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ON $\mathcal{Q}_p$ AND $\mathcal{D}_p$ SPACES OF HOLOMORPHIC FUNCTIONS IN $\mathbb{C}^n$

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Abstract. Following the line of Ouyang et al. in [Ouy2] to study the $\mathcal{Q}_p$ spaces of holomorphic functions in the unit ball of $\mathbb{C}^n$, we present in this paper several results and relations among $\mathcal{Q}_p(\mathbb{B}_n)$, the α-Bloch, the Dirichlet $\mathcal{D}_p$ and the little $\mathcal{Q}_{p,0}$ spaces.

1. Introduction. In 1998 C. Ouyang, W. Yang and R. Zhao in [Ouy2] introduced the $\mathcal{Q}_p$ and $\mathcal{Q}_{p,0}$ spaces associated with the Green function for the unit ball of $\mathbb{C}^n$ as an extension from the case of one complex variable, providing basic properties and relationships of these spaces with some others like Bloch and BMOA spaces. K. Stroethoff in 1989 [Str] was one of the first in generalizing those functions spaces to the unit ball in several complex variables working directly with the gradient of a holomorphic function.

J. S. Choa et al. in [Choa] and Ouyang et al. in [Ouy1] gave several characterizations of Bloch functions in $\mathbb{C}^n$ working with the invariant gradient. [Ouy2] includes mathematical

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tools according to the settings you must do to work practically with spaces in several complex variables such as the gradient and measure invariant. They present in that work many properties. In this paper we shall give additional properties since these spaces are plenty of material to study. For example the inclusion relations between the spaces $\mathcal{Q}_p$, α-Bloch and $\mathcal{D}_p$ spaces.

2. Preliminaries. The general reference for this section is the book of K. Zhu [Zh]. We adopt his notation.

The open unit ball in $\mathbb{C}^n$ ($n \geq 2$ throughout this paper) is the set

$$\mathbb{B}_n = \{z \in \mathbb{C}^n : |z| < 1\}.$$

The boundary of $\mathbb{B}_n$ will be denoted by $\mathbb{S}_n$. For $a \in \mathbb{B}_n \setminus \{0\}$ define the involutive automorphisms or Möbius transformation as

$$\phi_a(z) = \frac{a - \frac{\langle z,a\rangle}{|a|^2} a - \sqrt{1-|a|^2}\big(z - \frac{\langle z,a\rangle}{|a|^2}\, a\big)}{1 - \langle z, a\rangle} \qquad (z \in \mathbb{B}_n)\ ,$$

and $\phi_0(z) = -z$, where $\langle z, w\rangle = z_1\overline{w_1} + \ldots + z_n\overline{w_n}$ denotes the inner product. Then $\phi_a(0) = a$, $\phi_a(a) = 0$, $\phi_a = \phi_a^{-1}$ and $\phi_a \in \operatorname{Aut}(\mathbb{B}_n)$, where $\operatorname{Aut}(\mathbb{B}_n)$ is the group of biholomorphic authomorphisms of $\mathbb{B}_n$. It is known that

$$1 - |\phi_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\langle z,a\rangle|^2} = (1-|z|^2) J_R\phi_a(z)^{1/(n+1)} \tag{1}$$

where $J_R\phi_a(z)$ is the real Jacobian determinant of ϕ_a.

Let dv denote the normalized volume measure of $\mathbb{B}_n$, that is $v(\mathbb{B}_n) = 1$. The normalized surface measure on $\mathbb{S}_n$ will be denoted by $d\sigma$. Thus $\sigma(\mathbb{S}_n) = 1$ and we have in polar coordinates the formula

$$\int_{\mathbb{B}_n} f(z)\, dv(z) = 2n \int_0^1 r^{2n-1}\, dr \int_{\mathbb{S}_n} f(r\zeta)\, d\sigma(\zeta). \tag{2}$$

for all $f \in L^1(\mathbb{B}_n, dv)$. Let $d\lambda(z) = \frac{dv}{(1-|z|^2)^{n+1}}$, then $d\lambda$ is an $\mathcal{M}$-invariant measure, which means

$$\int_{\mathbb{B}_n} f(z)\, d\lambda(z) = \int_{\mathbb{B}_n} f \circ \phi(z)\, d\lambda(z)$$

for each $f \in L^1(\mathbb{B}_n, d\lambda)$ and $\phi \in \operatorname{Aut}(\mathbb{B}_n)$.

