FUNCTION SPACES X
BANACH CENTER PUBLICATIONS, VOLUME 102
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2014

ONE-WEIGHT WEAK TYPE ESTIMATES FOR FRACTIONAL AND SINGULAR INTEGRALS IN GRAND LEBESGUE SPACES

VAKHTANG KOKILASHVILI

Department of Mathematical Analysis, A. Razmadze Mathematical Institute
I. Javakhishvili Tbilisi State University, 2 University St., Tbilisi 0186;
and International Black Sea University,
3 Agmashenebeli Ave., Tbilisi 0131, Georgia
E-mail: kokil@rmi.ge

ALEXANDER MESKHI

Department of Mathematical Analysis, A. Razmadze Mathematical Institute
I. Javakhishvili Tbilisi State University, 2 University St., Tbilisi 0186, Georgia;
and Department of Mathematics, Faculty of Informatics and Control Systems
Georgian Technical University, 77 Kostava St., Tbilisi 0175, Georgia
E-mail: meskhi@rmi.ge

Abstract. We investigate weak type estimates for maximal functions, fractional and singular integrals in grand Lebesgue spaces. In particular, we show that for the one–weight weak type inequality it is necessary and sufficient that a weight function belongs to the appropriate Muckenhoupt class. The same problem is discussed for strong maximal functions, potentials and singular integrals with product kernels.

1. Introduction. In 1992 T. Iwaniec and C. Sbordone [11], in their studies regarding the integrability properties of the Jacobian in a bounded open set, introduced a new type of function spaces $L^{p)}(\Omega)$, called grand Lebesgue spaces. A generalized version of these spaces, $L^{p),\theta}(\Omega)$ appeared in the paper by L. Greco, T. Iwaniec and C. Sbordone [9], where the authors investigated the existence and uniqueness of the non-homogeneous n-harmonic equation of the form $\operatorname{div} A(x, \nabla u) = \mu$.

Harmonic analysis related to these spaces and their associate spaces (called small

2010 *Mathematics Subject Classification*: Primary 42B20, 42B25; Secondary 46E30.
Key words and phrases: Grand Lebesgue spaces; maximal operator, Hilbert transform, potential, weighted inequality.
The paper is in final form and no version of it will be published elsewhere.

DOI: 10.4064/bc102-0-9

Lebesgue spaces), was intensively studied during the last decade due to various applications (see e.g. the papers [4], [1], [2], [5], [6], [7], [15], [14], [18], [16] etc).

A version of weighted grand Lebesgue spaces adjusted for sets $\Omega \subseteq \mathbf{R}^n$ of infinite measure was introduced in [21], where the integrability of $|f(x)|^{p-\varepsilon}$ at infinity was controlled by means of a weight and where grand Lebesgue spaces were also considered along with the study of classical operators of harmonic analysis in such spaces.

Recently a necessary and sufficient condition guaranteeing the one-weight inequality for the Hardy–Littlewood maximal operator

$$Mf(x) = \sup_{I\subset[0,1],\ I\ni x} \frac{1}{|I|} \int\limits_I |f(t)|\,dt, \quad x \in [0,1],$$

in weighted grand Lebesgue spaces $L^{p)}_w([0,1])$ was established by A. Fiorenza, B. Gupta and P. Jain [5], while the same problem for the Hilbert transform

$$Hf(x) = p.v. \int\limits_0^1 \frac{f(t)}{x-t}\,dt, \quad x \in [0,1],$$

was studied by the authors of this paper in [15]. In particular, it turned out that the Hardy–Littlewood maximal operator (resp. the Hilbert transform) is bounded in $L^{p)}_w(I)$ if and only if the weight w belongs to the Muckenhoupt class $A_p(I)$. The one-weight theory for product kernel singular integrals and corresponding strong maximal functions was developed in [13].

Further, it was shown in [18] that

(a) the potential operator

$$I_\alpha f(x) = \int\limits_0^1 \frac{f(t)}{|x-t|^{1-\alpha}}\,dt, \quad x \in [0,1],$$

with the parameter α, $0<\alpha<1$, is not bounded from $L^{p),\theta_1}([0,1])$ to $L^{q),\theta_2}([0,1])$, where $1<p<\infty$, $q=\frac{p}{1-\alpha p}$, $\theta_2 < \frac{q\theta_1}{p}$;

(b) the one-weight inequality

$$\|I_\alpha(fw^\alpha)\|_{L^{q),\ \theta q/p}_w([0,1])} \le c\|f\|_{L^{p),\theta}_w([0,1])}$$

holds if and only if w belongs to the Muckenhoupt class $A_{1+q/p'}([0,1])$.

The same problems for fractional integrals with product kernels were investigated in [16].

Our aim is to establish weak type inequalities for these operators in grand Lebesgue spaces. In particular, we show that the weak type inequality for these operators holds if and only if the weight belongs to the Muckenhoupt class. Hence, in the frame of one-weight theory, weak type estimates are equivalent to strong type inequalities. In the classical Lebesgue spaces such theorems were proved in the well-known papers by B. Muckenhoupt [19], R. A. Hunt, B. Muckenhoupt and R. L. Wheeden [10], B. Muckenhoupt and R. L. Wheeden [20].

It should be also emphasized that regarding potential operators it turned out that the weak type estimate fails for the same values of the second parameter as in the strong type case.

