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EFFECTIVE ENERGY INTEGRAL FUNCTIONALS FOR THIN FILMS WITH BENDING MOMENT IN THE ORLICZ–SOBOLEV SPACE SETTING

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Abstract. In this paper we deal with the energy functionals for the elastic thin film $\omega \subset \mathbb{R}^2$ involving the bending moments. The effective energy functional is obtained by Γ-convergence and $3D$-$2D$ dimension reduction techniques. Then we prove the existence of minimizers of the film energy functional. These results are proved in the case when the energy density function has the growth prescribed by an Orlicz convex function M. Here M is assumed to be non-power-growth-type and to satisfy the conditions Δ_2 and ∇_2 (that is equivalent to the reflexivity of Orlicz and Orlicz–Sobolev spaces generated by M). These results extend results of G. Bouchitté, I. Fonseca and M. L. Mascarenhas for the case $M(t) = |t|^p$ for some $p \in (1, \infty)$.

1. Introduction. The mathematical theory of nonlinear elasticity has a long history with major contributions from L. Euler, J. Bernoulli, A. Cauchy, G. Kirchhoff, A. E. Love, T. von Karman and many modern authors (see [28, 6, 10, 18]). One of main problems in this research is to understand relations between three-dimensional and two-dimensional theories for thin domains.

We consider an elastic thin film as a bounded open subset $\omega \subset \mathbb{R}^2$ with Lipschitz

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boundary. The set $\Omega_\varepsilon := \omega \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}) \subset \mathbb{R}^3$ for a small thickness ε is considered as an elastic cylinder approximate to the film ω. A three-dimensional deformation $U_\varepsilon : \Omega_\varepsilon \to \mathbb{R}^3$ defined on the thin cylinder Ω_ε has the re-scaled elastic total energy represented by the difference of the re-scaled bulk and surface energies

$$\frac{1}{\varepsilon} \int_{\Omega_\varepsilon} W(DU_\varepsilon)\, dx - \frac{1}{\varepsilon} Q_\varepsilon(U_\varepsilon).$$

The purpose of this type of research is to investigate the limiting energies as $\varepsilon \to 0$ of the sequence of the above re-scaled elastic total energies and to understand the behavior as $\varepsilon \to 0$ of minimizers subject to appropriate boundary conditions.

Let the energy density function $W : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ have the growth prescribed by an Orlicz convex function M. We assume that M is non-power-growth-type and satisfies the conditions Δ_2 and ∇_2 (that is equivalent to the reflexivity of Orlicz and Orlicz–Sobolev spaces generated by M).

In our previous paper [26] we extend to the Orlicz–Sobolev space setting results established by H. Le Dret and A. Raoult in 1995 [27, Theorem 2, Theorem 8] (cf. [6, Theorem 12.2.1]) for the case of the above re-scaled total energy and thin films in the reflexive Sobolev space setting with $M(t) = |t|^p$ for some $p \in (1, \infty)$. In the famous case considered by H. Le Dret and A. Raoult the density function of the re-scaled surface energy $\frac{1}{\varepsilon} Q_\varepsilon(U_\varepsilon)$ of the re-scaled total energy is a function of the variable U_ε (independent on the scaled factor $\frac{1}{\varepsilon}$).

In the case considered in the recent work of G. Bouchitté, I. Fonseca and M. L. Mascarenhas in 2004 [7], the density function of the re-scaled surface energy $\frac{1}{\varepsilon} Q_\varepsilon(U_\varepsilon)$ is a function of the variable $\frac{1}{\varepsilon} U_\varepsilon$ (dependent explicitly in this way on the scaled factor $\frac{1}{\varepsilon}$), and this generates the bending moment of the film. Therefore, the different type of the limiting effective energy functional is obtained in [7].

The main purpose of the present paper (see Theorem 4.1 and Corollary 4.2) is to extend the results established by G. Bouchitté, I. Fonseca and M. L. Mascarenhas in 2004 [7, Theorem 1.2, Corollary 1.3] for the case of the above re-scaled total energy and thin films in the reflexive Sobolev space setting with $M(t) = |t|^p$ for some $p \in (1, \infty)$.

In Theorem 4.1, the effective energy functional for the thin film ω is obtained, by Γ-convergence and $3D$-$2D$ dimension reduction techniques applied to the sequence of the re-scaled total energy integral functionals of the elastic cylinders Ω_ε as the thickness ε goes to 0. In Corollary 4.2, the existence of minimizers of the energy functional for the thin film is established by showing that some sequence of re-scaled minimizers weakly converges in an appropriate Orlicz–Sobolev space to a minimizer of the film energy functional.

In Section 5, we give the proofs of Theorem 4.1 and Corollary 4.2. Our proof scheme extends the proof scheme of G. Bouchitté, I. Fonseca and M. L. Mascarenhas [7]. For these proofs we apply also results: for Orlicz convex functions [22, Proposition 4], for the Orlicz–Sobolev spaces [24, Theorem 5, Theorem 7] (cf. [13]), [19, Proposition 2.1], for differentiability properties of the Orlicz–Sobolev functions [3, Lemma 3.1, Lemma 3.2], for the sub-differential operator in Orlicz spaces [36, Lemma 1] and for quasiconvex integral functionals and quasiconvexification in the Orlicz–Sobolev space setting [16].

Recall that various concrete examples of M with $M \in \Delta_2 \cap \nabla_2$ can be found in [25, Theorem 7.1, pp. 58–59] and [29, 30]. Furthermore, the assumption $M \in \Delta_2 \cap \nabla_2$ is indispensable in the regularity study of minimizers of multiple variational integrals with the M-growth on Orlicz–Sobolev spaces (see discussions and references for many other concrete examples in [15]).

2. Some terminology and notation. From now on, unless stated to the contrary, $M : \mathbb{R} \to [0, \infty)$ is assumed to be a non-power-growth-type Orlicz N-function (i.e., even convex function satisfying $\lim_{t\to 0} \frac{M(t)}{t} = 0$ and $\lim_{t\to+\infty} \frac{M(t)}{t} = +\infty$).

We assume $M \in \Delta_2 \cap \nabla_2$. Here the condition $M \in \Delta_2$ means that $M(2t) \leq c\,M(t) \quad (t \geq t_0)$ for some $t_0 \in [0, \infty)$ and $c \in (0, \infty)$. The condition $M \in \nabla_2$ means that $\exists\, l > 1$, $\exists\, t_* \in [0, \infty)$ such that $M(t) \leq \frac{1}{2l} M(lt)$ for all $t \geq t_*$.

Let M^* be the complementary (conjugate) Orlicz N-function of M defined by

$$M^*(\tau) := \sup\{t\tau - M(t) : t \in \mathbb{R}\}.$$

It is known that the condition $M \in \nabla_2$ is equivalent to the condition $M^* \in \Delta_2$.

Denote by $|v|$ the Euclidean norm of v and by (u, v) the scalar product. Given an open bounded subset $G \subset \mathbb{R}^N$ with Lipschitz (e.g., C^2-smooth) boundary ∂G equipped with the $(N-1)$-dimensional Hausdorff measure $\mathcal{H}^{N-1}$. Denote by $L^M(G; \mathbb{R}^m)$ the Orlicz space of all (equivalent classes of) measurable functions $u : G \to \mathbb{R}^m$ equipped with the Luxemburg norm

$$\|u\|_{L^M(G;\mathbb{R}^m)} := \inf\Big\{\lambda > 0 : \int_\Omega M(|u(x)|/\lambda)\,dx \leq 1\Big\}.$$

It is known that $M \in \Delta_2 \cap \nabla_2$ is equivalent to the reflexivity of $L^M(G; \mathbb{R}^m)$.

Recall that the Orlicz–Sobolev space $W^{1,M}(G; \mathbb{R}^3)$ is defined as the Banach space of $\mathbb{R}^3$-valued functions u of $L^M(G; \mathbb{R}^3)$ with the Sobolev–Schwartz distributional derivative $Du \in L^M(G; \mathbb{R}^{3\times N})$ equipped with the norm

$$\|u\|_{W^{1,M}(G;\mathbb{R}^3)} := \|u\|_{L^M(G;\mathbb{R}^3)} + \|Du\|_{L^M(G;\mathbb{R}^{3\times N})} < \infty.$$

The subspace $W_0^{1,M}(G; \mathbb{R}^3)$ is defined as the closure in $\|\cdot\|_{W^{1,M}(G;\mathbb{R}^3)}$-norm of the set $C_0^\infty(G; \mathbb{R}^3)$ of C^∞-smooth $\mathbb{R}^3$-valued functions with compact support in G. Since ∂G is Lipschitz and $M, M^* \in \Delta_2$, by [17, Theorems 2.1, 2.3] there exists the bounded linear trace operator

$$\mathrm{Tr} : W^{1,M}(G; \mathbb{R}^3) \to L^M(\partial G; \mathbb{R}^3)$$

such that: (i) $\mathrm{Tr}(u) = u_{|\partial G}$ ($\forall u \in C^\infty(\overline{G})$) and (ii) $u \in W_0^{1,M}(G; \mathbb{R}^3)$ if and only if $\mathrm{Tr}(u) = 0$. So, for the simplicity of notation we will write "$u(x) = \varphi(x)$ on A" for $u \in W^{1,M}(G; \mathbb{R}^3)$ and $\varphi \in L^M(\partial G; \mathbb{R}^3)$ and $A \subset \partial G$ if $\mathrm{Tr}(u)(x) = \varphi(x)$ for almost every $x \in A$. Due to this reason, we also write "u on A" for "$\mathrm{Tr}(u)$ on A", etc.

By [2, Proof of Theorem 3.9] and [21, Proof of Lemma 2.2], given a normed subspace $(X, \|\cdot\|_{W^{1,M}(G;\mathbb{R}^3)})$ and $\Lambda \in X^*$, there exist $h_0, h_1, \ldots, h_N \in L^{M^*}(G; \mathbb{R}^3)$ such that

$$\Lambda(u) = \int_G (h_0, u)\,dx + \sum_{i=1}^{N} \int_G \Big(h_i, \frac{\partial u}{\partial x_i}\Big)\,dx \quad (u \in X). \tag{1}$$

Conversely, every functional Λ defined by (1) in the case $h_0, h_1, \dots, h_N \in L^{M^*}(G;\mathbb{R}^3)$, is an element of X^*.

3. Setup. Define $I := (-\frac{1}{2}, \frac{1}{2})$, $\Omega := \omega \times I$, $S^{\pm} := \omega \times \{\pm\frac{1}{2}\}$, $\Gamma := \partial\omega \times I$, and for each $\varepsilon > 0$, $S^{\pm}_{\varepsilon} := \omega \times \{\pm\frac{\varepsilon}{2}\}$, $\Gamma_\varepsilon := \partial\omega \times \varepsilon I$. Greek indexes will be used to distinguish the first two components of a vector, for instance (x_α) and (x_α, x_3), designates (x_1, x_2) and (x_1, x_2, x_3), respectively. We denote by $\mathbb{R}^{3\times 3}$ and $\mathbb{R}^{3\times 2}$ the vector spaces of respectively 3×3 and 3×2 real-valued matrices. Given $\bar{F} \in \mathbb{R}^{3\times 2}$ and $b \in \mathbb{R}^3$, denote by $(\bar{F}\,|\,b)$ the 3×3 matrix whose first two columns are those of $\bar{F}$ and the last column is b. By the analogous way, set $e_\alpha := (e_1\,|\,e_2) \in \mathbb{R}^{3\times 2}$ where $\{e_1, e_2, e_3\}$ is the standard basis of $\mathbb{R}^3$. Set $D_\alpha U := (\frac{\partial U}{\partial x_1}\,|\,\frac{\partial U}{\partial x_2})$, $D_3 U := \frac{\partial U}{\partial x_3}$, $DU := (D_\alpha U\,|\,D_3 U)$ for an $\mathbb{R}^3$-valued function U. Denote by C, $\widetilde{C}$ generic positive constants that may vary from line to line.

Let $W : \mathbb{R}^{3\times 3} \to \mathbb{R}$ be a continuous function satisfying the M-growth-type and coercivity conditions:

$$\frac{1}{C}(M(|F|) - 1) \le W(F) \le C(1 + M(|F|)) \quad (\forall F \in \mathbb{R}^{3\times 3}) \tag{2}$$

for some $C \in (0, \infty)$.

Set

$$\widetilde{\Psi}_\varepsilon := \big\{U \in W^{1,M}(\Omega_\varepsilon;\mathbb{R}^3) : U(\tilde{x}) = \tilde{x} \text{ on } \Gamma_\varepsilon\big\}.$$

We consider the variational integral functional $\widetilde{J}_\varepsilon : \widetilde{\Psi}_\varepsilon \to \mathbb{R}$, where $\widetilde{J}_\varepsilon(U)$ (the re-scaled total energy of the elastic cylinder Ω_ε under a deformation $U : \Omega_\varepsilon \to \mathbb{R}^3$) is represented by the difference of the re-scaled bulk and surface energies:

$$\widetilde{J}_\varepsilon(U) := \frac{1}{\varepsilon}\int_{\Omega_\varepsilon} W(DU)\,d\tilde{x} - \frac{1}{\varepsilon}\int_{\Omega_\varepsilon} (f_\varepsilon, U)\,d\tilde{x} \\ - \int_{S^+_\varepsilon} (g_0^+ + \frac{1}{\varepsilon}g, U)\,d\mathcal{H}^2 + \int_{S^-_\varepsilon} \Big(g_0^- + \frac{1}{\varepsilon}g, U\Big)\,d\mathcal{H}^2. \tag{3}$$

Here, $f_\varepsilon := f\big(\tilde{x}_\alpha, \frac{\tilde{x}_3}{\varepsilon}\big)$, $f \in L^{M^*}(\Omega;\mathbb{R}^3)$, $g_0^{\pm}, g \in L^{M^*}(\omega;\mathbb{R}^3)$ and $\mathcal{H}^2$ denotes the 2-dimensional Hausdorff measure in $\mathbb{R}^3$. Set

$$\overline{\Psi}_0 := \big\{\bar{u} \in W^{1,M}(\omega;\mathbb{R}^3) : \bar{u}(x_\alpha) = (x_\alpha, 0) \text{ on } \partial\omega\big\}.$$

Let $J_0 : \overline{\Psi}_0 \times L^M(\omega;\mathbb{R}^3) \to \mathbb{R}$ be defined by

$$J_0(\bar{u}, \bar{b}) := \int_\omega \mathcal{Q}^* W(D_\alpha \bar{u}\,|\,\bar{b})\,dx_\alpha - P_0(\bar{u}, \bar{b}), \tag{4}$$

where

$$\mathcal{Q}^* W(\bar{F}, z) := \inf\Big\{\int_Q W(\bar{F} + D_\alpha\varphi\,|\,\lambda D_3\varphi)\,dx : \lambda \in \mathbb{R},\ \varphi \in W^{1,M}(Q;\mathbb{R}^3), \\ \varphi(\cdot, x_3) \text{ is } Q'\text{-periodic } \mathcal{L}^1 \text{ a.e. } x_3 \in I,\ \lambda\int_Q D_3\varphi\,dx = z\Big\} \tag{5}$$

for every $\bar{F} \in \mathbb{R}^{3\times 2}$, $z \in \mathbb{R}^3$, with $Q' := I^2$, $Q := I^3$ and

$$P_0(\bar{u}, \bar{b}) := \int_\omega (\bar{f}, \bar{u})\,dx_\alpha + \int_\omega (g_0^+ - g_0^-, \bar{u})\,dx_\alpha + \int_\omega (g, \bar{b})\,dx_\alpha,$$

with $\bar{f}(x_\alpha) := \int_I f(x_\alpha, x_3)\,dx_3$.