The space of holomorphic functions in $\mathbb{B}_n$ will be denoted by $\mathrm{Hol}(\mathbb{B}_n)$. The complex gradient is denoted by $\nabla f = (\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_k})$ and $Rf = \sum_{k=1}^n z_k \frac{\partial f}{\partial z_k}$ is the radial derivative of f.

Let $f \in \mathrm{Hol}(\mathbb{B}_n)$, the invariant gradient of f is defined by $|\tilde{\nabla} f(z)| = |\nabla(f \circ \phi_z)(0)|$ which means

$$|\tilde{\nabla}(f \circ \phi)| = |(\tilde{\nabla} f) \circ \phi| \tag{3}$$

for all $\phi \in \operatorname{Aut}(\mathbb{B}_n)$.

For each $f \in \mathrm{Hol}(\mathbb{B}_n)$ we have

$$|\tilde{\nabla} f(z)|^2 = (1-|z|^2)(|\nabla f(z)|^2 - |Rf(z)|^2). \tag{4}$$

The Bloch space $\mathcal{B}$ of $\mathbb{B}_n$ is defined as the set of holomorphic functions $f \in \mathrm{Hol}(\mathbb{B}_n)$ such that

$$\|f\|_{\mathcal{B}} = \sup_{z\in\mathbb{B}_n} |\tilde{\nabla} f(z)| < \infty.$$

The function

$$g(z) = \frac{1}{2n}\int_{|z|}^{1} \frac{(1-t^2)^{n-1}}{t^{2n-1}}\, dt, \qquad z \in \mathbb{B}_n,$$

is called Green's function for the invariant Laplacian $\tilde{\Delta}$. The invariant Green's function of $\mathbb{B}_n$ is defined by $G(z,a) = g(\phi_a(z))$.

Proposition 2.1. *Let $n \geq 2$ be an integer, then there are positive constants C_1 and C_2 such that for all $z \in \mathbb{B}_n\backslash\{0\}$,*

$$C_1(1-|z|^2)^n|z|^{-2(n-1)} \leq g(z) \leq C_2(1-|z|^2)^n|z|^{-2(n-1)}. \tag{5}$$

The following definitions are due to Ouyang et al. [Ouy2]:

For $f \in \mathrm{Hol}(\mathbb{B}_n)$, $0<p<\infty$, $a \in \mathbb{B}_n$, let

$$I_p(f,a) = \int_{\mathbb{B}_n} |\tilde{\nabla} f(z)|^2 G^p(z,a)\, d\lambda(z)$$

$$J_p(f,a) = \int_{\mathbb{B}_n} |\tilde{\nabla} f(z)|^2 (1-|\phi_a(z)|^2)^{np}\, d\lambda(z).$$

They defined the $\mathcal{Q}_p(\mathbb{B}_n)$ space as

$$\mathcal{Q}_p(\mathbb{B}_n) = \{f \in \mathrm{Hol}(\mathbb{B}_n) : \sup_{a\in\mathbb{B}_n} I_p(f,a) < \infty\}$$

and its associated $\mathcal{Q}_{p,0}(\mathbb{B}_n)$ space as

$$\mathcal{Q}_{p,0}(\mathbb{B}_n) = \{f \in \mathrm{Hol}(\mathbb{B}_n) : \lim_{|a|\to 1^-} I_p(f,a) = 0\}.$$

They proved

$$\mathcal{Q}_p(\mathbb{B}_n) = \{f \in \mathrm{Hol}(\mathbb{B}_n) : \sup_{a\in\mathbb{B}_n} J_p(f,a) < \infty\}$$

and

$$\mathcal{Q}_{p,0}(\mathbb{B}_n) = \{f \in \mathrm{Hol}(\mathbb{B}_n) : \lim_{|a|\to 1^-} J_p(f,a) = 0\} \subset \mathcal{Q}_p(\mathbb{B}_n).$$

There are some basic differences with the case $n=1$:

- In general $|\tilde{\nabla} f(z)|^2$ is not a subharmonic function.
- $\int_{\mathbb{B}_n} G(z,a)\, d\lambda(z) = \infty$.
- When $0 < p \leq \frac{n-1}{n}$ or $\frac{n}{n-1} \leq p$, $\mathcal{Q}_p(\mathbb{B}_n)$ consists only of constant functions.