This paper can be considered as a natural continuation of investigations carried out by the authors in [13], [14], [15], [16], [18].

In the sequel the symbol $|E|$ will denote the Lebesgue measure of a measurable set E; $p' := \frac{p}{p-1}$, where $1 < p < \infty$; $a \approx b$ means that there are positive constants c_1 and c_2 such that $c_1 b \le a \le c_2 b$; the fact that a space X is continuously embedded in Y will be denoted by $X \hookrightarrow Y$; for a weight function ρ we define $\rho F := \int_F \rho(x)\,dx$; constants (often different constants in the same series of inequalities) will generally be denoted by c.

The paper is organized as follows: in Section 2 we recall some well-known facts regarding grand Lebesgue spaces and introduce weak grand Lebesgue space and discuss some properties of these spaces. In Section 3 we study the one-weight weak type problem for the Hardy–Littlewood maximal operator and the Hilbert transform defined on $[0,1]$, while Section 4 is devoted to the same problem for potentials. In Section 5 we derive one-weight criteria for integral operators with product kernels under the Muckenhoupt condition defined with respect to parallelepipeds.

2. Preliminaries. Let E be a bounded set in $\mathbf{R}^n$ with positive measure and let w be an almost everywhere positive integrable function (weight) on E. Suppose that $1 < p < \infty$ and φ is a continuous positive function on $(0, p-1]$ satisfying the condition $\lim_{x\to 0+} \varphi(x) = 0$.

The generalized weighted grand Lebesgue spaces $L_w^{p),\varphi(\cdot)}(E)$ is the class of those $f : E \to \mathbf{R}$ for which the norm

$$\|f\|_{L_w^{p),\varphi(\cdot)}(E)} = \sup_{0<\varepsilon\le p-1} \Big(\frac{\varphi(\varepsilon)}{|E|} \int_X |f(x)|^{p-\varepsilon} w(x)\,dx\Big)^{1/(p-\varepsilon)}$$

is finite.

If $\varphi(x) = x^\theta$, where θ is a positive number, then we denote $L_w^{p),\varphi(\cdot)}(E)$ by $L_w^{p),\theta}(E)$. If $w \equiv \text{const}$, then we use the symbol $L^{p),\theta}(E)$ for $L_w^{p),\theta}(E)$. In the case $\theta = 1$ the space $L_w^{p),\theta}(E)$ is usually denoted by $L_w^{p)}(E)$. For more properties of grand Lebesgue spaces, see, e.g., [4], [14].

Together with $L^{p),\theta}(E)$ spaces we are interested in weak grand Lebesgue spaces $WL_w^{p),\theta}(E)$ which we define by the quasinorm

$$\|f\|_{WL_w^{p),\theta}(E)} = \sup_{\lambda>0} \lambda \sup_{0<\varepsilon\le p-1} \big(\varepsilon^\theta w(\{x \in E : |f(x)| > \lambda\})\big)^{1/(p-\varepsilon)}.$$

It is easy to see that $L_w^{p),\theta} \hookrightarrow WL_w^{p),\theta}$. It can be also checked that, for example, the function $f(t) = t^{-1/p}$ belongs to the class $WL^{p),\theta}([0,1])$ but does not belong to $L^{p),\theta}([0,1])$ for $0 < \theta < 1$.

The following lemma was proved in [5] for $\theta = 1$ (see also [14] and [18] for $\theta > 0$).

Lemma 2.1. *Let $1 < p < \infty$, $\theta > 0$ and let w be a weight function on $[0,1]$. Then there is a positive constant c such that for arbitrary $f \in L_w^{p),\theta}([0,1])$ and all intervals $I \subset [0,1]$*

$$\|f\chi_I\|_{L_w^{p),\theta}} \le c(wI)^{-1/p} \|f\chi_I\|_{L_w^p} \cdot \|\chi_I\|_{L_w^{p),\theta}}.$$

Definition 2.2. Let $1 < p < \infty$. Suppose that w is a weight function on $[0,1]$. We say that $w \in A_p([0,1])$ if

$$\sup_{J\subset[0,1]} \frac{1}{|J|}\int_J w(x)\,dx\Big(\frac{1}{|J|}\int_J w^{1-p'}(x)\,dx\Big)^{p-1} < \infty,$$

where by J a subinterval of $[0,1]$ is denoted.

We refer, e.g., to [8] for essential properties of A_p classes.

3. Hardy–Littlewood maximal operator and Hilbert transform. Let M and H denote the Hardy–Littlewood maximal operator and the Hilbert transform, respectively, defined on $[0,1]$ (see Section 1 for definitions of M and H).

The first result of this section reads as follows:

Theorem 3.1. *Let $1 < p < \infty$, $\theta > 0$. Then the following conditions are equivalent*:

(i) *the operator M is bounded in $L^{p),\theta}_w([0,1])$;*
(ii) *M is bounded from $L^{p),\theta}_w([0,1])$ to $WL^{p),\theta}_w([0,1])$;*
(iii) $w \in A_p([0,1])$.

Proof. The equivalence of (i) and (iii) was proved in [5] (see also [14] for $\theta > 0$). Now we show that (ii) $\Rightarrow$ (iii).