4. The formulation of main results. Let $\mathcal{Z}$ be the space of membrane deformations defined by

$$\mathcal{Z} = \{z \in W^{1,M}(\Omega;\mathbb{R}^3) : D_3 z = 0,\ z(x) = (x_\alpha, 0) \text{ on } \Gamma\}. \tag{6}$$

Observe that $\mathcal{Z}$ is canonically isomorphic to $\overline{\Psi}_0$ [31, Theorem 1.1.3/1]. Let $\bar{z}$ denote the element of $\overline{\Psi}_0$ that is associated with $z \in \mathcal{Z}$ through this isomorphism:

$$z(x_\alpha, x_3) = \bar{z}(x_\alpha) \text{ a.e.} \tag{7}$$

Since we want to identify the sequence convergence with the thickness of our domain tending to zero, for simplicity we assume this thickness parameter ε takes its values in a sequence $\varepsilon_n \to 0$.

THEOREM 4.1. *Let $\widetilde{J}_\varepsilon$ be defined by* (3) *and J_0 be defined by* (4). *Assume $M \in \Delta_2 \cap \nabla_2$. Assume that the continuous function $W : \mathbb{R}^{3\times 3} \to \mathbb{R}$ satisfies the conditions* (2). *Let $\{U_\varepsilon\} \in \widetilde{\Psi}_\varepsilon$. For each $\varepsilon > 0$ and $\tilde{x} = (\tilde{x}_\alpha, \tilde{x}_3) \in \Omega_\varepsilon$ we associate $x = (x_\alpha, x_3) := \left(\tilde{x}_\alpha, \frac{1}{\varepsilon}\tilde{x}_3\right) \in \Omega$ and we set $z_\varepsilon(x_\alpha, x_3) := U_\varepsilon(\tilde{x}_\alpha, \tilde{x}_3)$.*

Then the sequence $\widetilde{J}_\varepsilon$ converges to J_0 in the following sense:

(i) (*lower bound*) *if $z_\varepsilon \rightharpoonup z$ weakly in $W^{1,M}(\Omega;\mathbb{R}^3)$, $\|z_\varepsilon\|_{W^{1,M}(\Omega;\mathbb{R}^3)} < +\infty$ and $z \in \mathcal{Z}$ with $z(x_\alpha, x_3) = \bar{z}(x_\alpha)$ through the isomorphism* (7) *and $\frac{1}{\varepsilon}\int_I D_3 z_\varepsilon\, dx_3 \rightharpoonup \bar{b}$ weakly in $L^M(\omega;\mathbb{R}^3)$ and $\|\frac{1}{\varepsilon} D_3 z_\varepsilon\|_{L^M(\Omega;\mathbb{R}^3)} < +\infty$, then*
$$\liminf_{\varepsilon\to 0} \widetilde{J}_\varepsilon(U_\varepsilon) \geq J_0(\bar{z}, \bar{b});$$

(ii) (*upper bound*) *for every pair $(\bar{z}, \bar{b}) \in \overline{\Psi}_0 \times L^M(\omega;\mathbb{R}^3)$, there exists a sequence $U_\varepsilon \in W^{1,M}(\Omega;\mathbb{R}^3)$ such that $z_\varepsilon \rightharpoonup z$ weakly in $W^{1,M}(\Omega;\mathbb{R}^3)$, $\|z_\varepsilon\|_{W^{1,M}(\Omega;\mathbb{R}^3)} < +\infty$ and $z \in \mathcal{Z}$ with $z(x_\alpha, x_3) = \bar{z}(x_\alpha)$ through the isomorphism* (7) *and $\frac{1}{\varepsilon}\int_I D_3 z_\varepsilon\, dx_3 \rightharpoonup \bar{b}$ weakly in $L^M(\omega;\mathbb{R}^3)$ and $\|\frac{1}{\varepsilon} D_3 z_\varepsilon\|_{L^M(\Omega;\mathbb{R}^3)} < +\infty$ and*
$$\lim_{\varepsilon\to 0} \widetilde{J}_\varepsilon(U_\varepsilon) = J_0(\bar{z}, \bar{b}).$$

Consider the asymptotic behavior of $U_\varepsilon \in \widetilde{\Psi}_\varepsilon$ such that

$$\widetilde{J}_\varepsilon(U_\varepsilon) \leq \inf_{U\in\widetilde{\Psi}_\varepsilon} \widetilde{J}_\varepsilon(U) + \gamma(\varepsilon), \tag{8}$$

where γ is a positive function such that $\gamma(\varepsilon) \to 0$ as $\varepsilon \to 0$.

COROLLARY 4.2 (The minimization problem). *Assume that $U_\varepsilon \in \widetilde{\Psi}_\varepsilon$ satisfies* (8). *Let the functions M, W and z_ε, $\bar{z}$ be such as in Theorem* 4.1. *Then:*

(i) *the sequence $\left(z_\varepsilon, \frac{1}{\varepsilon}\int_I D_3 z_\varepsilon\, dx_3\right)$ is relatively weakly compact in $W^{1,M}(\Omega;\mathbb{R}^3) \times L^M(\omega;\mathbb{R}^3)$;*

(ii) *the set $\mathcal{C}_{\mathrm{film}}$ of cluster points of the sequence $\left(z_\varepsilon, \frac{1}{\varepsilon}\int_I D_3 z_\varepsilon\, dx_3\right)$ in the weak topology is a non-empty subset of $\mathcal{Z} \times L^M(\omega;\mathbb{R}^3)$;*

(iii) *any point $(z_\infty, \bar{b})$ of $\mathcal{C}_{\mathrm{film}}$ can be identified with $(\bar{z}_\infty, \bar{b}) \in \overline{\Psi}_0 \times L^M(\omega;\mathbb{R}^3)$ by the 3D-2D dimension reduction isomorphism* (7) *and $(\bar{z}_\infty, \bar{b})$ is a solution of the minimization problem*

$$\inf_{\bar{u}\in\overline{\Psi}_0} \{J_0(\bar{u}, \bar{b}) : \bar{b} \in L^M(\omega;\mathbb{R}^3)\}. \tag{9}$$

5. The proofs of Theorem 4.1 and Corollary 4.2. We will reformulate Theorem 4.1 and Corollary 4.2 by the use of the following equivalent functionals $\bar{J}_\varepsilon^*$ and J_0^* (see the re-formulation in Theorem 5.1 and Corollary 5.2). Define

$$u_{0,\varepsilon}(x) := (x_\alpha, \varepsilon x_3), \quad u_{0,0}(x) := (x_\alpha, 0). \tag{10}$$

Notice that after the change of variables as in Theorem 4.1 with the association

$$x = (x_\alpha, x_3) := \Big(\tilde{x}_\alpha, \frac{1}{\varepsilon}\tilde{x}_3\Big), \quad u(x_\alpha, x_3) := U(\tilde{x}_\alpha, \tilde{x}_3), \tag{11}$$

the re-scaled energy $\widetilde{J}_\varepsilon(U)$ in (3) can be rewritten in the equivalent form

$$\begin{aligned} J_\varepsilon(u) = &\int_\Omega W\Big(D_\alpha U \,\Big|\, \frac{1}{\varepsilon} D_3 u\Big)\,dx - \int_\Omega (f,u)\,dx - \int_{S^+} (g_0^+, u)\,d\mathcal{H}^2 \\ &+ \int_{S^-} (g_0^-, u)\,d\mathcal{H}^2 - \int_\omega \Big(g, \frac{u^+ - u^-}{\varepsilon}\Big)\,dx_\alpha \\ = &\int_\Omega W\Big(D_\alpha U \,\Big|\, \frac{1}{\varepsilon} D_3 u\Big)\,dx - \int_\Omega (f,u)\,dx - \int_{S^+} (g_0^+, u)\,d\mathcal{H}^2 \\ &+ \int_{S^-} (g_0^-, u)\,d\mathcal{H}^2 - \int_\omega \Big(g, \frac{1}{\varepsilon}\int_I D_3 u\,dx_3\Big)\,dx_\alpha, \end{aligned} \tag{12}$$

where $u^\pm(x_\alpha) := \mathrm{Tr}_{S^\pm}(u)(x_\alpha)$ and u is an element of

$$\Psi_\varepsilon := \big\{u \in W^{1,M}(\Omega;\mathbb{R}^3) : u(x) = u_{0,\varepsilon}(x) \text{ on } \Gamma\big\}.$$

In order to individualize the new sequence $\frac{1}{\varepsilon}\int_I D_3 u\,dx_3$ it is needed to consider the new functional $\bar{J}_\varepsilon : W^{1,M}(\Omega;\mathbb{R}^3) \times L^M(\omega;\mathbb{R}^3) \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\bar{J}_\varepsilon(u, \bar{b}) := \begin{cases} \int_\Omega W(D_\alpha u \,|\, \frac{1}{\varepsilon} D_3 u)\,dx - P_\varepsilon(u) & \text{if } \frac{1}{\varepsilon}\int_I D_3 u\,dx_3 = \bar{b}(x_\alpha) \text{ and } u \in \Psi_\varepsilon \\ +\infty & \text{otherwise,} \end{cases} \tag{13}$$

where

$$P_\varepsilon(u) := \int_\Omega (f,u)\,dx - \int_{S^+} (g_0^+, u)\,d\mathcal{H}^2 + \int_{S^-} (g_0^-, u)\,d\mathcal{H}^2 + \int_\omega \Big(g, \frac{1}{\varepsilon}\int_I D_3 u\,dx_3\Big)\,dx_\alpha.$$

Observe that the re-scaled displacement $v = u - u_{0,\varepsilon}$ belongs to the set

$$V = W^{1,M}_\Gamma(\Omega;\mathbb{R}^3) := \big\{v \in W^{1,M}(\Omega;\mathbb{R}^3) : v(x) = 0 \text{ on } \Gamma\big\}$$

and

$$\begin{aligned} J_\varepsilon(v + u_{0,\varepsilon}) = &\int_\Omega W\Big(e_\alpha + D_\alpha V \,\Big|\, e_3 + \frac{1}{\varepsilon} D_3 v\Big)\,dx - \int_\Omega (f, v + u_{0,\varepsilon})\,dx \\ - &\int_{S^+} (g_0^+, v + u_{0,\varepsilon})\,d\mathcal{H}^2 + \int_{S^-} (g_0^-, v + u_{0,\varepsilon})\,d\mathcal{H}^2 - \int_\omega \Big(g, \frac{1}{\varepsilon}\int_I (D_3 v + \varepsilon \cdot e_3)\,dx_3\Big)\,dx_\alpha. \end{aligned}$$

Define $\bar{J}_\varepsilon^* : W^{1,M}(\Omega;\mathbb{R}^3) \times L^M(\omega;\mathbb{R}^3) \to \mathbb{R} \cup \{+\infty\}$ by

$$\bar{J}_\varepsilon^*(v, \bar{b}) := \begin{cases} \int_\Omega W(e_\alpha + D_\alpha v \,|\, e_3 + \frac{1}{\varepsilon} D_3 v)\,dx - P_\varepsilon(v + u_{0,\varepsilon}) \\ \qquad\qquad \text{if } \frac{1}{\varepsilon}\int_I D_3 v\,dx_3 + e_3 = \bar{b}(x_\alpha) \text{ and } v \in V, \\ +\infty \qquad\quad \text{otherwise.} \end{cases} \tag{14}$$

Let $\mathcal{V}$ be the space of membrane displacements defined by

$$\mathcal{V} = \big\{v \in W^{1,M}(\Omega;\mathbb{R}^3) : D_3 v = 0,\ v(x) = 0 \text{ on } \Gamma\big\} \subset V. \tag{15}$$

Similarly as in (6)–(7), $\mathcal{V}$ is canonically isomorphic to $W_0^{1,M}(\omega;\mathbb{R}^3)$ [31, Theorem 1.1.3/1]. Let $\bar{v}$ denote the element of $W_0^{1,M}(\omega;\mathbb{R}^3)$ that is associated with $v \in \mathcal{V}$ through the isomorphism

$$v(x_\alpha, x_3) = \bar{v}(x_\alpha) \text{ a.e.} \tag{16}$$

Analogously for $v \in \mathcal{V}$ and $\bar{b} \in L^M(\omega;\mathbb{R}^3)$ define the functional

$$J_0^*(v + u_{0,0}, \bar{b}) := \int_\omega \mathcal{Q}^* W(e_\alpha + D_\alpha \bar{v} \,|\, \bar{b} - e_3)\, dx_\alpha - P_0(\bar{v} + u_{0,0}, \bar{b} + e_3). \tag{17}$$