A very useful result is the following theorem [Zh, Theorem 1.12]:

Theorem 2.2. *Suppose c is real and $t > -1$. Then the integral*

$$J_{c,t}(z) = \int_{\mathbb{B}_n} \frac{(1-|w|^2)^t\, dv(w)}{|1-\langle z,w\rangle|^{n+1+t+c}}, \qquad z \in \mathbb{B}_n, \tag{6}$$

has the following asymptotic properties:

(1) *If $c<0$, then $J_{c,t}$ is bounded in $\mathbb{B}_n$.*
(2) *If $c=0$, then $J_{c,t} \sim \log\frac{1}{1-|z|^2}$ as $|z| \to 1^-$.*
(3) *If $c>0$, then $J_{c,t} \sim (1-|z|^2)^{-c}$ as $|z| \to 1^-$.*

3. Some properties. In this section we show some properties and relations between the $\mathcal{D}_p$, $\mathcal{Q}_p$ and $\mathcal{B}_\alpha$ spaces.

DEFINITION 3.1. Let $0 \leq p < \infty$. The space $\mathcal{D}_p$ is the set of holomorphic functions $f \in \mathrm{Hol}(\mathbb{B}_n)$ such that

$$\sup_{z \in \mathbb{B}_n} \int_{\mathbb{B}_n} |\tilde{\nabla} f(z)|^2 \, (1 - |z|^2)^{np} \, d\lambda(z) < \infty.$$

PROPOSITION 3.2. *Let $n \geq 2$ and $0 \leq p < \infty$. If $f \in \mathcal{D}_p$ with development in power series given by*

$$f(z) = \sum_\alpha a_\alpha z^\alpha$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a multi-index with $\alpha_i \in \mathbb{Z}$ for all $i = 1, 2, \ldots, n$. Then for all $N \in \mathbb{N}$

$$\sum_{|\alpha| \leq N} a_\alpha z^\alpha$$

belongs to $\mathcal{D}_p$.

Proof. Since $\mathcal{D}_p$ is a vectorial space ($\widetilde{\nabla}(\delta f + h) = \delta \widetilde{\nabla} f + \widetilde{\nabla} h$), it is enough to prove that, for each α, the monomial $a_\alpha z^\alpha = a_\alpha z_1^{\alpha_1} \cdots z_n^{\alpha_n} \in \mathcal{D}_p(\mathbb{B}_n)$.

It is easy to see that

$$F(z) = a_\alpha z^\alpha = \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} f(z_1 e^{i\theta_1}, \ldots, z_n e^{i\theta_n}) e^{-i(\alpha_1\theta_1 + \ldots + \alpha_n\theta_n)} \, d\theta_1 \cdots d\theta_n \,. \quad (7)$$

If U is the diagonal matrix $\mathrm{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n})$ then

$$f(z_1 e^{i\theta_1}, \ldots, z_n e^{i\theta_n}) = f \circ U(z_1, \ldots, z_n) = h(z).$$

By (7) and $\phi_z \in \mathrm{Aut}(\mathbb{B}_n)$,

$$(F \circ \phi_z)(w) = \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} (h \circ \phi_z)(w) e^{-i(\alpha_1\theta_1 + \ldots + \alpha_n\theta_n)} \, d\theta_1 \cdots d\theta_n.$$

Thus

$$\frac{\partial}{\partial w_j}(F \circ \phi_z)(w) = \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{\partial}{\partial w_j}(h \circ \phi_z)(w) e^{-i(\alpha_1\theta_1 + \ldots + \alpha_n\theta_n)} \, d\theta_1 \cdots d\theta_n \,.$$

Setting $w = 0$, we get

$$\tilde{\nabla}_j F(z) = \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \tilde{\nabla}_j h(z) e^{-i(\alpha_1\theta_1 + \ldots + \alpha_n\theta_n)} \, d\theta_1 \cdots d\theta_n \,.$$

By Jensen's inequality

$$|\tilde{\nabla}_j F(z)|^2 \leq \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} |\tilde{\nabla}_j h(z)|^2 \, d\theta_1 \cdots d\theta_n$$

and

$$\begin{aligned} |\tilde{\nabla} F(z)|^2 = \sum_{j=1}^n |\tilde{\nabla}_j F(z)|^2 &\leq \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \sum_{j=1}^n |\tilde{\nabla}_j h(z)|^2 \, d\theta_1 \cdots d\theta_n \\ &= \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} |\tilde{\nabla} h(z)|^2 \, d\theta_1 \cdots d\theta_n. \end{aligned}$$