Note that from (ii) it follows that

$$\int_I w^{1-p'}(x)\,dx < \infty \tag{1}$$

for arbitrary $I \subset [0,1]$. Indeed, if for some $I \subset [0,1]$,

$$\int_I w^{1-p'}(x)\,dx = \infty \tag{2}$$

i.e. $w^{-1/p} \notin L^{p'}(I)$, then there exists a non-negative function $h \in L^p(I)$ supported on I such that

$$\int_I w^{-1/p}(x)h(x)\,dx = \infty. \tag{3}$$

Let $f = w^{-1/p}h$. Then $f \in L^p_w(I)$ and, consequently, $f \in L^{p),\theta}_w(I)$ for arbitrary $\theta > 0$. Then for $x \in I$,

$$Mf(x) \ge \frac{1}{|I|}\int_I w^{-1/p}(y)h(y)\,dy = \infty.$$

Therefore (ii) yields

$$(wI)^{1/(p-\epsilon_0)} \le \frac{c}{\lambda}\,\|f\|_{L^{p),\theta}_w}$$

for arbitrary $\lambda > 0$ and arbitrary fixed ϵ_0, $0 < \epsilon_0 \le p-1$. Hence, $wI = 0$. This contradicts the definition of a weight function.

Thus, (1) is true for an arbitrary interval $I \subseteq [0,1]$.

Further, observe that if $I \subset [0,1]$ and $f \ge 0$, then

$$Mf(x) \ge \frac{1}{|I|}\int_I f(y)\,dy, \quad x \in I.$$

Putting the function $f(x) = w^{-1/(p-1)}\chi_I(x)$ and the number $\lambda = \frac{1}{|I|}\int_I w^{-1/(p-1)}(y)\,dy$ in the weak type inequality we find

$$|I|^{-1}\Big(\int_I w^{-1/(p-1)}\,dy\Big)\|\chi_I\|_{L^{p),\theta}_w} \le c\|w^{-1/(p-1)}\chi_I\|_{L^{p),\theta}_w}.$$

Applying Lemma 2.1 we obtain

$$|I|^{-1}\Big(\int_I w^{-1/(p-1)}\,dy\Big)\|\chi_I\|_{L^{p),\theta}_w} \le c\|w^{-1/(p-1)}\chi_I\|_{L^p_w}\|\chi_I\|_{L^{p),\theta}_w}\cdot(wI)^{-1/p}.$$

Hence,

$$\frac{1}{|I|}\int_I w(y)\,dy\Big(\frac{1}{|I|}\int_I w^{-1/(p-1)}(y)\,dy\Big)^{p-1} \le c,$$

where the constant c is independent of I. Since (i) $\Rightarrow$ (ii), the theorem is proved. ■

Theorem 3.2. *Let $1 < p < \infty$ and let $\theta > 0$. Then the following conditions are equivalent*:

(i) *the operator H is bounded in $L^{p),\theta}_w$*;
(ii) *H is bounded from $L^{p),\theta}_w([0,1])$ to $WL^{p),\theta}_w([0,1])$*;
(iii) *$w \in A_p([0,1])$*.

Proof. The equivalence of (i) and (iii) is proved in [15]. We claim that (ii) $\Rightarrow$ (iii). First of all let us show that (ii) yields (1) for an arbitrary interval $I \subset [0,1]$ with length less than or equal to $1/4$. Indeed, if (2) holds for such an interval I, then there is a non-negative function $h \in L^p(I)$ vanishing outside I such that (3) is true. Let $f = w^{-1/p}h$. Then $f \in L^p_w$. Consequently, $f \in L^{p),\theta}_w$.

For an interval I, let $J \subset [0,1]$ be one of two intervals satisfying the condition $|J| = |I|$ and having exactly one end point common with I. Then

$$|Hf(x)| \ge \frac{2}{|I|}\int_I w^{-1/p}(y)h(y)\,dy = \infty$$

for $x \in J$. Therefore due to (ii) we find

$$\big(\epsilon^\theta w\{x : |Hf(x)| > \lambda\}\big)^{1/(p-\epsilon)} \le \frac{c}{\lambda}\,\|f\|_{L^{p),\theta}}$$

for arbitrary $\lambda > 0$ and ϵ, $0 < \epsilon \le p-1$. Consequently,

$$(wJ)^{1/(p-\epsilon_0)} \le \frac{c}{\lambda}\,\|f\|_{L^{p),\theta}_w}$$

for some ϵ_0, $0 < \epsilon_0 \le p-1$ and arbitrary $\lambda > 0$.

Therefore $wJ = 0$; this contradicts the fact that w is positive almost everywhere.

Let, as above, I and J be two intervals in $[0,1]$ having equal lengths and only one common point.

Suppose that $f := w^{-1/(p-1)}\chi_I$. Then for arbitrary $x \in J$,

$$|Hf(x)| \ge \frac{1}{\pi}\Big|\int_I \frac{f(y)}{x-y}\,dy\Big| \ge \frac{2}{|I|}\int_I w^{-1/(p-1)}(y)\,dy.$$

Let

$$\lambda = \frac{2}{|I|}\int_I w^{-1/(p-1)}(y)\,dy.$$

Then due to (ii) we obtain

$$|I|^{-1}\Big(\int_I w^{-1/(p-1)}(y)\,dy\Big)\|\chi_J\|_{L^{p),\theta}_w} \le c\|w^{-1/(p-1)}\chi_I\|_{L^{p),\theta}_w}.$$