In this notation we have for $U_\varepsilon \in \widetilde{\Psi}_\varepsilon$

$$\widetilde{J}_\varepsilon(U_\varepsilon) = J_\varepsilon(u_\varepsilon) = J_\varepsilon(v_\varepsilon + u_{0,\varepsilon}),$$

where $u_\varepsilon \in \Psi_\varepsilon$, $v_\varepsilon \in V$ with $u_\varepsilon = v_\varepsilon + u_{0,\varepsilon}$ and

$$J_0(\bar{z}, \bar{b}) = J_0^*(v + u_{0,0}, \bar{b}) \quad (v \in \mathcal{V},\ \bar{z} = \bar{v} + u_{0,0} \in \overline{\Psi}_0).$$

Recall [12], [9, Definition 7.1] that a sequence of functions I_ε from a topological space X to $\overline{R}$ is said to Γ-converge to I_0 for the topology of X if the following conditions are satisfied for all $x \in X$:

$$\begin{cases} \forall x_\varepsilon \to x, & I_0(x) \le \liminf I_\varepsilon(x_\varepsilon), \\ \exists y_\varepsilon \to y, & I_\varepsilon(y_\varepsilon) \to I_0(y). \end{cases} \tag{18}$$

THEOREM 5.1. *Let $\bar{J}_\varepsilon^*$ be defined by* (14) *and J_0^* be defined by* (17). *Assume $M \in \Delta_2 \cap \nabla_2$. Suppose that the continuous function $W : \mathbb{R}^{3\times 3} \to \mathbb{R}$ satisfies the conditions* (2). *Then the sequence $\bar{J}_\varepsilon^*$ Γ-converges to J_0^* in the weak topology of $W^{1,M}(\Omega;\mathbb{R}^3) \times L^M(\omega;\mathbb{R}^3)$, as $\varepsilon \to 0$.*

Consider the asymptotic behavior of $u_\varepsilon \in \Psi_\varepsilon$ such that

$$J_\varepsilon(u_\varepsilon) \le \inf_{u \in \Psi_\varepsilon} J_\varepsilon(u) + \gamma(\varepsilon), \tag{19}$$

where γ is a positive function such that $\gamma(\varepsilon) \to 0$ as $\varepsilon \to 0$.

COROLLARY 5.2 (The minimization problem). *Assume that $u_\varepsilon \in \Psi_\varepsilon$ satisfies* (19). *Let the functions M and W be such as in Theorem* 5.1. *Then*:

(i) *the sequence $\left(u_\varepsilon, \frac{1}{\varepsilon}\int_I D_3 u_\varepsilon\, dx_3\right)$ is relatively weakly compact in $W^{1,M}(\Omega;\mathbb{R}^3) \times L^M(\omega;\mathbb{R}^3)$;*
(ii) *the set $\mathcal{C}_{\mathrm{film}}$ of cluster points of the sequence $\left(u_\varepsilon, \frac{1}{\varepsilon}\int_I D_3 u_\varepsilon\, dx_3\right)$ in the weak topology is a non-empty subset of $\mathcal{Z} \times L^M(\omega;\mathbb{R}^3)$;*
(iii) *any point $(u_\infty, \bar{b})$ of $\mathcal{C}_{\mathrm{film}}$ can be identified with $(\bar{u}_\infty, \bar{b}) \in \overline{\Psi}_0 \times L^M(\omega;\mathbb{R}^3)$ by the 3D-2D dimension reduction isomorphism* (7) *and $(\bar{u}_\infty, \bar{b})$ is a solution of the minimization problem*

$$\inf_{\bar{u} \in \overline{\Psi}_0} \left\{ J_0(\bar{u}, \bar{b}) : \bar{b} \in L^M(\omega;\mathbb{R}^3) \right\}.$$

We start the proofs of Theorem 5.1 and Corollary 5.2, with Lemmas 5.3–5.4.

We consider the condition

$$\exists\, i(M) \in [1,\infty),\ \exists\, c \in (0,\infty) \text{ such that } M(at) \le c\, a^{i(M)} M(t)\ (\forall t \ge 0,\ \forall a \le 1), \tag{20}$$

which is equivalent to the condition

$$\exists\, i(M) \in [1,\infty),\ \exists\, c \in (0,\infty) \text{ such that } \frac{1}{c}\, b^{i(M)} M(s) \le M(bs)\ (\forall s \ge 0,\ \forall b \ge 1). \tag{21}$$

Lemma 5.3 is a re-formulation of a part of [22, Proposition 4].

LEMMA 5.3. *Assume the dual Orlicz N-function M^* satisfies the condition Δ_2^{glob}, i.e. $M^*(2\tau) \le K\, M^*(\tau)$ for all $\tau \in [0,\infty)$ and for some $K \in (0,\infty)$.*

Then M satisfies the condition (20) for some $i(M) \in (1,\infty)$.

LEMMA 5.4 (Compactness). *Let M and W be such as in Theorem 5.1, let $v_\varepsilon \in W^{1,M}(\Omega;\mathbb{R}^3)$ and $\bar b_\varepsilon \in L^M(\omega;\mathbb{R}^3)$ be a sequence such that*

$$\sup_{\varepsilon\in(0,1)} \bar J_\varepsilon^*(v_\varepsilon, \bar b_\varepsilon) \le d < +\infty. \tag{22}$$

Then there exist $\bar d_1 > 0$ and $\bar d_2 > 0$ such that

(i) $$\sup_{\varepsilon\in(0,1)} \|v_\varepsilon\|_{W^{1,M}(\Omega;\mathbb{R}^3)} \le \bar d_1 < +\infty \tag{23}$$

and

$$\sup_{\varepsilon\in(0,1)} \Big\| \frac{1}{\varepsilon} D_3 v_\varepsilon \Big\|_{L^M(\Omega;\mathbb{R}^3)} \le \bar d_2 < +\infty \tag{24}$$

and the sequence $(v_\varepsilon, \frac{1}{\varepsilon}\int_I D_3 v_\varepsilon\, dx_3)$ is relatively weakly compact in $W^{1,M}(\Omega;\mathbb{R}^3) \times L^M(\omega;\mathbb{R}^3)$;

(ii) *the set of cluster points of the sequence $(v_\varepsilon, \frac{1}{\varepsilon}\int_I D_3 v_\varepsilon\, dx_3)$ in the weak topology of $W^{1,M}(\Omega;\mathbb{R}^3) \times L^M(\omega;\mathbb{R}^3)$ is a non-empty subset of $\mathcal{V} \times L^M(\omega;\mathbb{R}^3)$.*

Proof. We divide the proof into six steps, where in Steps 2–5 we assume additionally $M^* \in \Delta_2^{\mathrm{glob}}$.

Step 1. By (22) and (14) for $\bar J_\varepsilon^*$, $v_\varepsilon \in V$ for all $\varepsilon > 0$. Let $u_\varepsilon = v_\varepsilon + u_{0,\varepsilon}$. We claim that

$$\begin{aligned}
&\int_\Omega M\Big(\Big|\Big(D_\alpha u_\varepsilon \,\Big|\, \frac{D_3 u_\varepsilon}{\varepsilon}\Big)\Big|\Big)\, dx \le C_1 \\
&+ C_1\Big((\|f\|_{L^{M^*}(\Omega;\mathbb{R}^3)} + (\|g_0^+\|_{L^{M^*}(S^+;\mathbb{R}^3)} + \|g_0^-\|_{L^{M^*}(S^-;\mathbb{R}^3)})\|\operatorname{Tr}\|_{\mathcal{L}})\|Du_\varepsilon\|_{L^M(\Omega;\mathbb{R}^{3\times3})}\Big) \\
&\qquad + C_1\|g\|_{L^{M^*}(\omega;\mathbb{R}^3)} \Big\|\frac{1}{\varepsilon} D_3 u_\varepsilon\Big\|_{L^M(\Omega;\mathbb{R}^3)}
\end{aligned} \tag{25}$$

for some $C_1 \in (0,+\infty)$ and for all $\varepsilon \in (0,1)$. Here $\|\operatorname{Tr}\|_{\mathcal{L}} := N^+ + N^-$, where N^+ (resp., N^-) denotes the operator norm of the linear trace operator $\operatorname{Tr} : W^{1,M}(\Omega;\mathbb{R}^3) \to L^M(S^+;\mathbb{R}^3)$ (resp., $\operatorname{Tr} : W^{1,M}(\Omega;\mathbb{R}^3) \to L^M(S^-;\mathbb{R}^3)$).

For this, by the coercivity condition (2) together with (22), we infer that

$$\begin{aligned}
&\frac{1}{C}\Big(\int_\Omega M\Big(\Big|\Big(D_\alpha u_\varepsilon \,\Big|\, \frac{D_3 u_\varepsilon}{\varepsilon}\Big)\Big|\Big)\, dx - |\Omega|\Big) \le d + \Big|\int_\Omega (f, u_\varepsilon)\, dx\Big| + \Big|\int_{S^+} (g_0^+, u_\varepsilon)\, d\mathcal{H}^2\Big| \\
&\quad + \Big|\int_{S^-} (g_0^-, u_\varepsilon)\, d\mathcal{H}^2\Big| + \Big|\int_\omega \Big(g, \frac{u_\varepsilon^+ - u_\varepsilon^-}{\varepsilon}\Big)\, dx_\alpha\Big| = d + \Big|\int_\Omega (f, u_\varepsilon)\, dx\Big| \\
&\qquad + \Big|\int_{S^+} (g_0^+, u_\varepsilon)\, d\mathcal{H}^2\Big| + \Big|\int_{S^-} (g_0^-, u_\varepsilon)\, d\mathcal{H}^2\Big| + \Big|\int_\omega \Big(g, \frac{1}{\varepsilon}\int_I D_3 u_\varepsilon\, dx_3\Big)\, dx_\alpha\Big|.
\end{aligned}$$

By the generalized Hölder inequality (see, e.g., [33, Theorems 13.13, 13.11], [25, 38]) and Fubini Theorem, we deduce that

$$
\begin{aligned}
&\frac{1}{C}\Big(\int_\Omega M\Big(\Big|\Big(D_\alpha u_\varepsilon \,\Big|\, \frac{D_3 u_\varepsilon}{\varepsilon}\Big)\Big|\Big)\,dx - |\Omega|\Big)\\
&\le d + 2\|f\|_{L^{M^*}(\Omega;\mathbb{R}^3)}\|u_\varepsilon\|_{L^M(\Omega;\mathbb{R}^3)}\\
&\quad + 2\big(\|g_0^+\|_{L^{M^*}(S^+;\mathbb{R}^3)}\|u_\varepsilon^+\|_{L^M(S^+;\mathbb{R}^3)} + \|g_0^-\|_{L^{M^*}(S^-;\mathbb{R}^3)}\|u_\varepsilon^-\|_{L^M(S^-;\mathbb{R}^3)}\big)\\
&\quad + 2\|g\|_{L^{M^*}(\omega;\mathbb{R}^3)}\|\frac{1}{\varepsilon}D_3 u_\varepsilon\|_{L^M(\Omega;\mathbb{R}^3)} \qquad (26)\\
&\le d + 2\|f\|_{L^{M^*}(\Omega;\mathbb{R}^3)}\|u_\varepsilon\|_{L^M(\Omega;\mathbb{R}^3)}\\
&\quad + 2\big(\|g_0^+\|_{L^{M^*}(S^+;\mathbb{R}^3)} + \|g_0^-\|_{L^{M^*}(S^-;\mathbb{R}^3)}\big)\|\operatorname{Tr}\|_{\mathcal{L}}\big(\|u_\varepsilon\|_{L^M(\Omega;\mathbb{R}^3)} + \|Du_\varepsilon\|_{L^M(\Omega;\mathbb{R}^{3\times3})}\big)\\
&\quad + 2\|g\|_{L^{M^*}(\omega;\mathbb{R}^3)}\Big\|\frac{1}{\varepsilon}D_3 u_\varepsilon\Big\|_{L^M(\Omega;\mathbb{R}^3)}.
\end{aligned}
$$

By the $W^{1,M}$-generalization (see [24, Theorems 5 and 7] together with [13, Theorem 3.9], [20, Lemma 4.14], [19, Proposition 2.1]) for the Poincaré–Sobolev-type inequality (see [32, Theorem 3.6.4]), there exists $\widetilde{C} \in (0,\infty)$ such that

$$
\begin{aligned}
\|u_\varepsilon\|_{L^M(\Omega;\mathbb{R}^3)} &\le \widetilde{C}\Big(\|Du_\varepsilon\|_{L^M(\Omega;\mathbb{R}^{3\times3})} + \int_\Gamma |u_\varepsilon|\,d\mathcal{H}^2\Big)\\
&= \widetilde{C}\Big(\|Du_\varepsilon\|_{L^M(\Omega;\mathbb{R}^{3\times3})} + \int_\Gamma |u_{0,\varepsilon}|\,d\mathcal{H}^2\Big) \qquad (27)\\
&\le \widetilde{C}\big(\|Du_\varepsilon\|_{L^M(\Omega;\mathbb{R}^{3\times3})} + \mathcal{H}^2(\Gamma)\sup_{x\in\Omega}|x|\big) < \infty \quad (\forall\varepsilon\in(0,1)).
\end{aligned}
$$

Then (26)–(27) imply (25).

Step 2. By the additional assumption $M^* \in \Delta_2^{\mathrm{glob}}$, we may apply Lemma 5.3, and so M satisfies the condition (20) for some $i(M) \in (1,\infty)$.

We claim that

$$\|Du_\varepsilon\|_{L^M(\Omega;\mathbb{R}^{3\times3})} \le C_2 < \infty \quad (\forall\varepsilon\in(0,1)), \tag{28}$$

$$\|u_\varepsilon\|_{L^M(\Omega;\mathbb{R}^3)} \le C_3 < \infty \quad (\forall\varepsilon\in(0,1)), \tag{29}$$

$$\Big\|\frac{1}{\varepsilon}D_3 u_\varepsilon\Big\|_{L^M(\Omega;\mathbb{R}^3)} \le C_4 < \infty \quad (\forall\varepsilon\in(0,1)), \tag{30}$$

$$\int_\Omega M\Big(\Big|\Big(D_\alpha u_\varepsilon \,\Big|\, \frac{D_3 u_\varepsilon}{\varepsilon}\Big)\Big|\Big)\,dx \le C_5 < \infty \quad (\forall\varepsilon\in(0,1)) \tag{31}$$

for some C_2, C_3, C_4, C_5.