Now we apply Fubini's theorem. Since $U \in \operatorname{Aut}(\mathbb{B}_n)$ is unitary, we obtain by (3)

$$\begin{aligned}
&\int_{\mathbb{B}_n} |\tilde{\nabla} F(z)|^2 (1-|z|^2)^{np}\, d\lambda(z) \\
&\quad \le \int_{\mathbb{B}_n} \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} |\tilde{\nabla} h(z)|^2\, d\theta_1 \cdots d\theta_n\, (1-|z|^2)^{np}\, d\lambda(z) \\
&\quad = \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \int_{\mathbb{B}_n} |\tilde{\nabla}(f \circ U)(z)|^2 (1-|z|^2)^{np}\, d\lambda(z)\, d\theta_1 \cdots d\theta_n \\
&\quad = \int_{\mathbb{B}_n} |\tilde{\nabla}(f)(z)|^2 (1-|z|^2)^{np}\, d\lambda(z) < \infty . \quad \blacksquare
\end{aligned}$$

Proposition 3.3. *Let $n \ge 2$. Then*

(i) *If $0 \le p \le \frac{n-1}{n}$, $\mathcal{D}_p(\mathbb{B}_n)$ contains only the constant functions.*
(ii) *If $\frac{n-1}{n} < p \le 1$ then $\mathcal{Q}_p(\mathbb{B}_n) \subset \mathcal{D}_p(\mathbb{B}_n)$.*
(iii) *If $1 < p$, $\mathcal{Q}_p(\mathbb{B}_n) = \mathcal{B}(\mathbb{B}_n) \subset \mathcal{D}_p(\mathbb{B}_n)$.*

Proof. i) Let $0 \le p \le \frac{n-1}{n}$ and $f \in \mathcal{D}_p(\mathbb{B}_n)$ be a nonconstant function given by

$$f(z) = \sum_\alpha a_\alpha z^\alpha .$$

Choose $a_\alpha \ne 0$, then by Proposition 3.2, $z^\alpha \in \mathcal{D}_p$. A straightforward calculation shows that

$$\begin{aligned}
|\tilde{\nabla}(z^\alpha)|^2 &= (1-|z|^2)(|\nabla(z^\alpha)(z)|^2 - |R(z^\alpha)(z)|^2) \\
&= (1-|z|^2)\big(\alpha_1^2 |z_1^{\alpha_1-1} \cdots z_n^{\alpha_n}|^2 + \ldots + \alpha_n^2 |z_1^{\alpha_1} \cdots z_n^{\alpha_n-1}|^2 - |\alpha|^2 |z^\alpha|^2\big).
\end{aligned}$$

Thus

$$\begin{aligned}
&\int_{\mathbb{B}_n} |\tilde{\nabla}(z^\alpha)|^2 (1-|z|^2)^{np}\, d\lambda(z) \\
&= \int_{\mathbb{B}_n} \big(\alpha_1^2 |z_1^{\alpha_1-1} \cdots z_n^{\alpha_n}|^2 + \ldots + \alpha_n^2 |z_1^{\alpha_1} \cdots z_n^{\alpha_n-1}|^2 - |\alpha|^2 |z^\alpha|^2\big)(1-|z|^2)^{np-n}\, dv(z) \\
&= \int_0^1 r^{2n-1+2|\alpha|-2} (1-r^2)^{np-n} \\
&\qquad \cdot \int_{\mathbb{S}_n} \big(\alpha_1^2 |\zeta_1^{\alpha_1-1} \cdots \zeta_n^{\alpha_n}|^2 + \ldots + \alpha_n^2 |\zeta_1^{\alpha_1} \cdots \zeta_n^{\alpha_n-1}|^2 - |\alpha|^2 r^2 |\zeta^\alpha|^2\big)\, d\sigma(\zeta)\, dr \\
&\ge \int_0^1 r^{2(n-1+|\alpha|)-1} (1-r^2)^{np-n} \\
&\qquad \cdot \int_{\mathbb{S}_n} \big(\alpha_1^2 |\zeta_1^{\alpha_1-1} \cdots \zeta_n^{\alpha_n}|^2 + \ldots + \alpha_n^2 |\zeta_1^{\alpha_1} \cdots \zeta_n^{\alpha_n-1}|^2 - |\alpha|^2 |\zeta^\alpha|^2\big)\, d\sigma(\zeta)\, dr \\
&= \frac{(n-1)|\alpha|(n-1)!\alpha!}{(n-1+|\alpha|)!} \int_0^1 r^{2(n-1+|\alpha|)-1} (1-r^2)^{np-n}\, dr = \infty
\end{aligned}$$

because $np - n \le -1$. Therefore $f \notin \mathcal{Q}_p$.