Applying now Lemma 2.1 to the right-hand side of this inequality we get

$$|I|^{-1}\Big(\int_I w^{-1/(p-1)}(y)\,dy\Big)\|\chi_J\|_{L^{p),\theta}_w} \le c\Big(\int_I w^{-1/(p-1)}(y)\,dy\Big)^{1/p}\cdot (wI)^{-1/p}\|\chi_I\|_{L^{p),\theta}_w}.$$

Hence,

$$(wI)^{1/p}\Big(\int_I w^{-1/(p-1)}(y)\,dy\Big)^{1/p'}\|\chi_J\|_{L^{p),\theta}_w} \le c\|\chi_I\|_{L^{p),\theta}_w}\cdot|I|. \tag{4}$$

Analogously, it follows that

$$(wJ)^{1/p}\Big(\int_J w^{-1/(p-1)}(y)\,dy\Big)^{1/p'}\|\chi_I\|_{L^{p),\theta}_w} \le c\|\chi_J\|_{L^{p),\theta}_w}\cdot|I|. \tag{5}$$

Multiplying inequalities (4) and (5) we obtain

$$\begin{aligned}(wI)^{1/p}&\Big(\int_I w^{-1/(p-1)}(y)\,dy\Big)^{1/p'}\|\chi_J\|_{L^{p),\theta}_w}\\ &\times (wJ)^{1/p}\Big(\int_J w^{-1/(p-1)}(y)\,dy\Big)^{1/p'}\|\chi_I\|_{L^{p),\theta}_w} \le c_1\|\chi_J\|_{L^{p),\theta}_w}\cdot\|\chi_I\|_{L^{p),\theta}_w}\cdot|I|^2. \end{aligned}\tag{6}$$

On the other hand, due to Hölder's inequality we see that

$$|I| \le (wJ)^{1/p}\cdot\Big(\int_J w^{-1/(p-1)}(y)\,dy\Big)^{1/p'}.$$

Thus, (6) together with (4) and (5) yields that $w \in A_p([0,1])$. ■

4. Fractional integrals. Let $0 < \alpha < 1$. Suppose that I_α is the fractional integral operator defined on $[0,1]$ (see Section 1 for the definition of I_α). The corresponding fractional maximal operator is given by

$$M_\alpha f(x) = \sup_{J\ni x}\frac{1}{|J|^{1-\alpha}}\int_J |f(y)|\,dy, \quad x\in[0,1],$$

where J denotes a subinterval of $[0,1]$.

Obviously,

$$I_\alpha f \ge M_\alpha f, \qquad f\ge 0. \tag{7}$$

First we determine the range of the second parameter of the grand Lebesgue space for which the weak type inequality for M_α (consequently, for I_α) fails; namely the next statement is valid:

PROPOSITION 4.1. *Let $0<\alpha<1/p$, where $1<p<\infty$. We set $q=\frac{p}{1-\alpha p}$. Suppose that θ_1 and θ_2 are positive numbers such that $\theta_2 < \theta_1 q/p$. Then M_α (consequently, I_α) is not bounded from $L^{p),\theta_1}([0,1])$ to $WL^{q),\theta_2}([0,1])$.*

Proof. Observe that if J is any subinterval of $[0,1]$, then for $f=\chi_J$ and $x\in J$,

$$M_\alpha f(x) \ge \int_J \frac{dy}{|x-y|^{1-\alpha}} \ge |J|^\alpha.$$

If the inequality

$$\sup_{0<\epsilon\le p-1} \lambda\big(\varepsilon^{\theta_2} w\{x\in[0,1] : M_\alpha(x)>\lambda\}\big)^{1/(q-\varepsilon)} \le c\|f\|_{L^{p),\theta_1}_w} \tag{8}$$

holds, with a positive constant c independent of f and λ, then taking $\lambda = |J|^\alpha$ and $f=\chi_J$ in (8), we obtain

$$|J|^\alpha \sup_{0<\varepsilon\le p-1} \big(\varepsilon^{\theta_2}|J|\big)^{1/(q-\varepsilon)} \le c\|\chi_J\|_{L^{p),\theta_1}([0,1])}.$$

Hence,

$$|J|^\alpha \|\chi_J\|_{L^{q),\theta_2}([0,1])} \le c\|\chi_J\|_{L^{p),\theta_1}([0,1])}\,, \tag{9}$$

where the positive constant c does not depend on J. Now we argue as in the proof of Theorem 2.1 in [18] (see also [16]). Let us define the number ε_J which belongs to $(0,p-1]$ and satisfies the condition

$$\sup_{0<\varepsilon\le p-1} \big(\varepsilon^{\theta_1}|J|\big)^{1/(p-\varepsilon)} = \big(\varepsilon_J^{\theta_1}|J|\big)^{1/(p-\varepsilon_J)}. \tag{10}$$

Then (see the proof of Theorem 2.1 in [18]) $\lim_{|J|\to 0}\varepsilon_J = 0$.