For this, by (25) we infer that

$$\frac{1}{1 + \|Du_\varepsilon\|_{L^M(\Omega;\mathbb{R}^{3\times3})} + \|\frac{1}{\varepsilon}D_3 u_\varepsilon\|_{L^M(\Omega;\mathbb{R}^3)}}\int_\Omega M\Big(\Big|\Big(D_\alpha u_\varepsilon \,\Big|\, \frac{1}{\varepsilon}D_3 u_\varepsilon\Big)\Big|\Big)\,dx \le C_6 < \infty \tag{32}$$

for all $\varepsilon \in (0,1)$ and for some C_6.

Consider the case when $\|Du_\varepsilon\|_{L^M(\Omega;\mathbb{R}^{3\times3})}/2 \ge 1 > 0$ and $\|\frac{1}{\varepsilon}D_3 u_\varepsilon\|_{L^M(\Omega;\mathbb{R}^3)}/2 \ge 1 > 0$. Since

$$0 < \frac{\|Du_\varepsilon\|_{L^M(\Omega;\mathbb{R}^{3\times3})}}{2} < \|Du_\varepsilon\|_{L^M(\Omega;\mathbb{R}^{3\times3})}$$

and

$$0 < \frac{\|\frac{1}{\varepsilon}D_3u_\varepsilon\|_{L^M(\Omega;\mathbb{R}^3)}}{2} < \Big\|\frac{1}{\varepsilon}D_3u_\varepsilon\Big\|_{L^M(\Omega;\mathbb{R}^3)}$$

by the definition of the Luxemburg norm and by (20), we deduce that

$$\begin{aligned} 1 &< \int_\Omega M\Big(\frac{|Du_\varepsilon|}{\|Du_\varepsilon\|_{L^M(\Omega;\mathbb{R}^{3\times3})}/2}\Big)\,dx \\ &\le \Big(\frac{2}{\|Du_\varepsilon\|_{L^M(\Omega;\mathbb{R}^{3\times3})}}\Big)^{i(M)}\int_\Omega M(|Du_\varepsilon|)\,dx \quad (\forall\varepsilon\in(0,1)) \end{aligned} \tag{33}$$

and

$$\begin{aligned} 1 &< \int_\Omega M\Big(\frac{\frac{1}{\varepsilon}|D_3u_\varepsilon|}{\|\frac{1}{\varepsilon}D_3u_\varepsilon\|_{L^M(\Omega;\mathbb{R}^3)}/2}\Big)\,dx \\ &\le \Big(\frac{2}{\|\frac{1}{\varepsilon}D_3u_\varepsilon\|_{L^M(\Omega;\mathbb{R}^3)}}\Big)^{i(M)}\int_\Omega M\Big(\Big|\frac{1}{\varepsilon}D_3u_\varepsilon\Big|\Big)\,dx \qquad (\forall\varepsilon\in(0,1]). \end{aligned} \tag{34}$$

Obviously

$$\begin{aligned} &\int_\Omega M(|Du_\varepsilon|)\,dx + \int_\Omega M\Big(\Big|\frac{1}{\varepsilon}D_3u_\varepsilon\Big|\Big)\,dx \\ &\qquad\le 2\int_\Omega M\Big(\Big|\Big(D_\alpha u_\varepsilon \,\Big|\, \frac{1}{\varepsilon}D_3u_\varepsilon\Big)\Big|\Big)\,dx \qquad (\forall\varepsilon\in(0,1]). \end{aligned} \tag{35}$$

Therefore, (32), (33)–(34) and (35) imply

$$A\Big(\|Du_\varepsilon\|_{L^M(\Omega;\mathbb{R}^{3\times3})}, \Big\|\frac{1}{\varepsilon}D_3u_\varepsilon\Big\|_{L^M(\Omega;\mathbb{R}^3)}\Big) \le C_6 < \infty \tag{36}$$

whenever $\|Du_\varepsilon\|_{L^M(\Omega;\mathbb{R}^{3\times3})} \ge 2$ and $\|\frac{1}{\varepsilon}D_3u_\varepsilon\|_{L^M(\Omega;\mathbb{R}^3)} \ge 2$. Here

$$A(s,t) := \frac{1}{2}\cdot\frac{s^{i(M)}+t^{i(M)}}{2^{i(M)}(1+s+t)}.$$

Since $i(M) > 1$, $A(s,t) \to +\infty$ as $s \to +\infty$, $t \to +\infty$ and so there exists $C_7 \in (0,\infty)$ such that $A(s,t) > C_6$ $(\forall s,t > C_7)$. Hence, (36) implies the claims (28) and (30), where $C_2 = C_4 := \max\{C_7, 2\}$ $(\forall\varepsilon \in (0,1))$. By (27) and (25) we deduce the claims (29) and (31).

Step 3. Obviously,

$$C_7 := \sup_{\varepsilon\in(0,1)} \|u_{0,\varepsilon}\|_{W^{1,M}(\Omega;\mathbb{R}^3)} < +\infty.$$

Therefore, (28)–(29) imply (23):

$$\sup_{\varepsilon\in(0,1)} \|v_\varepsilon\|_{W^{1,M}(\Omega;\mathbb{R}^3)} \le \bar{d} := C_2 + C_3 + C_7 < \infty. \tag{37}$$

Step 4. We claim that

$$\lim_{\varepsilon\to0} \|D_3u_\varepsilon\|_{L^M(\Omega;\mathbb{R}^3)} = 0. \tag{38}$$

For this, by the convexity of M and $M(0) = 0$,

$$M(t) = M\Big(\frac{\varepsilon^{-1}t}{\varepsilon^{-1}}\Big) \le \frac{1}{\varepsilon^{-1}}M(\varepsilon^{-1}t) \quad (\forall\varepsilon\in(0,1)).$$

Since $|(z_\alpha\,|\,\varepsilon^{-1}z_3)| \geq \varepsilon^{-1}|z_3|$, we deduce, by (31) that

$$\begin{aligned} 0 &\leq \int_\Omega M(|D_3 u_\varepsilon|)\,dx \leq \varepsilon \int_\Omega M\big(\varepsilon^{-1}|D_3 u_\varepsilon|\big)\,dx \\ &\leq \varepsilon \int_\Omega M\big(|(D_\alpha u_\varepsilon\,|\,\varepsilon^{-1}D_3 u_\varepsilon)|\big)\,dx \leq \varepsilon \cdot C_4 < \infty \quad (\forall \varepsilon \in (0,1)). \end{aligned}$$

Hence

$$\lim_{\varepsilon\to 0}\int_\Omega M(|D_3 u_\varepsilon|)\,dx = 0. \tag{39}$$

It is known in the case of $M \in \Delta_2$ (see, e.g., [25, 33]) that (39) implies (38).

Step 5. It is known (see, e.g., [21, Theorems 1.1, 3.3]) that $W^{1,M}(\Omega;\mathbb{R}^3)$ is a separable reflexive Banach space as $M, M^* \in \Delta_2$. By the reflexivity and separability of the closed subspace $V = W^{1,M}_\Gamma(\Omega;\mathbb{R}^3)$ of $W^{1,M}(\Omega;\mathbb{R}^3)$, the Alaoglu–Bourbaki theorem together with [23, Theorem V.7.6] imply that any closed ball of V equipped with the weak topology is compact and metrizable. Similarly, any closed ball of $L^M(\omega;\mathbb{R}^3)$ equipped with the weak topology is compact and metrizable. Therefore, (23) and (24) imply the existence of some cluster point of the sequence $(v_\varepsilon, \frac{1}{\varepsilon}\int_I D_3 v_\varepsilon\,dx_3)$ in the weak topology of $V \times L^M(\omega;\mathbb{R}^3)$.

Now, let v be a cluster point in the weak topology $\sigma(V,V^*)$. Analogously, (28)–(29) imply that there exist $u \in W^{1,M}(\Omega;\mathbb{R}^3)$ and a subsequence (not relabeled) of the sequence u_ε such that u_ε converges weakly to u in $W^{1,M}(\Omega;\mathbb{R}^3)$. Then it is easy to check by the representation (1) that $v_\varepsilon = u_\varepsilon - u_{0,\varepsilon}$ converges weakly to $u - u_{0,0}$ in $W^{1,M}(\Omega;\mathbb{R}^3)$. Therefore, $u - u_{0,0} = v$ and $D_3 u_\varepsilon$ converges to $D_3 u$ in the weak topology $\sigma(L^M(\Omega;\mathbb{R}^3), L^{M^*}(\Omega;\mathbb{R}^3))$. By (38) and the generalized Hölder inequality [33, Theorems 13.13, 13.11], for every $y \in L^{M^*}(\Omega;\mathbb{R}^3)$ we deduce that

$$\Big|\int_\Omega (y, D_3 u)\,dx\Big| = \lim_{\varepsilon\to 0}\Big|\int_\Omega (y, D_3 u_\varepsilon)\,dx\Big| \leq \lim_{\varepsilon\to 0} 2\|y\|_{L^{M^*}(\Omega;\mathbb{R}^3)}\|D_3 u_\varepsilon\|_{L^M(\Omega;\mathbb{R}^3)} = 0.$$

Therefore, $\int_\Omega (y, D_3 u)\,dx = 0$ for every $y \in L^{M^*}(\Omega;\mathbb{R}^3)$, and so $D_3 u = 0$ a.e. Since $D_3 u_{0,0} = 0$, $D_3 v = 0$ follows, and so $v \in \mathcal{V}$.

Step 6. Now consider the general assumption $M, M^* \in \Delta_2$. By [25, (4.5) in p. 24], there exists some Orlicz N-function $N_1 \in \Delta_2^{\mathrm{glob}}$ such that

$$N_1(\tau) = M^*(\tau) \quad (\forall \tau \geq \tau_0)$$

for some $\tau_0 \in (0,\infty)$. Let $M_1 := N_1^*$. By known results of the theory of N-functions and Orlicz spaces [25, 37, 33], we deduce the following assertions: $(M^*)^* = M$, $M_1^* = (N_1^*)^* = N_1 \in \Delta_2^{\mathrm{glob}}$, $L_{M^*} = L_{N_1}$ and $L_M = L_{(M^*)^*} \cong (L_{M^*})^* = (L_{N_1})^* \cong L_{N_1^*}$ with equivalent norms, $L_M = L_{N_1^*} = L_{M_1}$ and $M_1 = N_1^* \in \Delta_2$ and $(L_M)^* = (L_{M_1})^* \cong L_{M^*} = L_{M_1^*}$ with equivalent norms.

So, $M_1 \in \Delta_2$, $M_1^* \in \Delta_2^{\mathrm{glob}}$, $W_0^{1,M}(\Omega;\mathbb{R}^3) = W_0^{1,M_1}(\Omega;\mathbb{R}^3)$ and $W^{1,M}(\Omega;\mathbb{R}^3) = W^{1,M_1}(\Omega;\mathbb{R}^3)$ with equivalent norms.

Furthermore, we deduce that the continuous function $W : \mathbb{R}^{3\times 3} \to \mathbb{R}$ satisfying the conditions (2) with respect to M, satisfies the conditions (2) with respect to M_1:

$$\frac{1}{C'}(M_1(|F|) - 1) \leq W(F) \leq C'(1 + M_1(|F|)) \quad (\forall F \in \mathbb{R}^{3\times 3})$$

for some $C' \in (0,\infty)$.

Therefore, we can apply the results of Steps 1–5 with respect to M_1 in place of M. Then by the above assertions for relations between M, M^* and M_1, M_1^*, we deduce all assertions of Lemma 5.4 with respect to M under the general assumption $M, M^* \in \Delta_2$. ∎

Remind that the quasiconvex envelope $\mathcal{Q}g : \mathbb{R}^{m\times n} \to \mathbb{R}$ of a continuous function $g : \mathbb{R}^{m\times n} \to \mathbb{R}$ is defined (see [9, Definition 6.3], [11, Theorem 6.9]) by

$$\mathcal{Q}g(E) := \inf\Big\{ \frac{1}{\operatorname{meas}(B)} \int_B g(E + D\varphi)\,dx : \varphi \in C_0^\infty(B;\mathbb{R}^m)\Big\}$$

for all $E \in \mathbb{R}^{m\times n}$ where B is the open unit ball of $\mathbb{R}^n$.

Proposition 5.5. *Let $\mathcal{Q}^*W$ be defined by* (5) *and let $W : \mathbb{R}^{3\times 3} \to \mathbb{R}$ be a continuous function satisfying the condition* (2)*. Then*

$$\mathcal{Q}^*\mathcal{Q}W(\bar{F}\,|\,z) = \mathcal{Q}^*W(\bar{F}\,|\,z), \tag{40}$$

where $\mathcal{Q}W$ denotes the quasiconvex envelope of W.