ii) With $a = 0$

$$\int_{\mathbb{B}_n} |\tilde{\nabla} f(z)|^2 (1-|\phi_a(z)|^2)^{np}\, d\lambda(z) = \int_{\mathbb{B}_n} |\tilde{\nabla} f(z)|^2 (1-|z|^2)^{np}\, d\lambda(z).$$

iii) If $f \in \mathcal{B}(\mathbb{B}_n)$ then

$$\int_{\mathbb{B}_n} |\tilde{\nabla} f(z)|^2 (1-|z|^2)^{np}\, d\lambda(z) \leq \sup_{z\in\mathbb{B}} |\tilde{\nabla} f(z)|^2 \int_0^1 r^{2n-1}(1-r^2)^{np-n-1}\, dr < \infty,$$

since $1 < p$. ∎

Proposition 3.4. *If $0 < p < \infty$ and $J_p(f,a) < \infty$ for all $a \in \mathbb{B}_n$, then $J_p(f,a)$ is a continuous function as a function of a.*

Proof. If f is constant on $\mathbb{B}_n$ it is clear that $J_p(f,a)$ is continuous for all $a \in \mathbb{B}_n$. Therefore suppose that f is not constant, in particular $J_p(f,a) \neq 0$. Let $a \in \mathbb{B}_n$ be fixed and let $\delta > 0$ be such that $\overline{B}(a,\delta) = \{z \in \mathbb{C}^n : |z-a| \leq \delta\} \subset \mathbb{B}_n$. The function $l : \overline{\mathbb{B}}_n \times \overline{B}(a,\delta) \to \mathbb{R}$ defined by

$$(z,\zeta) \mapsto \frac{(1-|\zeta|^2)^{np}}{|1-\langle \zeta, z\rangle|^{2np}}$$

is uniformly continuous on $\overline{\mathbb{B}}_n \times \overline{B}(a,\delta)$. Then given $\epsilon > 0$, there exists $\rho > 0$ such that if $|z'-z| < \rho$ and $|\zeta' - \zeta| < \rho$ then

$$|l(z',\zeta') - l(z,\zeta)| < \frac{\epsilon}{J_p(f,0)}.$$

Then if $|a-b| < \rho$,

$$|J_p(f,a) - J_p(f,b)| \leq \int_{\mathbb{B}_n} |\widetilde{\nabla} f(z)|^2 (1-|z|^2)^{np}\, |l(z,a) - l(z,b)|\, d\lambda(z) < \epsilon. \quad ∎$$

Corollary 3.5. *For $\frac{n-1}{n} < p < \frac{n}{n-1}$, the following inclusions are true*:

$$\mathcal{Q}_{p,0}(\mathbb{B}_n) \subset \mathcal{Q}_p(\mathbb{B}_n) \subset \mathcal{D}_p(\mathbb{B}_n).$$

Proof. Let $f \in \mathcal{Q}_{p,0}$. We can extend $J_p(f,a)$ continuously to $\overline{\mathbb{B}}_n$ by setting

$$\widetilde{J}_p(f,a) = \begin{cases} J_p(f,a) & \text{if } a \in \mathbb{B}_n \\ 0 & \text{if } a \in \partial\mathbb{B}_n. \end{cases}$$

Then $\widetilde{J}_p(f,a)$ is uniformly continuous on $\overline{\mathbb{B}}_n$ and therefore

$$\max_{a\in\overline{\mathbb{B}}_n} \widetilde{J}_p(f,a) = \max_{a\in\mathbb{B}_n} J_p(f,a) = J_p(f,b),$$

for some $b \in \mathbb{B}_n$. Finally $f \in \mathcal{Q}_p$. ∎

The inclusion $\mathcal{Q}_{p,0} \subset \mathcal{Q}_p$ was proved in [Ouy2], however the previous corollary gives a shorter proof.