For $J\subset[0,1]$ with sufficiently small length, let us choose η_J so that

$$\alpha = \frac{1}{p-\varepsilon_J} - \frac{1}{q-\eta_J}\,. \tag{11}$$

Due to (10) we find

$$|J|^\alpha \eta_J^{\theta_2/(q-\eta_J)}|J|^{1/(q-\eta_J)} \le c\varepsilon_J^{\theta_1/(p-\varepsilon_J)}|J|^{1/(p-\varepsilon_J)}. \tag{12}$$

Hence,

$$\eta_J^{\theta_2/(q-\eta_J)}\varepsilon_J^{-\theta_1/(p-\varepsilon_J)} \le c. \tag{13}$$

Further, (11) and (13) imply

$$\left(\frac{q-\frac{p-\varepsilon_J}{1-\alpha(p-\varepsilon_J)}}{\varepsilon_J}\right)^{\theta_2/(p-\varepsilon_J)-\alpha\theta_2} \varepsilon_J^{-\theta_1/(p-\varepsilon_J)+\theta_2/(p-\varepsilon_J)-\alpha\theta_2} \le c. \tag{14}$$

Passing now to the limit as $|J|\to 0$ we see that the left-hand side of (14) tends to $+\infty$ because the limit of the first factor is $\big[(1-\alpha p)^{-2}\big]^{\theta_2/p-\alpha\theta_2}$, and

$$\lim_{|J|\to 0}\varepsilon_J^{(\theta_2-\theta_1)/(p-\varepsilon_J)-\alpha\theta_2} = \lim_{|J|\to 0}\varepsilon_J^{(\theta_2-\theta_1)/p-\alpha\theta_2} = \infty.$$

The result for I_α follows from (7). ■

THEOREM 4.2. *Let $0<\alpha<1/p$, where $1<p<\infty$. We set $q=\frac{p}{1-\alpha p}$. Then the following conditions are equivalent*:

(i) $\|I_\alpha(fw^\alpha)\|_{L^{q),\theta q/p}_w([0,1])} \le c\|f\|_{L^{p),\theta}_w([0,1])}$ (*one-weight strong type inequality*);
(ii) $\|I_\alpha(fw^\alpha)\|_{WL^{q),\theta q/p}_w([0,1])} \le c\|f\|_{L^{p),\theta}_w([0,1])}$ (*weak type inequality*);
(iii) $\|M_\alpha(fw^\alpha)\|_{L^{q),\theta q/p}_w([0,1])} \le c\|f\|_{L^{p),\theta}_w([0,1])}$;
(iv) $\|M_\alpha(fw^\alpha)\|_{WL^{q),\theta q/p}_w([0,1])} \le c\|f\|_{L^{p),\theta}_w([0,1])}$;
(v) $w\in A_{1+q/p'}([0,1])$.

Proof. The fact that (i) $\Leftrightarrow$ (v) was proved in [18] (see also [16]). The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iv) and (i) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are obvious. The theorem will be proved if we show that (iv) $\Rightarrow$ (v).

Further, we show that (iv) implies

$$A := \int_0^1 w^{-p'/q}(x)\,dx = \|w^{\alpha-1}\|_{L^{p'}_w} < \infty.$$

Indeed, if we assume that $A = \infty$, then there is a non-negative function $g \in L^p_w([0,1])$ such that $\int_0^1 g(x)w^\alpha(x)\,dx = \infty$. On the other hand, it is easy to check that

$$M_\alpha(gw^\alpha)(x) \geq \int_0^1 g(t)w^\alpha(t)\,dt = \infty, \quad x \in [0,1].$$

This together with Lemma 2.1 yields

$$(wI)^{1/(q-\varepsilon_0)} \leq \frac{c}{\lambda}\,\|g\|_{L^{p),\theta}_w([0,1])} \leq \frac{c}{\lambda}\,\|g\|_{L^p_w([0,1])} < \infty$$

for all $\lambda > 0$ and fixed ε_0, $0 < \varepsilon_0 \leq q-1$. Hence $wI = 0$ a.e. on $[0,1]$. This contradicts the definition of a weight function.

Further, let us observe that (iv) is equivalent to

(iv$'$) M_α is bounded from $L^{p),\theta}_w([0,1])$ to $WL^{q),\psi(\cdot)}_w([0,1])$,

where

$$\psi(x) = \varphi(x^\theta), \quad \varphi(x) = \Big[\frac{x-q}{1-\alpha(x-q)} + p\Big]^{1-(x-q)\alpha}. \tag{15}$$

This follows from the fact that $\varphi(x) \approx x^{q/p}$ near 0.

Let J be a subinterval of $[0,1]$ and let $f := \chi_J w^{-\alpha-p'/q}$. Then for $x \in J$,

$$M_\alpha(w^\alpha f)(x) \geq |J|^{\alpha-1}\int_J w^{-p'/q}(x)\,dx.$$

Hence, for $\lambda = |J|^{\alpha-1}\int_J w^{-p'/q}(x)\,dx$, by Lemma 2.1 we obtain

$$\begin{aligned}
|J|^{\alpha-1}\Big(\int_J w^{-p'/q}\Big)\|\chi_J\|_{L^{q),\psi(x)}_w([0,1])} &\leq c\|f\|_{L^{p),\theta}([0,1])} \\
&\leq c(w(J))^{-1/p}\Big(\int_J |f(t)|^p w(t)\,dt\Big)^{1/p}\|\chi_J\|_{L^{p),\theta}_w([0,1])} \\
&= cw(J)^{-1/p}\Big(\int_J w^{-p'/q}\Big)^{1/p}\|\chi_J\|_{L^{p),\theta}_w([0,1])}.
\end{aligned}$$