Proof. The proof of (40) is the same such as in [7, Proposition 1.1]. It suffices to apply also the following facts. Obviously

$$\mathcal{C}W \le \mathcal{Q}W \le W,$$

where $\mathcal{C}W$ denotes the convex envelope of W. Therefore, by convexity of M the function $\mathcal{Q}W$ satisfies the growth and coercivity conditions (2). By Focardi [16, Proposition 3.2] $\mathcal{Q}W$ is M-Lipschitz continuous in the sense

$$|\mathcal{Q}W(F_1) - \mathcal{Q}W(F_2)| \le \operatorname{const}\big(1 + h(|F_1|) + h(|F_2|)\big)|F_1 - F_2| \qquad (\forall F_1, F_2 \in \mathbb{R}^{3\times 3}), \tag{41}$$

where h denotes the right derivative of M. By (41), $|\mathcal{Q}W(F)| \le \operatorname{const}\big(1 + h(|F|)\cdot|F|\big)$ for all $F \in \mathbb{R}^{3\times 3}$. By the Płuciennik–Tian–Wang Lemma (see [36, Lemma 1]), for $u \in L^M(Q;\mathbb{R}^{3\times 3})$, $h(|u|) \in L^{M^*}(Q)$ and so $h(|F|)|F| \in L^1(Q)$ by the L^M-Hölder inequality [25]. $\mathcal{Q}W$ is continuous, hence (see e.g. [25], [34]), the superposition operator $N_{\mathcal{Q}W}$ mapping $L^M(Q,\mathbb{R}^{3\times 3})$ into $L^1(Q)$ is continuous. ∎

Let $\mathcal{A}(\omega)$ be a family of all open subsets of ω. According to (13) define the functional $E_\varepsilon : W^{1,M}(\Omega;\mathbb{R}^3) \times L^M(\omega;\mathbb{R}^3) \times \mathcal{A}(\omega) \to \mathbb{R} \cup \{+\infty\}$ by

$$E_\varepsilon(u,\bar{b},A) = \begin{cases} \int_{A\times I} W(D_\alpha u\,|\,\frac{1}{\varepsilon}D_3 u)\,dx & \text{if } \frac{1}{\varepsilon}\int_I D_3 u\,dx_3 = \bar{b}(x_\alpha) \text{ and } u \in \Psi_\varepsilon \\ +\infty & \text{otherwise.} \end{cases}$$

Denote by $E_0 : \mathcal{Z} \times L^M(\omega;\mathbb{R}^3) \times \mathcal{A}(\omega) \to \mathbb{R} \cup \{+\infty\}$ the Γ-lower limit (see [12]) of E_ε, i.e.

$$\begin{aligned} E_0(u,\bar{b},A) := \inf\Big\{ &\liminf_{n\to+\infty} \int_{A\times I} W(D_\alpha u_n\,|\,\lambda_n D_3 u_n)\,dx : u_n \rightharpoonup u \\ &\text{weakly in } W^{1,M}(A\times I;\mathbb{R}^3),\quad \lambda_n \int_I D_3 u_n\,dx_3 \rightharpoonup \bar{b} \text{ weakly in } L^M(A;\mathbb{R}^3)\Big\}, \end{aligned} \tag{42}$$

where $\lambda_n := (\varepsilon_n)^{-1}$.

Later on, we say that $u_n \to u$ in $L^M_{\mathrm{loc}}(A\times I;\mathbb{R}^3)$ if $u_n \to u$ in $L^M(D\times I;\mathbb{R}^3)$-norm for any $D \Subset A$.

LEMMA 5.6. *Let the functions M and W be such as in Theorem* 5.1 *and E_0 be defined by* (42). *Then for any sequence $\lambda_n \to 0$, there exists a subsequence λ_{n_k} such that for each $(u, \bar{b}) \in \mathcal{Z} \times L^M(\omega; \mathbb{R}^3)$, the set function $E_0(u, b, \cdot)$ is a trace of a Radon measure, absolutely continuous with respect to the 2-dimensional Lebesgue measure.*

The proof of Lemma 5.6 is the same as that of Lemma 2.1 in [7].

LEMMA 5.7. *Let the functions M and W be such as in Theorem* 5.1. *Let $A \in \mathcal{A}(\omega)$, $L \in \mathbb{R}$, $u \in \mathcal{Z}$ and consider the sequences $u_n \in W^{1,M}(A \times I; \mathbb{R}^3)$ and $\lambda_n \in \mathbb{R}$ such that $u_n \to u$ in $L^M_{\mathrm{loc}}(A \times I; \mathbb{R}^3)$-norm, $\lambda_n \int_I D_3 u_n \, dx_3 \rightharpoonup \bar{b}$ weakly in $L^M(A; \mathbb{R}^3)$ and*

$$\lim_{n\to+\infty} \int_{A\times I} W\big(D_\alpha u_n \,|\, \lambda_n D_3 u_n\big)\, dx = L.$$

Then there exist a subsequence λ_{n_k} of λ_n and a sequence $\tilde{u}_k \in W^{1,M}(A \times I; \mathbb{R}^3)$ such that $\tilde{u}_k = u$ on $\Theta_k(\partial A) \times I$ for some neighborhood $\Theta_k(\partial A)$, $\tilde{u}_k \to u$ in $L^M_{\mathrm{loc}}(A \times I; \mathbb{R}^3)$-norm, $\lambda_n \int_I D_3 \tilde{u}_k \, dx_3 \rightharpoonup \bar{b}$ weakly in $L^M(A; \mathbb{R}^3)$ and

$$\limsup_{k\to+\infty} \int_{A\times I} W(D_\alpha \tilde{u}_k \,|\, \lambda_n D_3 \tilde{u}_k)\, dx \le L.$$

The proof of Lemma 5.7 is the same as that of Lemma 2.2 in [7].

LEMMA 5.8. *The infimum in* (42) *for E_0 remains unchanged if we replace W by its quasiconvex envelope $\mathcal{Q}W$.*

Proof. Fix $(u, \bar{b}, A) \in \mathcal{Z} \times L^M(\omega; \mathbb{R}^3) \times \mathcal{A}(\omega)$ and define

$$\begin{aligned}\widehat{\mathcal{Q}}E_0(u, \bar{b}, A) := \inf\Big\{ &\liminf_{n\to+\infty} \int_{A\times I} \mathcal{Q}W(D_\alpha u_n \,|\, \lambda_n D_3 u_n)\, dx : u_n \rightharpoonup u \\ &\text{weakly in } W^{1,M}(A \times I; \mathbb{R}^3), \quad \lambda_n \int_I D_3 u_n \, dx_3 \rightharpoonup \bar{b} \text{ weakly in } L^M(A; \mathbb{R}^3)\Big\}.\end{aligned}$$

Since $W(\bar{F} \,|\, z) \ge \mathcal{Q}W(\bar{F} \,|\, z)$ for all $\bar{F} \in \mathbb{R}^{3\times 2}$ and $z \in \mathbb{R}^3$, $E_0(u, \bar{b}, A) \ge \widehat{\mathcal{Q}}E_0(u, \bar{b}, A)$.

To prove the opposite inequality, fix $\delta > 0$ and let $u_n \in W^{1,M}(A \times I; \mathbb{R}^3)$ be such that $u_n \rightharpoonup u$ weakly in $W^{1,M}(A \times I; \mathbb{R}^3)$, $\lambda_n \int_I D_3 u_n \, dx_3 \rightharpoonup \bar{b}$ weakly in $L^M(A; \mathbb{R}^3)$ and

$$\widehat{\mathcal{Q}}E_0(u, \bar{b}, A) \ge \lim_{n\to+\infty} \int_{A\times I} \mathcal{Q}W(D_\alpha u_n \,|\, \lambda_n D_3 u_n)\, dx - \delta. \tag{43}$$

By [7, (2.2) in Proof of Proposition 1.1] and by the Focardi $W^{1,M}$-generalization in [16, Theorem 3.1] of the Acerbi–Fusco weak l.s.c. theorem together with the Acerbi–Fusco $W^{1,\infty}$-relaxation theorem [1], for each n, there exists a sequence $\{u_{n,k}\}$ converging to u_n weakly in $W^{1,M}(A \times I; \mathbb{R}^3)$ such that

$$\int_{A\times I} \widehat{\mathcal{Q}}W(D_\alpha u_n \,|\, \lambda_n D_3 u_n)\, dx = \lim_{k\to+\infty} \int_{A\times I} W(D_\alpha u_{n,k} \,|\, \lambda_n D_3 u_{n,k})\, dx. \tag{44}$$

From (43) and (44) we have

$$\widehat{\mathcal{Q}}E_0(u, \bar{b}, A) \ge \lim_{n\to+\infty} \lim_{k\to+\infty} \int_{A\times I} W(D_\alpha u_{n,k} \,|\, \lambda_n D_3 u_{n,k})\, dx - \delta. \tag{45}$$

Since $W^{1,M}(A \times I;\mathbb{R}^3) \hookrightarrow\hookrightarrow L^M_{\mathrm{loc}}(A \times I;\mathbb{R}^3)$ compactly (see Donaldson–Trudinger [13, Theorem 3.9] together with Gossez [20, Proposition 4.3]), $u_{n,k} \to u_n$ in $L^M_{\mathrm{loc}}(A \times I;\mathbb{R}^3)$-norm,

$$\lim_{n\to+\infty}\lim_{k\to+\infty} \|u_{n,k} - u\|_{L^M(D\times I;\mathbb{R}^3)} = 0 \qquad (\forall D \Subset A) \tag{46}$$

and for the weak topology of $L^M(A;\mathbb{R}^3)$,

$$\lim_{n\to+\infty}\lim_{k\to+\infty} \lambda_n \int_I D_3 u_{n,k}\, dx_3 = \bar{b}. \tag{47}$$

By (43) together with the coercivity condition (2) for $\mathcal{Q}W$, we have

$$\sup_{n,k}\Big\|\lambda_n \int_I D_3 u_{n,k}\, dx_3\Big\|_{L^M(A;\mathbb{R}^3)} < +\infty. \tag{48}$$

It is known (see, e.g., [21]) that $W^{1,M}(\Omega;\mathbb{R}^3)$ is separable and reflexive for the case when $M, M^* \in \Delta_2$. By the reflexivity and separability of $W^{1,M}(\Omega;\mathbb{R}^3)$, the Alaoglu–Bourbaki theorem together with [23, Theorem V.7.6] imply that any closed ball equipped with the weak topology is compact and metrizable. By (45), (46), (47), (48) and by using the Moore Lemma [14, Lemma I.7.6] (on double limits of sequence with respect metrizable topologies) we can find a subsequence u_{n,k_n} of $u_{n,k}$ satisfying $u_{n,k_n} \to u$ in $L^M(D_q \times I;\mathbb{R}^3)$-norm for any fixed sequence D_q with $D_q \Subset D_{q+1}$ and $\bigcup_{q\in\mathbb{N}} = A$ (and so in $L^M_{\mathrm{loc}}(A \times I;\mathbb{R}^3)$-norm), $\lambda_n \int_I D_3 u_{n,k_n}\, dx_3 \rightharpoonup \bar{b}$ weakly in $L^M(A;\mathbb{R}^3)$ and realizing the double limit in the right hand side of (45). Consequently we have

$$\widehat{\mathcal{Q}}E_0(u,\bar{b},A) \geq \lim_{n\to+\infty} \int_{A\times I} W\big(D_\alpha u_{n,k_n} \,|\, \lambda_n D_3 u_{n,k_n}\big)\, dx - \delta \geq E_0(u,\bar{b},A) - \delta.$$

Letting $\delta \to 0$, we obtain the conclusion. ■

Notice that by Proposition 5.5 and Lemma 5.8 we may assume without loss of generality that W is quasiconvex. Therefore by the condition (2), $M \in \Delta_2$, together with Focardi [16, Proposition 3.2], W satisfies

$$|W(\xi_1) - W(\xi_2)| \leq C\big(1 + h(1 + |\xi_1| + |\xi_2|)\big)|\xi_1 - \xi_2| \tag{49}$$

for some $C \in (0,+\infty)$ and for all $\xi_1, \xi_2 \in \mathbb{R}^{3\times 3}$, where h denotes the right derivative of M. Define

$$\begin{aligned} W^\lambda(\bar{F}\,|\,b) := \inf\Big\{\int_Q W(\bar{F} + D_\alpha\varphi \,|\, \lambda D_3\varphi)\, dx : \varphi \in W^{1,M}(Q;\mathbb{R}^3),\\ \varphi(\cdot,x_3) \ \text{ is } Q'\text{-periodic } \mathcal{L}^1 \text{ a.e. } x_3 \in I,\ \lambda \int_Q D_3\varphi\, dx = b\Big\} \end{aligned} \tag{50}$$

and

$$W_k(\bar{F}\,|\,b) := \inf_{|\lambda|\leq k} W^\lambda(\bar{F}\,|\,b), \tag{51}$$

where $\lambda \in \mathbb{R}$ and $(\bar{F}, b) \in \mathbb{R}^{3\times 2} \times \mathbb{R}^3$.

PROPOSITION 5.9. *Assume that a quasiconvex function $W : \mathbb{R}^{3\times 3} \to [0,+\infty)$ satisfies the conditions* (2) *and $M \in \Delta_2$. Then the functions W^λ, W_k and $\mathcal{Q}^*W$ satisfy the condition*:

$$\begin{aligned}&|W^\lambda(\bar F\,|\,b) - W^\lambda(\bar F'\,|\,b')|, |W_k(\bar F\,|\,b) - W_k(\bar F'\,|\,b')|, |\mathcal{Q}^*W(\bar F\,|\,b) - \mathcal{Q}^*W(\bar F'\,|\,b')|\\ &\le h_*\big(|\bar F| + |b| + |\bar F'| + |b'|\big)\cdot(|\bar F - \bar F'| + |b - b'|) \qquad (\forall\ (\bar F\,|\,b), (\bar F'\,|\,b') \in \mathbb{R}^{3\times 2}\times\mathbb{R}^3)\end{aligned}$$

for some nondecreasing function $h_* : [0,+\infty) \to [0,+\infty)$.

Proof. Fix $(\bar F, b), (\bar F', b') \in \mathbb{R}^{3\times 2}\times\mathbb{R}^3$. Let φ_n be a infimizing sequence in the definition of $W^\lambda(\bar F\,|\,b)$, and consider the sequence $\psi_n := \varphi_n + \big(\frac{b'-b}{\lambda}\big)x_3$. We may assume that

$$\int_Q W\big(\bar F + D_\alpha\varphi_n\,|\,\lambda D_3\varphi_n\big)\,dx \le W^\lambda(\bar F, b) + 1 \le W(\bar F, b) + 1. \tag{52}$$

Since $\psi_n(\cdot, x_3)$ is Q'-periodic $\mathcal{L}^1$ a.e. $x_3 \in I$ and $\lambda\int_Q D_3\psi_n\,dx = b'$, ψ_n is an admissible function in the definition of $W^\lambda(\bar F'\,|\,b')$.