Definition 3.6. Let $\mathcal{B}_\nabla(\mathbb{B}_n)$ be the set of holomorphic functions such that

$$\mathcal{B}_\nabla(\mathbb{B}_n) = \{f \in \mathrm{Hol}(\mathbb{B}_n) : \sup_{z\in\mathbb{B}_n} |\nabla f(z)| < \infty\}.$$

Proposition 3.7. *The following inclusion is true*

$$\mathcal{B}_\nabla(\mathbb{B}_n) \subset \bigcap\Big\{\mathcal{Q}_{p,0}(\mathbb{B}_n) : \frac{n-1}{n} < p < \frac{n}{n-1}\Big\}.$$

Proof. Let $0 < r < 1$ be fixed. Since $f \in \mathcal{B}_\nabla(\mathbb{B}_n)$, by (4) we have

$$|\tilde{\nabla} f(z)|^2 \leq (1-|z|^2)|\nabla f(z)|^2 \leq (1-|z|^2)C(f).$$

In this way

$$\int_{\mathbb{B}_n} |\widetilde{\nabla} f(z)|^2\, G^p(z,a)\, d\lambda(z) \leq C_1(f) \int_{\mathbb{B}_n} (1-|z|^2)\, \frac{(1-|\phi_a(z)|^2)^{np}}{|\phi_a(z)|^{2(n-1)p}}\, d\lambda(z).$$

By the change of variable formula and (1)

$$\begin{aligned}\int_{\mathbb{B}_n} (1-|z|^2)\, \frac{(1-|\phi_a(z)|^2)^{np}}{|\phi_a(z)|^{2(n-1)p}}\, d\lambda(z) &= \int_{\mathbb{B}_n} (1-|\phi_a(z)|^2)\, \frac{(1-|z|^2)^{np}}{|z|^{2(n-1)p}}\, d\lambda(z)\\ &= (1-|a|^2) \int_{\mathbb{B}_n} \frac{(1-|z|^2)^{np+1}}{|z|^{2(n-1)p}|1-\langle z,a\rangle|^2}\, d\lambda(z).\end{aligned}$$

Since $1-|z| < |1-\langle z,a\rangle|$ and $p < \frac{n}{n-1}$ we have

$$\begin{aligned}C_1(f)(1-|a|^2) &\int_{\mathbb{B}_n(0,r)} \frac{(1-|z|^2)^{np+1}}{|z|^{2(n-1)p}|1-\langle z,a\rangle|^2}\, d\lambda(z)\\ &\leq C_1(f) \frac{1-|a|^2}{(1-r^2)^2} \int_{\mathbb{B}_n(0,r)} \frac{d\lambda(z)}{|z|^{2(n-1)p}} = C' < \infty. \qquad (8)\end{aligned}$$

On the other hand, since $\frac{n-1}{n} < p$, we have $np-n > -1$. Thus by Theorem 2.2

$$\begin{aligned}C_1(f)(1-|a|^2) &\int_{\mathbb{B}_n \setminus \mathbb{B}_n(0,r)} \frac{(1-|z|^2)^{np+1}}{|z|^{2(n-1)p}|1-\langle z,a\rangle|^2}\, d\lambda(z)\\ &\leq C_1(f)\, \frac{1-|a|^2}{r^{2(n-1)p}} \int_{\mathbb{B}_n} \frac{(1-|z|^2)^{np-n}}{|1-\langle z,a\rangle|^2}\, dv(z) \leq C(r,f)(1-|a|^2). \qquad (9)\end{aligned}$$

The claim follows from these estimations by taking the limit when $|a| \to 1^-$ in (8) and (9). ∎

The proof of Proposition 3.7 suggests the following definition.

Definition 3.8. Let $\alpha \in \mathbb{R}$. The space $\mathcal{B}_\alpha(\mathbb{B}_n)$ of $\mathbb{B}_n$ is the set of holomorphic functions $f \in \mathrm{Hol}(\mathbb{B}_n)$ such that

$$\|f\|_{\mathcal{B},\alpha} = \sup_{z\in\mathbb{B}_n} (1-|z|^2)^\alpha |\tilde{\nabla} f(z)| < \infty.$$

Proposition 3.9. *Let $-1 < \alpha < 0$. Then*

$$\mathcal{B}_\alpha(\mathbb{B}_n) \subset \bigcap \Big\{ \mathcal{Q}_{p,0}(\mathbb{B}_n) : \frac{n-1}{n} < p < \frac{n}{n-1} \Big\}.$$

Proof. Let $0 < r < 1$ be fixed and $-1 < \alpha < 0$. Setting $\beta = 1-\alpha > 1$, by (4) we have

$$|\tilde{\nabla} f(z)|^2 \leq (1-|z|^2)^{1-\alpha} \sup_{z\in\mathbb{B}_n} (1-|z|^2)^\alpha (|\nabla f(z)|^2 - |Rf(z)|^2) = (1-|z|^2)^\beta \|f\|_{\mathcal{B},\alpha}.$$