It is easy to see that there is a number η_J depending on J such that $0 < \eta_J \leq p-1$ and

$$|J|^{\alpha-1}w(J)^{1/p}\Big(\int_J w^{-p'/q}\Big)^{1/p'}\|\chi_J\|_{L^{q),\psi(x)}_w([0,1])} \leq c\big(\eta_J w(J)\big)^{1/(p-\eta_J)}.$$

For such an η_J we choose ε_J so that

$$\frac{1}{p-\eta_J} - \frac{1}{q-\varepsilon_J} = \alpha. \tag{16}$$

Then $0 < \varepsilon_J \leq q-1$ and

$$|J|^{\alpha-1}w(J)^{1/p-1/(p-\eta_J)}\eta_J^{-\theta/(p-\eta_J)}\psi(\varepsilon_J)^{1/(q-\varepsilon_J)}w(J)^{1/(q-\varepsilon_J)}\Big(\int_J w^{-p'/q}\Big)^{1/p'} \leq c.$$

Observe now that by (16) and the definition of the function ψ we have

$$\eta_J^{-\theta/(p-\eta_J)}\psi(\varepsilon_J)^{1/(q-\varepsilon_J)} = \eta_J^{-\theta/(p-\eta_J)}\varphi(\varepsilon_J^{\theta})^{1/(q-\varepsilon_J)} \approx \eta_J^{-\theta/(p-\eta_J)}\varepsilon_J^{\theta(1+\alpha q)/(q-\varepsilon_J)}$$
$$= \big(\eta_J^{-1/(p-\eta_J)}\varepsilon_J^{(1+\alpha q)/(q-\varepsilon_J)}\big)^{\theta} \approx \big(\eta_J^{-1/(p-\eta_J)}\varphi(\varepsilon_J)^{1/(q-\varepsilon_J)}\big)^{\theta} = 1$$

and also,

$$\frac{1}{p} - \frac{1}{p-\eta_J} + \frac{1}{q-\varepsilon_J} = \frac{1}{p} - \alpha = \frac{1}{q}\,.$$

Finally, we conclude that

$$|J|^{\alpha-1}w(J)^{1/q}\Big(\int_J w^{-p'/q}\Big)^{1/p'} \le c$$

for all intervals $J \subseteq [0,1]$. ∎

5. Integral operators with product kernels. Let $\mathcal{R} := I_0 \times \ldots \times I_n$, where I_k are fixed bounded intervals in $\mathbf{R}$. Suppose also that $0 < \alpha < 1$. Suppose also that $n \ge 2$. This section is devoted to the weak type estimates for the operators:

$$\mathcal{H}^{(n)}f(x,y) = \int_{\mathcal{R}} \frac{f(t_1,\ldots,t_n)}{\prod_{i=1}^n (x_i - t_i)}\,dt_1\cdots dt_n,$$
$$\mathcal{M}_\alpha^{(n)}f(x) = \sup_{J_1\times\ldots\times J_n \ni (x_1,\ldots,x_n)} \frac{1}{\big(\prod_{i=1}^n |J_i|\big)^{1-\alpha}} \int_{J_1\times\ldots\times J_n} |f(y_1,\ldots,y_n)|\,dy_1\cdots dy_n,$$
$$\mathcal{I}_\alpha^{(n)}f(x) = \int_{\mathcal{R}} \frac{f(t_1,\ldots,t_n)}{\prod_{i=1}^n |x_i - t_i|^{1-\alpha}}\,dt_1\cdots dt_n,$$

where $x = (x_1,\ldots,x_n) \in \mathcal{R}$. In the definition of $\mathcal{M}_\alpha^{(n)}$ the supremum is taken over all parallelepipeds $J_1 \times \ldots \times J_n \subseteq \mathcal{R}$ with sides parallel to the coordinate axes.

If $\alpha = 0$, then $\mathcal{M}_\alpha^{(n)}$ is the strong Hardy–Littlewood maximal operator denoted by $\mathcal{M}^{(n)}$.

It is easy to verify that

$$\mathcal{I}_\alpha^{(n)}f \ge \mathcal{M}_\alpha^{(n)}f, \quad f \ge 0. \tag{17}$$

The one-weight problem for these operators in the classical Lebesgue spaces was studied by the first author in the papers [12]. For the weight theory regarding integral operators with product kernels in classical Lebesgue spaces we refer also to the monograph [17].

DEFINITION 5.1. Let $1 < r < \infty$. We say that a weight function w *belongs to the Muckenhoupt class* $\mathcal{A}_r(\mathcal{R})$ ($w \in \mathcal{A}_r(\mathcal{R})$) if

$$\mathcal{A}_r(\mathcal{R}) := \sup_{R\subset\mathcal{R}} \frac{1}{|R|}\int_R w\Big(\frac{1}{|R|}\int_R w^{1-r'}\Big)^{r-1} < \infty,$$

where the supremum is taken over all n-dimensional subintervals $R \subset \mathcal{R}$ with sides parallel to the coordinate axes.

Our aim is to establish criteria for the weak type inequality for these operators under the $\mathcal{A}_p$ condition.