By the condition (49) and by the Hölder inequality in Orlicz spaces (see [25]) we deduce that

$$\begin{aligned}&\Big|\int_Q W(\bar F' + D_\alpha\psi_n\,|\,\lambda D_3\psi_n)\,dx - \int_Q W(\bar F + D_\alpha\varphi_n\,|\,\lambda D_3\varphi_n)\,dx\Big|\\ &\le C\int_Q \big(1 + h\big(1 + \big|(\bar F' + D_\alpha\psi_n\,|\,\lambda D_3\psi_n)\big| + \big|(\bar F + D_\alpha\varphi_n\,|\,\lambda D_3\varphi_n)\big|\big)\big)\\ &\qquad\cdot\big|(\bar F' + D_\alpha\psi_n\,|\,\lambda D_3\psi_n) - (\bar F + D_\alpha\varphi_n\,|\,\lambda D_3\varphi_n)\big|\,dx\\ &\le 2C\big\|1 + h(1 + |(\bar F' + D_\alpha\varphi_n\,|\,D_3\varphi_n + b' - b)| + |(\bar F + D_\alpha\varphi_n\,|\,\lambda D_3\varphi_n)|\big\|_{L^M(Q)}\\ &\qquad\cdot\big\|(|\bar F' - \bar F| + |b' - b|)\big\|_{L^{M^*}(Q)}.\end{aligned} \tag{53}$$

By the coercivity condition in (2), (52) implies that

$$\begin{aligned}\infty > W(\bar F, b) + 1 &\ge \int_Q W(\bar F + D_\alpha\varphi_n\,|\,\lambda D_3\varphi_n)\,dx\\ &\ge \frac{1}{C_2}\Big(\int_Q M(|\bar F + D_\alpha\varphi_n\,|\,\lambda D_3\varphi_n|)\,dx - 1\Big)\end{aligned}$$

for some $C_2 \in (0,+\infty)$ and so

$$\sup_n \int_Q M\big(|(\bar F + D_\alpha\varphi_n\,|\,\lambda D_3\varphi_n)|\big) < C_2(W(\bar F\,|\,b) + 1) + 1.$$

Hence

$$\sup_n\big\|(\bar F + D_\alpha\varphi_n\,|\,\lambda D_3\varphi_n)\big\|_{L^M(Q;\mathbb{R}^3)} \le C_2(W(\bar F\,|\,b) + 1) + 1 \tag{54}$$

and

$$\begin{aligned}&\big\|1 + \big|(\bar F' + D_\alpha\varphi_n\,|\,D_3\varphi_n + b' - b)\big| + \big|(\bar F + D_\alpha\varphi_n\,|\,\lambda D_3\varphi_n)\big|\big\|_{L^M(Q)}\\ &\quad\le C_3\big(1 + |\bar F'| + |\bar F| + |b| + |b'|\big) + 2\big(C_2(W(\bar F\,|\,b) + 1) + 1\big) =: C_4(b, b', \bar F, \bar F'),\end{aligned} \tag{55}$$

where $C_3 := \|1\|_{L^M(Q)} < +\infty$. By the Płuciennik–Tian–Wang Lemma (see [36, Lemma 1]) for $M \in \Delta_2$, there exists a function $r : [0,+\infty) \to [0,+\infty)$ such that $\|z\|_{L^M(Q)} \le a \Rightarrow \|h(|z|)\|_{L^{M^*}(Q)} \le r(a)$. Define

$$r_M(a) = \sup\big\{\|h(|z|)\|_{L^{M^*}(Q)} : \|z\|_{L^M(Q)} \le a\big\}. \tag{56}$$

Then $0 \le r_M(a) \le r(a) < +\infty$ and r_M is nondecreasing. Therefore (53) and (55) imply that

$$\Big|\int_Q W(\bar F' + D_\alpha\psi_n \,|\, \lambda D_3\psi_n)\,dx - \int_Q W(\bar F + D_\alpha\varphi_n \,|\, \lambda D_3\varphi_n)\,dx\Big| \\ \le 2C\big(C_5 + r_M(C_4(b,b',\bar F,\bar F'))\big)\cdot(|\bar F' - \bar F| + |b'-b|) < +\infty, \quad (57)$$

where $C_5 := \|1\|_{L^{M^*}(Q)} < +\infty$. By the upper bound condition in (2) for $W \ge 0$ and $M \in \Delta_2$,

$$\begin{aligned} W(\bar F, b) &\le W(\bar F, b) + W(\bar F', b') \\ &\le C_6(1 + M(|\bar F|) + M(|b|) + M(|\bar F'|) + M(|b'|)) \end{aligned} \quad (58)$$

for some $C_6 \in (0,+\infty)$ and for all $(\bar F, b), (\bar F', b') \in \mathbb{R}^{3\times 2} \times \mathbb{R}^3$. Hence (57), (58) and the definition C_4 in (55) imply the existence of some nondecreasing function $h_* : [0,+\infty) \to [0,+\infty)$ such that

$$\Big|\int_Q W(\bar F' + D_\alpha\psi_n \,|\, \lambda D_3\psi_n)\,dx - \int_Q W(\bar F + D_\alpha\varphi_n \,|\, \lambda D_3\varphi_n)\,dx\Big| \\ \le h_*\big(|\bar F| + |b| + |\bar F'| + |b'|\big)\cdot(|\bar F - \bar F'| + |b - b'|) =: \widetilde C(b,b',\bar F,\bar F') \quad (59)$$

for all $(\bar F\,|\,b), (\bar F'\,|\,b') \in \mathbb{R}^{3\times 2} \times \mathbb{R}^3$. By the definition of $W^\lambda(\bar F'\,|\,b')$, (59) implies that

$$\begin{aligned} W^\lambda(\bar F'\,|\,b') &\le \int_Q W(\bar F' + D_\alpha\psi_n \,|\, \lambda D_3\psi_n)\,dx \\ &\le \int_Q W(\bar F + D_\alpha\varphi_n \,|\, \lambda D_3\varphi_n)\,dx + \widetilde C(b,b',\bar F,\bar F'), \end{aligned} \quad (60)$$

and letting $n \to +\infty$, we infer that

$$W^\lambda(\bar F'\,|\,b') \le W^\lambda(\bar F\,|\,b) + \widetilde C(b,b',\bar F,\bar F'). \quad (61)$$

Using the same arguments for the pair $(\bar F'\,|\,b')$ in place of $(\bar F\,|\,b)$, we deduce that

$$W^\lambda(\bar F\,|\,b)| \le W^\lambda(\bar F'\,|\,b') + \widetilde C(b,b',\bar F,\bar F'). \quad (62)$$

Taking infimum over $|\lambda| \le k$ in (61), (62), we infer that

$$\begin{aligned} W_k(\bar F'\,|\,b') &\le W_k(\bar F\,|\,b) + \widetilde C(b,b',\bar F,\bar F'), \\ W_k(\bar F\,|\,b) &\le W_k(\bar F'\,|\,b') + \widetilde C(b,b',\bar F,\bar F'). \end{aligned} \quad (63)$$

Since $W_k(\bar F\,|\,b) \uparrow \mathcal{Q}^*W(\bar F\,|\,b)$ as $k \to +\infty$, (63) implies that

$$|\mathcal{Q}^*W(\bar F'\,|\,b') - \mathcal{Q}^*W(\bar F\,|\,b)| \le \widetilde C(b,b',\bar F,\bar F'). \quad \blacksquare$$

LEMMA 5.10. *Let W be a quasiconvex continuous function satisfying* (2) *and $M \in \Delta_2 \cap \nabla_2$. Consider the Γ-lower limit E_0 defined in* (42)*. Then*

$$E_0(u,\bar b,A) \ge \int_A \mathcal{Q}^*W(D_\alpha u\,|\,\bar b)\,dx_\alpha \quad (64)$$

for all $(u,\bar b,A) \in \mathcal{Z} \times L^M(\omega;\mathbb{R}^3) \times \mathcal{A}(\omega)$.

Proof. By Proposition 5.9, $\mathcal{Q}^*W(D_\alpha u\,|\,\bar b) : A \to [0,+\infty)$ is measurable.

Step 1. Let $A = Q'$ and $u(x) := \bar{F}x_\alpha + u_0$ with $\bar{F} \in \mathbb{R}^{3\times 2}$ and $u_0, \bar{b} \in \mathbb{R}^3$. Assume that

$$E_0(u, \bar{b}, Q') < +\infty.$$

By Lemma 5.7 we may restrict ourselves, in (42) to sequences having the same trace as their limit. Consider the sequence

$$\psi_n(x) := \varphi_n + (\bar{F}x_\alpha + u_0),$$

where $\varphi_n \in W^{1,M}(Q;\mathbb{R}^3)$ is such that $\varphi_n = 0$ on $\partial Q' \times I$, $\varphi_n \rightharpoonup 0$ weakly in $W^{1,M}(Q;\mathbb{R}^3)$ (so φ_n is bounded in $W^{1,M}(Q;\mathbb{R}^3)$) and $\lambda_n \int_I D_3\varphi_n\, dx_3 \rightharpoonup \bar{b}$ weakly in $L^M(Q';\mathbb{R}^3)$ and

$$\lim_{n\to+\infty} \int_Q W(D_\alpha\varphi_n \,|\, \lambda_n D_3\varphi_n)\, dx < +\infty. \tag{65}$$

Define

$$\tilde{\varphi}_n := \varphi_n + x_3\Big(\frac{\bar{b}}{\lambda_n} - \int_Q D_3\varphi_n\, dx\Big).$$

By (65) and the coercivity condition in (2), we deduce that, by the same arguments for proving (54),

$$\sup_n \big\|(D_\alpha\varphi_n \,|\, \lambda_n D_3\varphi_n)\big\|_{L^M(Q;\mathbb{R}^3)} < +\infty \tag{66}$$

and so by the Hölder inequality,

$$\sup_n \Big|\int_Q \lambda_n D_3\varphi_n\, dx\Big| \le 2\sup_n \|\lambda_n D_3\varphi_n\|_{L^M(Q;\mathbb{R}^3)} \cdot \|1\|_{L^{M^*}(Q)} < +\infty.$$

Hence, we deduce that $\tilde{\varphi}_n$ is bounded in $W^{1,M}(Q;\mathbb{R}^3)$, $\lambda_n \int_Q D_3\tilde{\varphi}_n\, dx_3 = \bar{b}$, $D_\alpha\varphi_n = D_\alpha\tilde{\varphi}_n$ and

$$\sup_n \big\|(D_\alpha\tilde{\varphi}_n \,|\, \lambda_n D_3\tilde{\varphi}_n)\big\|_{L^M(Q;\mathbb{R}^3)} < +\infty \tag{67}$$

and since $\varphi_n = 0$ on $\partial Q' \times I$, $\tilde{\varphi}_n(\cdot, x_3)$ is Q'-periodic. Thus $\tilde{\varphi}_n$ are admissible functions for the definition of $\mathcal{Q}^*W$ and we have

$$\int_Q W(\bar{F} + D_\alpha\tilde{\varphi}_n \,|\, \lambda_n D_3\tilde{\varphi}_n)\, dx \ge \mathcal{Q}^*W(\bar{F}\,|\,\bar{b}). \tag{68}$$

On the other hand, by the weak continuity of the Lebesgue integral,

$$\begin{aligned}
&\lim_{n\to+\infty} \big\|(\bar{F} + D_\alpha\varphi_n \,|\, \lambda_n D_3\varphi_n) - (\bar{F} + D_\alpha\tilde{\varphi}_n \,|\, \lambda_n D_3\tilde{\varphi}_n)\big\|_{L^\infty(Q;\mathbb{R}^3)} \\
&\quad = \lim_{n\to+\infty} \Big|\int_Q \lambda_n D_3\varphi_n\, dx - \bar{b}\Big| = \lim_{n\to+\infty} \Big|\int_{Q'} \Big(\int_I \lambda_n D_3\varphi_n\, dx - \bar{b}\Big)\, dx_\alpha\Big| = 0.
\end{aligned} \tag{69}$$

Since W satisfies (49), we have

$$\begin{aligned}
&\big|(W(\bar{F} + D_\alpha\varphi_n \,|\, \lambda_n D_3\varphi_n) - W(\bar{F} + D_\alpha\tilde{\varphi}_n \,|\, \lambda_n D_3\tilde{\varphi}_n))\big| \\
&\qquad \le C\big(1 + h\big(1 + |(\bar{F} + D_\alpha\varphi_n \,|\, \lambda_n D_3\varphi_n)| + |(\bar{F} + D_\alpha\tilde{\varphi}_n \,|\, \lambda_n D_3\tilde{\varphi}_n)|\big)\big) \\
&\qquad\qquad \cdot \big|(\bar{F} + D_\alpha\varphi_n \,|\, \lambda_n D_3\varphi_n) - (\bar{F} + D_\alpha\tilde{\varphi}_n \,|\, \lambda_n D_3\tilde{\varphi}_n)\big|.
\end{aligned} \tag{70}$$

By (66), (67) we deduce, by the Płuciennik–Tian–Wang Lemma (see [36, Lemma 1]) the existence of $C \in (0, +\infty)$ such that

$$\sup_n \big\|h\big(1 + |(\bar{F} + D_\alpha\varphi_n \,|\, \lambda_n D_3\varphi_n)| + |(\bar{F} + D_\alpha\tilde{\varphi}_n \,|\, \lambda_n D_3\tilde{\varphi}_n)|\big)\big\|_{L^{M^*}(Q)} \le C. \tag{71}$$

By the boundedness of the embedding $L^{M^*}(Q) \hookrightarrow L^1(Q)$ (see, e.g., [25]), (71) implies that

$$\sup_n \big\| h\big(1 + |(\bar{F} + D_\alpha \varphi_n \,|\, \lambda_n D_3 \varphi_n)| + |(\bar{F} + D_\alpha \tilde{\varphi}_n \,|\, \lambda_n D_3 \tilde{\varphi}_n)|\big) \big\|_{L^1(Q)} < +\infty. \tag{72}$$