Since $(1-|z|^2)^\beta \leq 1-|z|^2$ it follows from the proof of Proposition 3.7 that

$$\lim_{|a|\to 1^-} \int_{\mathbb{B}_n} (1-|z|^2)^\beta \frac{(1-|\phi_a(z)|^2)^{np}}{|\phi_a(z)|^{2(n-1)p}}\, d\lambda(z) = 0. \quad ∎$$

Proposition 3.10. *Let $0 < \alpha < 1$. Then*

$$\mathcal{B}_\alpha(\mathbb{B}_n) \subset \bigcap \Big\{ \mathcal{Q}_{p,0}(\mathbb{B}_n) : \frac{n+\alpha-1}{n} < p < \frac{n}{n-1} \Big\}.$$

Proof. Let $0 < r < 1$ be fixed and $0 < \alpha < 1$. Setting $\beta = 1 - \alpha > 0$, by (4) we have

$$|\tilde{\nabla} f(z)|^2 \le (1-|z|^2)^{1-\alpha} \sup_{z \in \mathbb{B}_n} (1-|z|^2)^\alpha (|\nabla f(z)|^2 - |Rf(z)|^2) = (1-|z|^2)^\beta \|f\|_{\mathcal{B},\alpha}$$

with $0 < \beta < 1$.

In this way

$$\int_{\mathbb{B}_n} |\tilde{\nabla} f(z)|^2 \, G^p(z,a) \, d\lambda(z) \le C_1(p) \, \|f\|_{\mathcal{B},\alpha} \int_{\mathbb{B}_n} (1-|z|^2)^\beta \frac{(1-|\phi_a(z)|^2)^{np}}{|\phi_a(z)|^{2(n-1)p}} \, d\lambda(z).$$

By the change of variable formula

$$\begin{aligned} \int_{\mathbb{B}_n} (1-|z|^2)^\beta \frac{(1-|\phi_a(z)|^2)^{np}}{|\phi_a(z)|^{2(n-1)p}} \, d\lambda(z) &= \int_{\mathbb{B}_n} (1-|\phi_a(z)|^2)^\beta \frac{(1-|z|^2)^{np}}{|z|^{2(n-1)p}} \, d\lambda(z) \\ &= (1-|a|^2)^\beta \int_{\mathbb{B}_n} \frac{(1-|z|^2)^{np+\beta}}{|z|^{2(n-1)p} |1-\langle z,a\rangle|^{2\beta}} \, d\lambda(z). \end{aligned}$$

Since $1 - |z| < |1 - \langle z,a\rangle|$ and $p < \frac{n}{n-1}$ we have

$$\begin{aligned} (1-|a|^2)^\beta & \int_{\mathbb{B}_n(0,r)} \frac{(1-|z|^2)^{np+\beta}}{|z|^{2(n-1)p} |1-\langle z,a\rangle|^{2\beta}} \, d\lambda(z) \\ & \le \frac{(1-|a|^2)^\beta}{(1-r^2)^{2\beta}} \int_{\mathbb{B}_n(0,r)} \frac{d\lambda(z)}{|z|^{2(n-1)p}} = C' < \infty. \end{aligned} \tag{10}$$

On the other hand, since $\frac{n+\alpha-1}{n} < p$, we have $-1 < np - \alpha - n$. Hence by Theorem 2.2

$$\begin{aligned} (1-|a|^2)^\beta & \int_{\mathbb{B}_n \setminus \mathbb{B}_n(0,1/2)} \frac{(1-|z|^2)^{np+\beta}}{|z|^{2(n-1)p} |1-\langle z,a\rangle|^{2\beta}} \, d\lambda(z) \\ & \le \frac{(1-|a|^2)^\beta}{r^{2(n-1)p}} \int_{\mathbb{B}_n} \frac{(1-|z|^2)^{np+\beta-n-1}}{|1-\langle z,a\rangle|^{2\beta}} \, dv(z) < \infty. \end{aligned} \tag{11}$$

Since $\frac{n+\alpha-1}{n} < p < \frac{n}{n-1}$, (11) has only the asymptotic properties (1) and (2) in Theorem 2.2, nevertheless, the property (3) is never satisfied because of the range of the parameter p.

