dimensions, as usual. Then it remains to consider $\{2^{-\nu\delta}\}_\nu \in \ell_{q^*}$. For $q^* = \infty$, i.e., $q_1 \le q_2$, we need $\delta \ge 0$. Otherwise, for $q_1 > q_2$, $\delta > 0$ is required. Altogether the embedding $B^{s_1}_{p_1,q_1} \hookrightarrow B^{s_2}_{p_2,q_2}$ is continuous, if

$$p_1 \le p_2, \quad s_2 \le s_1, \quad \begin{cases} \delta \ge 0, & \text{if} \quad q_1 \le q_2, \\ \delta > 0, & \text{if} \quad q_1 > q_2. \end{cases}$$

Example 3.5. Sometimes it is interesting to consider the case where only the source space is weighted and the target space is unweighted, i.e., we have $w_1 = w$ and $w_2 \equiv 1$. Then Theorem 3.2 implies that $B^{s_1}_{p_1,q_1}(w) \hookrightarrow B^{s_2}_{p_2,q_2}$ if

$$\Big\{2^{-\nu(s_1-s_2+n/p_2)}\big\|\{w(Q_{\nu,m})^{-1/p_1}\}_m\,|\,\ell_{p^*}\big\|\Big\}_\nu \in \ell_{q^*},$$

where p^* and q^* are given by (23).

Example 3.6. Also interesting is the case where both spaces are weighted in the same way, i.e., $w_1 = w_2 = w$. Here we get in (24) terms of type

$$2^{-\nu(s_1-s_2)}\big\|\{w(Q_{\nu,m})^{1/p_2-1/p_1}\}_m\,|\,\ell_{p^*}\big\|. \tag{26}$$

When $p^* < \infty$, i.e., $p_1 > p_2$, we have $0 < \frac{1}{p_2} - \frac{1}{p_1} = \frac{1}{p^*}$. For any fixed $\nu \in \mathbb{N}_0$ we have

$$\begin{aligned}\big\|\{w(Q_{\nu,m})^{1/p_2-1/p_1}\}_m\,|\,\ell_{p^*}\big\| &= \Big(\sum_{m\in\mathbb{Z}^n}\big|w(Q_{\nu,m})^{1/p^*}\big|^{p^*}\Big)^{1/p^*} = \Big(\sum_{m\in\mathbb{Z}^n} w(Q_{\nu,m})\Big)^{1/p^*} \\ &= \Big(\sum_{m\in\mathbb{Z}^n}\int_{Q_{\nu,m}} w(y)\,\mathrm{d}y\Big)^{1/p^*} = \Big(\int_{\mathbb{R}^n} w(y)\,\mathrm{d}y\Big)^{1/p^*}.\end{aligned}$$

So we have to demand for our weight that $\int_{\mathbb{R}^n} w(y)\,\mathrm{d}y < \infty$. But this impossible for a doubling weight, recall Proposition 2.12. So we are left to consider the case $p^* = \infty$, i.e., $p_1 \le p_2$. Then $\frac{1}{p_2} - \frac{1}{p_1} < 0$. For a fixed $\nu \in \mathbb{N}_0$ we have to deal with expressions of type

$$\sup_{m\in\mathbb{Z}^n} w(Q_{\nu,m})^\kappa$$

for an arbitrary $\kappa < 0$.

Now we want to simplify the expression in (24) a little bit using the doubling property. For every cube $Q_{\nu,m}$, $\nu \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, we get the cube $Q_{0,2^{-\nu}m}$ via ν-times application of (11)

$$w(Q_{0,2^{-\nu}m}) \le 2^{\nu n\gamma} w(Q_{\nu,m}).$$

So, for any $\kappa < 0$, we have

$$w(Q_{\nu,m})^{\kappa} \le 2^{-\nu n\gamma\kappa} w(Q_{0,2^{-\nu}m})^{\kappa}.$$

For $p^* = \infty$, i.e., $p_1 \le p_2$, we get

$$\begin{aligned}\big\|\{w(Q_{\nu,m})^{\kappa}\}_m \,|\, \ell_\infty\big\| &= \sup_{m\in\mathbb{Z}^n} w(Q_{\nu,m})^{\kappa} \le 2^{-\nu n\gamma\kappa} \sup_{m\in\mathbb{Z}^n} \left(w(Q_{0,2^{-\nu}m})^{\kappa}\right)\\ &= 2^{-\nu n\gamma\kappa}\big(\inf_{m\in\mathbb{Z}^n} w(Q_{0,2^{-\nu}m})\big)^{\kappa} \le 2^{-\nu n\gamma\kappa}\big(\inf_{x\in\mathbb{Q}^n} w(Q_{0,x})\big)^{\kappa}\end{aligned}$$

and we have to require

$$\inf_{x\in\mathbb{Q}^n} w(Q_{0,x}) \ge c > 0 \tag{27}$$

for our weight.