By (69), (70) and (72), we deduce that

$$\begin{aligned} 0 &\le \limsup_{n\to+\infty} \big| \big(W(\bar{F} + D_\alpha \varphi_n \,|\, \lambda_n D_3 \varphi_n) - W(\bar{F} + D_\alpha \tilde{\varphi}_n \,|\, \lambda_n D_3 \tilde{\varphi}_n) \big) \big| \\ &\le 2C \limsup_{n\to+\infty} \big\| (\bar{F} + D_\alpha \varphi_n \,|\, \lambda_n D_3 \varphi_n) - (\bar{F} + D_\alpha \tilde{\varphi}_n \,|\, \lambda_n D_3 \tilde{\varphi}_n) \big\|_{L^\infty(Q;\mathbb{R}^3)} \\ &\quad \cdot \sup_n \big\| h\big(1 + |(\bar{F} + D_\alpha \varphi_n \,|\, \lambda_n D_3 \varphi_n)| + |(\bar{F} + D_\alpha \tilde{\varphi}_n \,|\, \lambda_n D_3 \tilde{\varphi}_n)|\big) \big\|_{L^1(Q;\mathbb{R}^3)} = 0. \end{aligned} \tag{73}$$

From (68) and (73), we infer that

$$\liminf_{n\to+\infty} \int_Q W(\bar{F} + D_\alpha \varphi_n \,|\, \lambda_n D_3 \varphi_n)\, dx \ge \mathcal{Q}^* W(\bar{F} \,|\, \bar{b}). \tag{74}$$

We complete the proof of (64) for the case in Step 1 by taking the infimum over all admissible sequences in (74), and then we get the inequality

$$E_0(\bar{F} x_\alpha + u_0, \bar{b}, Q') \ge \int_{Q'} \mathcal{Q}^* W(\bar{F} \,|\, \bar{b})\, dx_\alpha = \mathcal{L}^2(Q') \cdot \mathcal{Q}^* W(\bar{F} \,|\, \bar{b}) = \mathcal{Q}^* W(\bar{F} \,|\, \bar{b}). \tag{75}$$

Step 2. Fix $(u, \bar{b}, A) \in \mathcal{Z} \times L^M(\omega;\mathbb{R}^3) \times \mathcal{A}(\omega)$. Let $u_n \in W^{1,M}(A \times I;\mathbb{R}^3)$ be such that $u_n \rightharpoonup u$ weakly in $W^{1,M}(A \times I;\mathbb{R}^3)$, $\lambda_n \int_I D_3 u_n \, dx_3 \rightharpoonup \bar{b}$ weakly in $L^M(A;\mathbb{R}^3)$ and

$$+\infty > E_0(u, \bar{b}, A) = \lim_{n\to+\infty} \int_{A\times I} W(D_\alpha u_n \,|\, \lambda_n D_3 u_n)\, dx. \tag{76}$$

Define the sequence of measures

$$\mu_n := \Big(\int_I W(D_\alpha u_n \,|\, \lambda_n D_3 u_n)\, dx_3 \Big) \mathcal{L}^2 \lfloor A.$$

By (76) and [4, Theorem 1.59]) we can find a subsequence (not relabeled) $\{\mu_n\}$ weakly* converging to some nonnegative measure μ. Denote by ρ the density of the absolutely continuous part of μ with respect to the 2-dimensional Lebesgue measure. In order to prove (64) it suffices to show that, for a.e. $x_0 \in A$,

$$\rho(x_0) \ge \mathcal{Q}^* W(D_\alpha u(x_0) \,|\, \bar{b}(x_0)). \tag{77}$$

By the Besicovitch derivation theorem [4, Theorem 2.22], for a.e. $x_0 \in A$,

$$\rho(x_0) = \lim_{\varepsilon\to 0} \frac{\mu(x_0 + \varepsilon Q')}{\varepsilon^2}. \tag{78}$$

By [3, Lemma 3.1, Lemma 3.2] we deduce that for a.e. $x_0 \in A$,

$$\begin{aligned} 0 &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2} \int_{x_0+\varepsilon Q'} M\Big(\Big| \frac{u(x) - u(x_0) - \langle \nabla u(x_0), x - x_0 \rangle}{\varepsilon} \Big| \Big)\, dx_\alpha \\ &= \lim_{\varepsilon\to 0} \int_{Q'} M\Big(\Big| \frac{u(x_0 + \varepsilon y_\alpha) - u(x_0) - \varepsilon \langle \nabla u(x_0), y_\alpha \rangle}{\varepsilon} \Big| \Big)\, dy_\alpha \end{aligned} \tag{79}$$

and

$$0 = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2} \int_{x_0+\varepsilon Q'} M(|\bar{b}(x) - \bar{b}(x_0)|)\, dx_\alpha = \lim_{\varepsilon\to 0} \int_{Q'} M(|\bar{b}(x_0 + \varepsilon y) - \bar{b}(x_0)|)\, dy_\alpha. \tag{80}$$

Fix x_0 satisfying (78), (79), (80) and let $\varepsilon \to 0$ be a sequence such that

$$\mu(\partial(x_0 + \varepsilon Q')) = 0 \tag{81}$$

for all $\varepsilon > 0$ (this sequence exists due to [4, Proposition 1.62, Example 1.63]). By $M \in \Delta_2$ and [25], (79), (80) imply that

$$\lim_{\varepsilon\to 0}\int_{Q'} M\Big(d\Big|\frac{u(x_0+\varepsilon y_\alpha) - u(x_0) - \varepsilon\langle\nabla u(x_0), y_\alpha\rangle}{\varepsilon}\Big|\Big)\,dy_\alpha = 0 \quad (\forall d \in (0,+\infty)) \tag{82}$$

and

$$\big\|\bar b(x_0 + \varepsilon(\cdot)) - \bar b(x_0)\big\|_{L^M(Q';\mathbb{R}^3)} \to 0 \qquad (\varepsilon \to 0). \tag{83}$$

Using (78), (81) and the definition of μ, we infer that

$$\begin{aligned}
\rho(x_0) &= \lim_{\varepsilon\to 0}\lim_{n\to+\infty}\frac{1}{\varepsilon^2}\int_{(x_0+\varepsilon Q')\times I} W(D_\alpha u_n\,|\,\lambda_n D_3 u_n)\,dx \\
&= \lim_{\varepsilon\to 0}\lim_{n\to+\infty}\int_Q W\big(D_\alpha u_n(x_0+\varepsilon y_\alpha, y_3)\,|\,\lambda_n D_3 u_n(x_0+\varepsilon y_\alpha, y_3)\big)\,dy \\
&= \lim_{\varepsilon\to 0}\lim_{n\to+\infty}\int_Q W(D_\alpha u_{n,\varepsilon}\,|\,\lambda_n\varepsilon D_3 u_{n,\varepsilon})\,dy,
\end{aligned} \tag{84}$$

where

$$u_{n,\varepsilon}(y) := \frac{u_n(x_0+\varepsilon y_\alpha, y_3) - u(x_0)}{\varepsilon}\,.$$

Since $W^{1,M}(\Omega;\mathbb{R}^3) \hookrightarrow\hookrightarrow L^M(\Omega;\mathbb{R}^3)$ compactly (see Donaldson–Trudinger [13, Theorem 3.9] together with Gossez [20, Proposition 4.13]), $u_n \to u$ in $L^M_{\mathrm{loc}}(A\times I;\mathbb{R}^3)$, and so $u_n \to u$ in $L^M((x_0+\varepsilon Q')\times I;\mathbb{R}^3)$ for $x_0+\varepsilon Q' \Subset A$. By the convexity of M and $M \in \Delta_2$, we have

$$\begin{aligned}
&\int_Q M(|u_{n,\varepsilon}(y) - \langle\nabla u(x_0), y_\alpha\rangle|)\,dx \\
&\quad= \int_Q M\Big(\frac{|u_n(x_0+\varepsilon y_\alpha, y_3) - u(x_0) - \varepsilon\langle\nabla u(x_0), y_\alpha\rangle|}{\varepsilon}\Big)\,dx \\
&\quad= \frac{1}{\varepsilon^2}\int_{(x_0+\varepsilon Q')\times I} M\Big(\frac{|u_n(x) - u(x_0) - \langle\nabla u(x_0), x_\alpha - x_0\rangle|}{\varepsilon}\Big)\,dx \\
&\quad\le \frac{1}{\varepsilon^2}\cdot\frac{1}{2}\int_{(x_0+\varepsilon Q')\times I} M\Big(2\frac{|u_n(x) - u(x)|}{\varepsilon}\Big)\,dx \\
&\qquad+ \frac{1}{\varepsilon^2}\cdot\frac{1}{2}\int_{(x_0+\varepsilon Q')\times I} M\Big(2\frac{|u(x) - u(x_0) - \langle\nabla u(x_0), x_\alpha - x_0\rangle|}{\varepsilon}\Big)\,dx.
\end{aligned} \tag{85}$$

By (82) and (85)

$$\lim_{\varepsilon\to 0}\lim_{n\to+\infty}\int_Q M(|u_{n,\varepsilon}(y) - \langle\nabla u(x_0), y_\alpha\rangle|)\,dx = 0.$$

By $M \in \Delta_2$ together with [25], we infer that

$$\lim_{\varepsilon\to 0}\lim_{n\to+\infty}\big\|u_{n,\varepsilon}(y) - \langle\nabla u(x_0), y_\alpha\rangle\big\|_{L^M(Q;\mathbb{R}^3)} = 0. \tag{86}$$

By (83) and the Hölder inequality for any $\varphi \in L^{M^*}(Q';\mathbb{R}^3)$,

$$\lim_{\varepsilon\to 0}\Big|\int_{Q'} (\bar{b}(x_0+\varepsilon y_\alpha)-\bar{b}(x_0))\varphi(y_\alpha)\,dy_\alpha\Big| \le \lim_{\varepsilon\to 0} 2\big\|\bar{b}(x_0+\varepsilon(\cdot))-\bar{b}(x_0)\big\|_{L^M(Q';\mathbb{R}^3)}\|\varphi\|_{L^{M^*}(Q';\mathbb{R}^3)} = 0.$$

Hence by $\lambda_n \int_I D_3 u_n\,dx_3 \rightharpoonup \bar{b}$ weakly in $L^M(A;\mathbb{R}^3)$, we infer that, for $\varphi \in L^{M^*}(Q';\mathbb{R}^3)$,

$$\begin{aligned}\lim_{\varepsilon\to 0}\lim_{n\to+\infty}&\int_Q \lambda_n\varepsilon D_3 u_{n,\varepsilon}(y)\varphi(y)\,dy\\ &=\lim_{\varepsilon\to 0}\lim_{n\to+\infty}\frac{1}{\varepsilon^2}\int_{(x_0+\varepsilon Q')\times I}\lambda_n D_3u_n(x)\varphi\Big(\frac{x_\alpha-x_0}{\varepsilon}\Big)\,dx\\ &=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_{(x_0+\varepsilon Q')\times I}\bar{b}(x_\alpha)\varphi\Big(\frac{x_\alpha-x_0}{\varepsilon}\Big)\,dx_\alpha\\ &=\lim_{\varepsilon\to 0}\int_{Q'}\bar{b}(x_0+\varepsilon y_\alpha)\varphi(y_\alpha)\,dy_\alpha=\int_{Q'}\bar{b}(x_0)\varphi(y_\alpha)\,dy_\alpha.\end{aligned}\tag{87}$$

By the Moore Lemma (see [14, Lemma I.7.6]) from (84), (86) and (87), we construct $\tilde{u}_k := u_{\varepsilon_k,n_k}$ and λ_{n_k} such that

$$\tilde{u}_k(x)\to D_\alpha u(x_0)(x)\quad\text{in } L^M(Q;\mathbb{R}^3),$$

where $D_\alpha u(x_0)(x) := D_\alpha u(x_0)x_\alpha$ and

$$\lambda_{n_k}\int_I \varepsilon_k D_3\tilde{u}_k\,dy_3 \rightharpoonup \bar{b}(x_0)\quad\text{weakly in}\quad L^M(A;\mathbb{R}^3)$$

and

$$\rho(x_0)=\lim_{k\to+\infty}\int_Q W(D_\alpha\tilde{u}_k\,|\,\lambda_{n_k}\varepsilon_k D_3\tilde{u}_k)\,dy.$$

By the definition of E_0,

$$\lim_{k\to+\infty}\int_Q W(D_\alpha\tilde{u}_k\,|\,\lambda_{n_k}\varepsilon_k D_3\tilde{u}_k)\,dy \ge E_0\big(D_\alpha u(x_0)(\cdot),\bar{b}(x_0),Q'\big),\tag{88}$$

and so the claim (77) follows from inequality (88) and the inequality (75) proved in Step 1. ∎

Lemma 5.11. *Under the hypothesis of Lemma* 5.10, *we have*

$$E_0(u,\bar{b},A)\le\int_A \mathcal{Q}^*W(D_\alpha u|\bar{b})\,dx_\alpha\tag{89}$$

for all $(u,\bar{b},A)\in\mathcal{Z}\times L^M(\omega;\mathbb{R}^3)\times\mathcal{A}(\omega)$.

Proof. By Proposition 5.9, $\mathcal{Q}^*W(D_\alpha u|\bar{b}) : A\to[0,+\infty)$ and $W_k(D_\alpha u|\bar{b}) : A\to[0,+\infty)$ are measurable.

We claim that for each fixed $k\in\mathbb{N}$ and for all $(u,\bar{b},A)\in\mathcal{Z}\times L^M(\omega,\mathbb{R}^3)\times\mathcal{A}(\omega)$,

$$E_0(u,\bar{b},A)\le\int_A W_k(D_\alpha u|\bar{b})\,dx_\alpha.\tag{90}$$

This claim will be proven in two steps.

Step 1. Let u be an affine function, i.e. $u = \bar{F}x_\alpha$ and $b \in \mathbb{R}$. Let φ be admissible for the definition of $W^\lambda(\bar{F}|b)$ in (50). Extending Q'-periodically the Q'-periodic function φ, we define $\varphi_n : \mathbb{R}^2 \times I \to \mathbb{R}^3$ by

$$\varphi_n(x) := \frac{\lambda}{\lambda_n}\varphi\Big(\frac{\lambda_n}{\lambda}x_\alpha, x_3\Big).$$

Then, $\varphi_n \in W^{1,M}(A \times I;\mathbb{R}^3)$ and $\varphi_n \to 0$ in $L^M(A \times I;\mathbb{R}^3)$-norm.