The following lemma will be useful for us (see [13], [16]):

Lemma 5.2. *Let $1 < p < \infty$ and let w be a weight on $\mathcal{R}$. Then there is a positive constant c such that for all $f \in L^p_w(\mathcal{R})$ and all parallelepipeds $P \subset \mathcal{R}$*

$$\|f\chi_P\|_{L^{p),\theta}_w(\mathcal{R})} \le cw(P)^{-1/p}\|f\chi_P\|_{L^p_w(\mathcal{R})}\|\chi_P\|_{L^{p)}_w(\mathcal{R})}.$$

Theorem 5.3. *Let $1 < p < \infty$ and $\theta > 0$. Then the following conditions are equivalent*:

(i) *the operator $\mathcal{M}^{(n)}$ is bounded in $L^{p),\theta}_w(\mathcal{R})$;*
(ii) *$\mathcal{M}^{(n)}$ is bounded from $L^{p),\theta}_w(\mathcal{R})$ to $WL^{p),\theta}_w(\mathcal{R})$;*
(iii) *the operator $\mathcal{H}^{(n)}$ is bounded in $L^{p),\theta}_w(\mathcal{R})$;*
(iv) *$\mathcal{H}^{(n)}$ is bounded from $L^{p),\theta}_w(\mathcal{R})$ to $WL^{p),\theta}_w(\mathcal{R})$;*
(v) *$w \in \mathcal{A}_p(\mathcal{R})$.*

Proof. The implications (i) $\Leftrightarrow$ (v), (iii) $\Leftrightarrow$ (v) are known (see [13]). The implication (ii) $\Rightarrow$ (v) follows just in the same manner as in the case of Theorem 3.1; we need only to substitute one-dimensional intervals by n-dimensional ones. Now we show some details of the implication (iv) $\Rightarrow$ (v).

For simplicity let us consider the case $n = 2$. Following the proof of Theorem 1.2 in [13] let $\mathcal{R} = [c_1, d_1; c_2, d_2]$. Suppose that $J = [a_1, b_1; a_2, b_2]$ is an arbitrary rectangle belonging to $\mathcal{R}$ with the conditions $b_i - a_i < \frac{d_i - c_i}{4}$, $i = 1, 2$. Then there exists a rectangle $J' = [a'_1, b'_1; a'_2, b'_2] \subset \mathcal{R}$ such that $b'_i - a'_i = b_i - a_i$, $i = 1, 2$, having only one vertex coinciding with at least one vertex of J.

Let us now take the test function $f = w^{1-p'}\chi_J$. Then for $(x_1, x_2) \in J'$, we have

$$|\mathcal{H}^{(n)} f(x)| \ge \frac{b}{|J|}\int_J w^{1-p'}(t_1, t_2)\, dt_1\, dt_2$$

for some positive constant b. Hence, assuming that $\lambda = \frac{b}{|J|}\int_J w^{1-p'}(t_1, t_2)\, dt_1\, dt_2$ in the weak type inequality and taking Lemma 5.2 into account we find that

$$\begin{aligned}\frac{1}{|J|}\, w^{1-p'}(J)\|\chi_{J'}\|_{L^{p),\theta}_w(\mathcal{R})} &\le c\|\mathcal{H}^{(n)} f\|_{L^{p),\theta}_w(\mathcal{R})} \le c\|f\|_{L^{p),\theta}_w(\mathcal{R})} \\ &\le c(w(J))^{-1/p}\Big(\int_R w^{1-p'}(t_1, t_2)\, dt_1\, dt_2\Big)^{1/p}\|\chi_J\|_{L^{p),\theta}_w(\mathcal{R})}.\end{aligned}$$

Analogously we have

$$\frac{1}{|J'|}w^{1-p'}(J')\|\chi_J\|_{L^{p),\theta}_w(\mathcal{R})} \le c(w(J'))^{-1/p}\Big(\int_{J'} w^{1-p'}(t_1, t_2)\, dt_1\, dt_2\Big)^{1/p}\|\chi_{J'}\|_{L^{p),\theta}_w(\mathcal{R})}.$$

Now the result follows by using Hölder's inequality and arguing as in the case $n = 1$ (see Section 3).

The fact that $w^{1-p'}(J) < \infty$ for arbitrary n-dimensional subinterval of $\mathcal{R}$ follows in the same way as in the case of $n = 1$ using the construction of n-dimensional subintervals J and J' introduced and used above; details are omitted. ∎

Now we discuss the operator $\mathcal{I}^{(n)}_\alpha$. First we formulate the following statement which follows in the same manner as Proposition 4.1 was proved (see also [16] for the strong type case); details are omitted.

PROPOSITION 5.4. *Let $0 < \alpha < 1$, $1 < p < \frac{1}{\alpha}$, θ_1 and θ_2 be positive numbers such that $\theta_2 < \theta_1 q/p$, where $q = \frac{p}{1-\alpha p}$. Then the operator $\mathcal{K}_\alpha$ is not bounded from $L^{p),\theta_1}(R_0)$ to $WL^{q),\theta_2}(R_0)$, where $\mathcal{K}_\alpha$ is $\mathcal{I}_\alpha^{(n)}$ or $\mathcal{M}_\alpha^{(n)}$.*

THEOREM 5.5. *Let $0 < \alpha < 1/p$, where $1 < p < \infty$. Let $q = \frac{p}{1-\alpha p}$. Then the following conditions are equivalent*:

(i) $\|\mathcal{I}_\alpha(fw^\alpha)\|_{L_w^{q),\theta q/p}(\mathcal{R})} \le c\|f\|_{L_w^{p),\theta}(\mathcal{R})}$ (*one-weight inequality*);
(ii) $\|\mathcal{I}_\alpha(fw^\alpha)\|_{WL_w^{q),\theta q/p}(\mathcal{R})} \le c\|f\|_{L_w^{p),\theta}(\mathcal{R})}$ (*one-weight weak type inequality*);
(iii) $\|\mathcal{M}_\alpha(fw^\alpha)\|_{L_w^{q),\theta q/p}(\mathcal{R})} \le c\|f\|_{L_w^{p),\theta}(\mathcal{R})}$;
(iv) $\|\mathcal{M}_\alpha(fw^\alpha)\|_{WL_w^{q),\theta q/p}(\mathcal{R})} \le c\|f\|_{L_w^{p),\theta}(\mathcal{R})}$;
(v) $w \in \mathcal{A}_{1+q/p'}(\mathcal{R})$.