Now the claim follows from these estimations by taking the limit when $|a| \to 1^-$ in (10) and (11). ∎

The following corollary shows an interesting relation among the α-Bloch spaces and the hyperbolic $\mathcal{Q}_p^*$ classes of holomorphic functions. The classes of this kind are introduced in [JRT]. These classes are defined below:

For a holomorphic function $f : (\mathbb{B}_n, |\cdot|_{\mathrm{Euc}}) \to (\mathbb{D}, |\cdot|_{\mathrm{Hyp}})$, $0 < p < \infty$, $a \in \mathbb{B}_n$, define the invariant hyperbolic gradient of f as

$$|\tilde{\nabla}^* f(z)| = \frac{|\tilde{\nabla} f(z)|}{1 - |f(z)|^2} \,. \tag{12}$$

Let

$$I_p^*(f,a) = \int_{\mathbb{B}_n} |\tilde{\nabla}^* f(z)|^2 G^p(z,a)\, d\lambda(z)\,,$$

and define the $\mathcal{Q}_p^*(\mathbb{B}_n)$ class as

$$\mathcal{Q}_p^*(\mathbb{B}_n) = \{f \in \mathrm{Hol}(\mathbb{B}_n) : \sup_{a \in \mathbb{B}_n} I_p^*(f,a) < \infty\}$$

and its respective little class

$$\mathcal{Q}_{p,0}^*(\mathbb{B}_n) = \{f \in \mathrm{Hol}(\mathbb{B}_n) : \lim_{|a| \to 1^-} I_p^*(f,a) = 0\}.$$

Definition 3.11. The class $B^\dagger$ is the set of holomorphic functions such that

$$B^\dagger = \{f \in \mathcal{B}_\nabla(\mathbb{B}_n) : f(\mathbb{B}_n) \subset \mathbb{D} \text{ is a compact set}\}\,.$$

Corollary 3.12. *The following inclusions are true*:

(i) $$B^\dagger(\overline{\mathbb{B}}_n) \subset \bigcap \Big\{ \mathcal{Q}_{p,0}^*(\mathbb{B}_n) : \frac{n-1}{n} < p < \frac{n}{n-1} \Big\};$$

(ii) *if* $-1 < \alpha < 0$, *then* $\mathcal{B}_\alpha(\mathbb{B}_n) \cap B^\dagger(\overline{\mathbb{B}}_n) \subset \bigcap \Big\{ \mathcal{Q}_{p,0}^*(\mathbb{B}_n) : \frac{n+\alpha}{n} < p < \frac{n}{n-1} \Big\}$;

(iii) *if* $0 < \alpha < 1$, *then* $\mathcal{B}_\alpha(\mathbb{B}_n) \cap B^\dagger(\overline{\mathbb{B}}_n) \subset \bigcap \Big\{ \mathcal{Q}_{p,0}^*(\mathbb{B}_n) : \frac{n+\alpha-1}{n} < p < \frac{n}{n-1} \Big\}$.

Proof. If $f \in B^\dagger(\overline{\mathbb{B}}_n)$ then there exists $M > 0$ such that

$$\frac{1}{(1-|f(z)|^2)^2} \le M \quad \text{and} \quad |\tilde{\nabla}^* f(z)|^2 \le M |\tilde{\nabla} f(z)|^2.$$

The result follows from this inequality and the previous proofs. ∎

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References

[Choa] J. S. Choa, H. O. Kim, Y. Y. Park, *A Bergman–Carleson measure characterization of Bloch functions in the unit ball of* $\mathbb{C}^n$, Bull. Korean Math. Soc. 29 (1992), 285–293.

[JRT] D. D. Jiménez S., L. F. Reséndis O., L. M. Tovar S. *On hyperbolic classes in the unit ball of* $\mathbb{C}_n$, in preparation.

[Ouy1] C. Ouyang, W. S. Yang, R. H. Zhao, *Characterizations of Bergman spaces and Bloch space in the unit ball of* $\mathbb{C}^n$, Trans. Amer. Math. Soc. 347 (1995), 4301–4313.

[Ouy2] C. Ouyang, W. Yang, R. Zhao, *Möbius invariant* $\mathcal{Q}_p$ *spaces associated with the Green's function on the unit ball of* $\mathbb{C}^n$, Pacific J. Math. 182 (1998), 69–99.

[Str] K. Stroethoff, *Besov-type characterization for the Bloch space*, Bull. Austral. Math. Soc. 39 (1989), 405–420.

[Zh] K. Zhu, *Space of Holomorphic Functions in the Unit Ball*, Grad. Texts in Math. 226, Springer, New York 2005.