The function $y_\alpha \mapsto \lambda \int_I D_3\varphi\, dx_3$ is Q'-periodic and belongs to $L^M(Q';\mathbb{R}^3)$, since by the Jensen inequality and $M \in \Delta_2$

$$\begin{aligned}\int_{Q'} M\Big(\Big|\int_I D_3\varphi(y_\alpha, x_3)\, dx_3\Big|\Big)\, dy_\alpha &\le \int_{Q'} M\Big(\int_I |D_3\varphi(y_\alpha, x_3)|\, dx_3\Big)\, dy_\alpha \\ &\le \int_{Q'}\int_I M\big(|D_3\varphi(y_\alpha, x_3)|\big)\, dx_3\, dy_\alpha < \infty.\end{aligned}$$

By the $L^M(Q')$-generalization (see [35, Homogenization Theorem 7.1, Remark p. 121]) for the Riemann–Lebesgue Lemma in $L^p(Q')$-spaces (see, e.g., [11]) we infer that

$$\begin{aligned}\lambda_n \int_I D_3\varphi_n\, dx &= \lambda \int_I D_3\varphi\Big(\frac{\lambda_n}{\lambda}x_\alpha, x_3\Big)\, dx_3 \\ &\rightharpoonup \lambda \int_I\int_{Q'} D_3\varphi(y_\alpha, x_3)\, dy_\alpha\, dx_3 = \bar{b} \;\text{ weakly in }\; L^M(Q',\mathbb{R}^3).\end{aligned}$$

Define

$$\begin{aligned}H(x_\alpha, x_3) &:= \big(\bar{F} + D_\alpha\varphi(x_\alpha, x_3)\,|\,\lambda D_3\varphi(x_\alpha, x_3)\big),\\ \widetilde{W}(x_\alpha) &:= \int_I W(H(x_\alpha, x_3))\, dx_3.\end{aligned}$$

Since $H \in L^M(Q;\mathbb{R}^3)$ and $M \in \Delta_2$, by the condition (2)

$$\int_{Q'} |\widetilde{W}(x_\alpha)|\, dx_\alpha \le \int_Q C(1 + M(H(x_\alpha, x_3)))\, dx_\alpha\, dx_3 < \infty$$

and so $\widetilde{W} \in L^1(Q';\mathbb{R}^3)$. Using the $L^1(Q')$ Riemann–Lebesgue Lemma (see, e.g., [11]), we deduce that

$$\begin{aligned}E_0(u, \bar{b}, A) &\le \lim_{n\to+\infty}\int_{A\times I} W(\bar{F} + D_\alpha\varphi_n\,|\,\lambda D_3\varphi_n)\, dx = \lim_{n\to+\infty}\int_{Q'} 1_A(x_\alpha)\cdot\widetilde{W}\Big(\frac{\lambda_n}{\lambda}x_\alpha\Big)\, dx_\alpha \\ &= \int_{Q'} 1_A(x_\alpha)\Big(\int_{Q'}\widetilde{W}(y_\alpha)\, dy_\alpha\Big)\, dx_\alpha = \mathcal{L}^2(A)\int_Q W(\bar{F} + D_\alpha\varphi\,|\,\lambda D_3\varphi)\, dx. \quad (91)\end{aligned}$$

Taking the infimum over all admissible φ and $|\lambda| \le k$, we obtain

$$E_0(u, \bar{b}, A) \le \mathcal{L}^2(A)W_k(\bar{F}\,|\,b) = \int_A W_k(\bar{F}\,|\,b)\, dx_\alpha.$$

Step 2. Let u be a piecewise affine function and $\bar{b}$ be a piecewise constant. Let $\{A_i\}_{i=1,\dots,l} \subset \mathcal{A}(\omega)$ be a finite and measurable partition of A such that u and $\bar{b}$ are affine and constant, respectively on each A_i, $i = 1, \dots, l$. By Step 1 for all $i = 1, \dots, l$

$$E_0(u, \bar{b}, A_i) \le \int_{A_i} W_k(D_\alpha u\,|\,\bar{b})\, dx_\alpha.$$

By Lemma 5.6, $E_0(u,\bar b,\cdot)$ is a measure and so

$$E_0(u,\bar b,A)=\sum_{i=1}^{l}E_0(u,\bar b,A_i)\le\sum_{i=1}^{l}\int_{A_i}W_k(D_\alpha u\,|\,\bar b)dx_\alpha=\int_A W_k(D_\alpha u\,|\,\bar b)\,dx_\alpha.$$

Step 3. Let $(u,\bar b,A)\in W^{1,M}(\omega;\mathbb{R}^3)\times L^M(\omega;\mathbb{R}^3)\times\mathcal{A}(\omega)$ and let $\{(u_n,\bar b_n)\}$ be a sequence such that u_n are piecewise affine functions, $\bar b_n$ are piecewise constants and $u_n\to u$ in $W^{1,M}(A;\mathbb{R}^3)$-norm and a.e., $\bar b_n\to\bar b$ in $L^M(A;\mathbb{R}^3)$-norm and a.e. Since $E_0(\cdot,\cdot,A)$ is a l.s.c. function, we have

$$E_0(u,\bar b,A)\le\liminf_{n\to+\infty}E_0(u_n,\bar b_n,A)\le\liminf_{n\to+\infty}\int_A W_k(D_\alpha u_n\,|\,\bar b_n)\,dx_\alpha. \tag{92}$$

Since W_k is continuous (see Proposition 5.9) and satisfies $0\le W_k(F)\le W(F)\le C(1+M(|F|))$, the superposition operator $N_{W_k}:L^M(\Omega;\mathbb{R}^3)\to L^1(\Omega;\mathbb{R}^3)$ is continuous (see, e.g., [5, Theorem 3], [34, Theorem 3.2]), and so

$$\lim_{n\to+\infty}\int_A W_k(D_\alpha u_n\,|\,\bar b_n)\,dx_\alpha=\int_A W_k(D_\alpha u\,|\,\bar b)\,dx_\alpha.$$

Therefore, (92) implies that

$$E_0(u,\bar b,A)\le\int_A W_k(D_\alpha u\,|\,\bar b)\,dx_\alpha\qquad(\forall k\in\mathbb{N}).$$

By Lemma 5.6, $E_0(u,\bar b,\cdot)$ is a measure which is absolutely continuous with respect to the 2-dimensional Lebesgue measure, and so by the Radon–Nikodym theorem, we can write $E_0(u,\bar b,\cdot)=\rho\mathcal{L}^2\lfloor\omega$ for some $\rho\in L^1(\omega)$. Let $x_0\in\omega$ be a Lebesgue point for ρ, $D_\alpha u$ and $\bar b$. Then from the definition of $\mathcal{Q}^*W$ and W_k,

$$\mathcal{Q}^*W(D_\alpha u(x_0)\,|\,\bar b(x_0))=\lim_{k\to+\infty}W_k(D_\alpha u(x_0)\,|\,\bar b(x_0)).$$

By (90) and by the Radon–Nikodym theorem for $k\in\mathbb{N}$, we infer that

$$\rho(x_\alpha)\le W_k(D_\alpha u(x_\alpha)\,|\,\bar b(x_\alpha))\quad\text{for }\mathcal{L}^2\text{ a.e. }x_\alpha\in\omega\text{ and for all }k\in\mathbb{N}. \tag{93}$$

Therefore,

$$\rho(x_0)\le\mathcal{Q}^*W(D_\alpha u(x_0)\,|\,\bar b(x_0))\quad\text{for }\mathcal{L}^2\text{ a.e. }x_0\in\omega$$

and so

$$E_0(u,\bar b,A)=\int_A\rho(x_\alpha)\,dx_\alpha\le\int_A\mathcal{Q}^*W(D_\alpha u(x_\alpha)\,|\,\bar b(x_\alpha))\,dx_\alpha.\quad\blacksquare$$

Proof of Theorem 5.1. Let $u_\varepsilon\in\Psi_\varepsilon$ be such that $u_\varepsilon\rightharpoonup\bar u$ weakly in $W^{1,M}(\Omega;\mathbb{R}^3)$, $\frac{1}{\varepsilon}\int_I D_3u_\varepsilon\,dx_3\rightharpoonup\bar b$ weakly in $L^M(\omega;\mathbb{R}^3)$. It is easy to check by the representation (1), the isomorphism (16) and by the Fubini theorem that $P_\varepsilon(u_\varepsilon)\to P_0(\bar u,\bar b)$ and $P_\varepsilon(v_\varepsilon+u_{0,\varepsilon})\to P_0(\bar v+u_{0,0},\bar b+e_3)$ as $\varepsilon\to 0$, with $u_\varepsilon=v_\varepsilon+u_{0,\varepsilon}$ and $\bar u=\bar v+u_{0,0}$, where $v_\varepsilon\in V$. By the Kuratowski Compactness Theorem (see [12]) in order to show that $\bar J_\varepsilon$ Γ-converges to J_0 it is enough to prove that the Γ-lower limit E_0 of any subsequence of E_ε coincides with J_0. Therefore the assertions of Theorem 5.1 follow from Lemmas 5.10 and 5.11 applied to the sequence $u_\varepsilon=v_\varepsilon+u_{0,\varepsilon}$. $\blacksquare$

Proof of Corollary 5.2. Observe that $v_\varepsilon:=u_\varepsilon-u_{0,\varepsilon}$ belongs to V. By (19),

$$J_\varepsilon(u_\varepsilon)=J_\varepsilon(v_\varepsilon+u_{0,\varepsilon})\le\inf_{v\in V}J_\varepsilon(v+u_{0,\varepsilon})+\gamma(\varepsilon). \tag{94}$$

Therefore

$$\bar{J}^*_\varepsilon(v_\varepsilon, \bar{b}_\varepsilon) \leq \inf_{v \in V} \bar{J}^*_\varepsilon(v, \bar{b}) + \gamma(\varepsilon), \tag{95}$$

where $\frac{1}{\varepsilon}\int_I D_3 v_\varepsilon \, dx_3 + e_3 = \bar{b}_\varepsilon(x_\alpha)$ and $\frac{1}{\varepsilon}\int_I D_3 v \, dx_3 + e_3 = \bar{b}(x_\alpha)$. It is easy to check that

$$\begin{aligned} J_\varepsilon(u_{0,\varepsilon}) = \int_\Omega W(e_\alpha \,|\, e_3) \, dx - \int_\Omega (f, u_{0,\varepsilon}) \, dx - \int_{S^+} (g_0^+, u_{0,\varepsilon}) \, d\mathcal{H}^2 \\ + \int_{S^-} (g_0^-, u_{0,\varepsilon}) \, d\mathcal{H}^2 - \int_\omega \Big(g, \frac{1}{\varepsilon} \int_I \varepsilon \cdot e_3\Big) \, dx_\alpha \leq C < +\infty \end{aligned}$$

for some C and for all $\varepsilon \in (0,1)$. Hence (95) implies that $\sup_{\varepsilon \in (0,1)} \bar{J}^*_\varepsilon(v_\varepsilon, \bar{b}_\varepsilon) < +\infty$. Therefore by Lemma 5.4 the sequence $(v_\varepsilon, \bar{b}_\varepsilon)$ is bounded, weakly compact in $W^{1,M}(\Omega; \mathbb{R}^3) \times L^M(\omega; \mathbb{R}^3)$ and any cluster point $(v_*, \bar{b}_*)$ belongs to $\mathcal{V} \times L^M(\omega; \mathbb{R}^3)$.

Fix $\tilde{v} \in W^{1,M}(\Omega; \mathbb{R}^3)$, $\tilde{b} \in L^M(\omega; \mathbb{R}^3)$ and $J^*_0(\tilde{v} + u_{0,0}, \tilde{b}) < +\infty$. By Theorem 5.1 there exists a sequence $\tilde{v}_\varepsilon = \tilde{u}_\varepsilon - u_{0,\varepsilon} \in W^{1,M}(\Omega; \mathbb{R}^3)$ such that $\tilde{v}_\varepsilon \rightharpoonup \tilde{v} = \tilde{u} - u_{0,0}$ weakly in $W^{1,M}(\Omega; \mathbb{R}^3)$ and $\tilde{b}_\varepsilon = \frac{1}{\varepsilon}\int_I D_3 \tilde{v}_\varepsilon \, dx_3 + e_3 \rightharpoonup \tilde{b}$ weakly in $L^M(\omega; \mathbb{R}^3)$ and $\bar{J}^*_\varepsilon(\tilde{v}_\varepsilon, \tilde{b}_\varepsilon) \to J^*_0(\tilde{v} + u_{0,0}, \tilde{b})$. Therefore, applying Theorem 5.1, (18) and the assumption $\gamma(\varepsilon) \to 0$ as $\varepsilon \to 0$, we infer that

$$\begin{aligned} J_0(u_*, \bar{b}_*) = J^*_0(v_* + u_{0,0}, \bar{b}_*) \leq \liminf_{\varepsilon \to 0} \bar{J}^*_\varepsilon(v_\varepsilon, \bar{b}_\varepsilon) \\ \leq \liminf_{\varepsilon \to 0} (\bar{J}^*_\varepsilon(\tilde{v}_\varepsilon, \tilde{b}_\varepsilon) + \gamma(\varepsilon)) = J^*_0(\tilde{v} + u_{0,0}, \tilde{b}) = J_0(\tilde{u}, \tilde{b}), \end{aligned}$$

where $u_* = v_* + u_{0,0}$. Using the isomorphism (16) and the representation (1), we re-write the statements obtained above for v_ε and v_*. By this way, we deduce all statements of Corollary 5.2. ∎

Let us inform that we have recently obtained results in the setting of the Orlicz–Sobolev space $W^{1,M}$ that extend other known results for thin films in the case $M(t) = |t|^p$ for some $p \in (1, \infty)$. In particular, our results extend the results obtained in 2009 by G. Bouchitté, I. Fonseca and M. L. Mascarenhas [8] for thin films with bending moment depending also on the third thickness variable. Their proofs require other techniques and we will discuss these issues in our forthcoming papers.

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