Proof. The implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iv) are obvious. By the estimate (17) we have also (i) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iv). The equivalence (i) $\Leftrightarrow$ (v) was derived in [16]. The implication (iv) $\Rightarrow$ (v) follows in the same manner as in the proof of Theorem 4.2; we need only to take n-dimensional intervals instead of one-dimensional ones and use Lemma 5.2 instead of Lemma 2.1 (see also [16] for some details). ∎

Acknowledgments. The authors were supported by the Rustaveli National Science Foundation Grant (Contract numbers D/13-23 and 31/47). The authors are grateful to Mr. Jan Kowalski for a careful reading of the manuscript and corrections.

References

[1] C. Capone, A. Fiorenza, *On small Lebesgue spaces*, J. Funct. Spaces Appl. 3 (2005), 73–89.
[2] G. Di Fratta, A. Fiorenza, *A direct approach to the duality of grand and small Lebesgue spaces*, Nonlinear Anal. 70 (2009), 2582–2592.
[3] D. E. Edmunds, V. Kokilashvili, A. Meskhi, *Bounded and Compact Integral Operators*, Math. Appl. 543, Kluwer Acad. Publ., Dordrecht, 2002.
[4] A. Fiorenza, *Duality and reflexivity in grand Lebesgue spaces*, Collect. Math. 51 (2000), 131–148.
[5] A. Fiorenza, B. Gupta, P. Jain, *The maximal theorem in weighted grand Lebesgue spaces*, Studia Math. 188 (2008), 123–133.
[6] A. Fiorenza, G. E. Karadzhov, *Grand and small Lebesgue spaces and their analogs*, Z. Anal. Anwend. 23 (2004), 657–681.
[7] A. Fiorenza, J.-M. Rakotoson, *Petits espaces de Lebesgue et quelques applications*, C. R. Acad. Sci. Paris 334 (2002), 23–26.
[8] J. García-Cuerva, J. L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland Math. Stud. 116, North-Holland, Amsterdam 1985.
[9] L. Greco, T. Iwaniec, C. Sbordone, *Inverting the p-harmonic operator*, Manuscripta Math. 92 (1997), 249–258.
[10] R. A. Hunt, B. Muckenhoupt, R. Wheeden, *Weighted norm inequalities for the conjugate function and Hilbert transform*, Trans. Amer. Math. Soc. 176 (1973), 227–251.
[11] T. Iwaniec, C. Sbordone, *On the integrability of the Jacobian under minimal hypotheses*, Arch. Rational Mech. Anal. 119 (1992), 129–143.

[12] V. Kokilashvili, *Weighted Lizorkin–Triebel spaces. Singular integrals, multipliers, imbedding theorems* (Russian), Trudy Mat. Inst. Steklov. 161 (1983), 125–149; English transl.: Proc. Steklov Inst. Mat. 161 (1984), 135–162.

[13] V. Kokilashvili, *Singular integrals and strong maximal functions in weighted grand Lebesgue spaces*, in: Nonlinear Analysis, Function Spaces and Applications, vol. 9, Acad. Sci. Czech Repub. Inst. Math., Prague 2011.

[14] V. Kokilashvili, *Weighted problems for operators of harmonic analysis in some Banach function spaces*, Lecture course of the Summer School and Workshop "Harmonic Analysis and Related Topics" (HART2010), Lisbon, June 21–25, 2010; http://www.math.ist.utl.pt/~hart2010/kokilashvili.pdf.

[15] V. Kokilashvili, A. Meskhi, *A note on the boundedness of the Hilbert tranform in weighted grand Lebesgue spaces*, Georgian Math. J. 16 (2009), 547–551.

[16] V. Kokilashvili, A. Meskhi, *Potentials with product kernels in grand Lebesgue spaces: one-weight criteria*, Lith. Math. J. 53 (2013), 27–39.

[17] V. Kokilashvili, A. Meskhi, L.-E. Persson, *Weighted Norm Inequalities for Integral Transforms with Product Kernels*, Nova Science Publ., New York 2010.

[18] A. Meskhi, *Criteria for the boundedness of potential operators in grand Lebesgue spaces*, arXiv:1007.1185v1 [math.FA] 7 Jul 2010.

[19] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. 165 (1972), 207–226.

[20] B. Muckenhoupt, R. Wheeden, *Weighted norm inequalities for fractional integrals*, Trans. Amer. Math. Soc. 192 (1974), 261–276.

[21] S. G. Samko, S. M. Umarkhadzhiev, *On Iwaniec–Sbordone spaces on sets which may have infinite measure*, Azerb. J. Math. 1 (2010), 67–84.