Corollary 3.7. *Let $-\infty < s_2 \le s_1 < \infty$, $0 < p_1 \le p_2 < \infty$, $0 < q_1, q_2 \le \infty$ and let w be a doubling weight with the corresponding doubling constant γ. The embedding $B^{s_1}_{p_1,q_1}(w) \hookrightarrow B^{s_2}_{p_2,q_2}(w)$ is continuous, if*

$$\inf_{x\in\mathbb{Q}^n} w(Q_{0,x}) \ge c > 0, \tag{28}$$

$$\begin{cases}\delta > n(\gamma-1)(\frac{1}{p_1}-\frac{1}{p_2}), & \text{if } \ q^* < \infty,\\ \delta \ge n(\gamma-1)(\frac{1}{p_1}-\frac{1}{p_2}), & \text{if } \ q^* = \infty,\end{cases} \tag{29}$$

where δ is given by (25).

Proof. As in Example 3.6 we have $\kappa = \frac{1}{p_2} - \frac{1}{p_1} \le 0$. In view of our above considerations (24) can be reduced to

$$\begin{aligned}&2^{-\nu(s_1-s_2)}\big\|\{w_1(Q_{\nu,m})^{-1/p_1} w_2(Q_{\nu,m})^{1/p_2}\}_m \,|\, \ell_{p^*}\big\|\\ &\quad\le 2^{-\nu(s_1-s_2)} 2^{-\nu n\gamma(1/p_2-1/p_1)}\big(\inf_{x\in\mathbb{Q}^n} w(Q_{0,x})\big)^{1/p_2-1/p_1}\\ &\quad\le c\, 2^{-\nu(s_1-s_2+n\gamma(1/p_2-1/p_1))}\end{aligned}$$

in view of (28). This converges in ℓ_{q^*} for $(s_1-s_2)+n\gamma(\frac{1}{p_2}-\frac{1}{p_1}) = \delta+n(\gamma-1)(\frac{1}{p_2}-\frac{1}{p_1}) > 0$, where we can admit, in addition, the limiting case $\delta = n(\gamma-1)(\frac{1}{p_1}-\frac{1}{p_2})$ if $q^* = \infty$, that is, when $q_1 \le q_2$. ∎

Corollary 3.8. *Let $-\infty < s_2 \le s_1 < \infty$, $0 < p_1 \le p_2 < \infty$, $0 < q_1, q_2 \le \infty$ and let w be a doubling weight with the corresponding doubling constant γ. The embedding $B^{s_1}_{p_1,q_1}(w) \hookrightarrow B^{s_2}_{p_2,q_2}$ is continuous, if*

$$\inf_{x\in\mathbb{Q}^n} w(Q_{0,x}) \ge c > 0, \tag{30}$$

$$\begin{cases}\delta > \frac{n}{p_1}(\gamma-1), & \text{if } \ q^* < \infty,\\ \delta \ge \frac{n}{p_1}(\gamma-1), & \text{if } \ q^* = \infty.\end{cases} \tag{31}$$

Proof. We obtain in (24) similarly as above,

$$\begin{aligned} & 2^{-\nu(s_1-s_2)}\big\|\{w_1(Q_{\nu,m})^{-1/p_1}w_2(Q_{\nu,m})^{1/p_2}\}_m \mid \ell_{p^*}\big\| \\ & \quad = 2^{-\nu(s_1-s_2)}2^{-\nu n/p_2}\sup_{m\in\mathbb{Z}^n} w(Q_{\nu,m})^{-1/p_1} \\ & \quad \le 2^{-\nu(s_1-s_2+n/p_2-n\gamma/p_1)}\Big(\inf_{x\in\mathbb{Q}^n} w(Q_{0,x})\Big)^{-1/p_1} \le c\,2^{-\nu(\delta-n(\gamma-1)/p_1)} \end{aligned}$$

using (28). This converges in ℓ_{q^*}, if $\delta-\frac{n}{p_1}(\gamma-1)>0$, where we may again admit, in addition, $\delta=\frac{n}{p_1}(\gamma-1)$ when $q_1\le q_2$. ■

Remark 3.9. Here we followed corresponding arguments from [11, 13] for Muckenhoupt weights.

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