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ON THE DEGREE OF STRONG APPROXIMATION OF INTEGRABLE FUNCTIONS IN STEPANOV SENSE

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Abstract. Considering the class of almost periodic functions integrable in the Stepanov sense we extend and generalize the results of the first author [8], as well as the results of L. Leindler [6] and P. Chandra [4, 5].

1. Introduction. Let S^p $(1 < p \le \infty)$ be the class of all almost periodic functions integrable in the Stepanov sense with the norm

$$\|f\|_{S^p} := \begin{cases} \sup\limits_u \Big\{ \dfrac{1}{\pi} \displaystyle\int_u^{u+\pi} |f(t)|^p \, dt \Big\}^{1/p} & \text{when } \ 1 < p < \infty, \\ \sup\limits_u |f(u)| & \text{when } \ p = \infty. \end{cases}$$

Suppose that the Fourier series of $f \in S^p$ has the form

$$Sf(x) = \sum_{\nu=-\infty}^{\infty} A_\nu(f) e^{i\lambda_\nu x}, \qquad \text{where } \ A_\nu(f) = \lim_{L\to\infty} \frac{1}{L} \int_0^L f(t) e^{-i\lambda_\nu t} \, dt,$$

with the partial sums

$$S_{\gamma_k} f(x) = \sum_{|\lambda_\nu| \le \gamma_k} A_\nu(f) e^{i\lambda_\nu x}$$

and that $0 = \lambda_0 < \lambda_\nu < \lambda_{\nu+1}$ if $\nu \in \mathbb{N} = \{1, 2, 3, \dots\}$, $\lim_{\nu\to\infty} \lambda_\nu = \infty$, $\lambda_{-\nu} = -\lambda_\nu$, $|A_\nu| + |A_{-\nu}| > 0$. Let $\Omega_{\alpha,p}$, with some fixed positive α, be the set of functions of class S^p bounded on $U = (-\infty, \infty)$ whose Fourier exponents satisfy the condition

$$\lambda_{\nu+1} - \lambda_\nu \ge \alpha \quad (\nu \in \mathbb{N}).$$

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In case $f \in \Omega_{\alpha,p}$

$$S_{\lambda_k} f(x) = \int_0^{\infty} \big\{ f(x+t) + f(x-t) \big\} \Psi_{\lambda_k, \lambda_k+\alpha}(t)\, dt,$$

where

$$\Psi_{\lambda,\eta}(t) = \frac{2 \sin \frac{(\eta-\lambda)t}{2} \sin \frac{(\eta+\lambda)t}{2}}{\pi(\eta-\lambda)t^2} \quad (0 < \lambda < \eta,\ |t| > 0).$$

Let $A := (a_{n,k})$ $(k, n = 0, 1, \dots)$ be a lower triangular infinite matrix of real numbers satisfying the condition

$$a_{n,k} \geq 0 \ (k, n = 0, 1, \dots), \quad a_{n,k} = 0 \ (k > n) \quad \text{and} \ \sum_{k=0}^{n} a_{n,k} = 1. \tag{1.1}$$

Let us consider the strong mean

$$H^q_{n,A,\gamma} f(x) = \Big\{ \sum_{k=0}^{n} a_{n,k} |S_{\gamma_k} f(x) - f(x)|^q \Big\}^{1/q} \quad (q > 0). \tag{1.2}$$

As measures of approximation by the quantity (1.2), we use the best approximation of f by entire functions g_σ of exponential type σ bounded on the real axis, shortly $g_\sigma \in B_\sigma$, and the moduli of continuity of f defined by the formulas

$$E_\sigma(f)_{S^p} = \inf_{g_\sigma} \|f - g_\sigma\|_{S^p},$$

$$\omega f(\delta)_{S^p} = \sup_{|t| \leq \delta} \|f(\cdot + t) - f(\cdot)\|_{S^p}$$

and

$$w_x f(\delta)_p := \Big\{ \frac{1}{\delta} \int_0^{\delta} |\varphi_x(t)|^p \, dt \Big\}^{1/p} \quad \text{with } 1 < p < \infty,$$

where $\varphi_x(t) := f(x+t) + f(x-t) - 2f(x)$.

A sequence $c := (c_n)$ of nonnegative numbers tending to zero is called the *Rest Bounded Variation Sequence*, or briefly $c \in RBVS$, if it has the property

$$\sum_{n=m}^{\infty} |c_n - c_{n+1}| \leq K(c) c_m \tag{1.3}$$

for all natural numbers m, where $K(c)$ is a constant depending only on c.

A sequence $c := (c_n)$ of nonnegative numbers will be called the *Head Bounded Variation Sequence*, or briefly $c \in HBVS$, if it has the property

$$\sum_{n=0}^{m-1} |c_n - c_{n+1}| \leq K(c) c_m \tag{1.4}$$

for all natural numbers m, or only for all $m \leq N$ if the sequence c has only a finite number of nonzero terms and the last nonzero term is c_N.

Therefore we assume that the sequence $(K(\alpha_n))_{n=0}^{\infty}$ is bounded, that is, that there exists a constant K such that

$$0 \leq K(\alpha_n) \leq K$$

for all n, where $K(\alpha_n)$ denotes the sequence of constants appearing in the inequalities (1.3) or (1.4) for the sequence $\alpha_n := (a_{n,k})_{k=0}^{\infty}$. Now we can give the conditions to be used later on. We assume that for all n and $0 \leq m \leq n$

$$\sum_{k=m}^{\infty} |a_{n,k} - a_{n,k+1}| \leq K a_{n,m} \tag{1.5}$$

and

$$\sum_{k=0}^{m-1} |a_{n,k} - a_{n,k+1}| \leq K a_{n,m} \tag{1.6}$$

if $\alpha_n := (a_{n,k})_{k=0}^{\infty}$ belongs to $RBVS$ or $HBVS$, respectively.

The C-norm of the deviation $|\sum_{k=0}^{n} a_{n,k}[S_k f(x) - f(x)]|$, with the partial sums $S_k f$ of classical trigonometric Fourier series, was estimated by P. Chandra [4, 5] for monotonic sequences $(a_{n,k})$ and by L. Leindler [6] for the sequences of bounded variation. These results were generalized by W. Łenski [8] who considered the strong means $H^q_{n,A}$, also in the classical case, and the functions belonging to L^p. In the present paper we shall consider the almost periodic functions integrable in the Stepanov sense giving similarly estimations for the strong means $H^q_{n,A}$ in individual points and in norms.

We shall write $I_1 \ll I_2$ if there exists a positive constant C such that $I_1 \leq C I_2$.

2. Main results. Let us consider a function w_x of modulus of continuity type on the interval $[0, +\infty)$, i.e. a nondecreasing continuous function having the following properties: $w_x(0) = 0$, $w_x(\delta_1 + \delta_2) \leq w_x(\delta_1) + w_x(\delta_2)$ for any $\delta_1, \delta_2 \geq 0$ with x such that the set

$$\Omega_{\alpha,p}(w_x) = \Big\{ f \in \Omega_{\alpha,p} : \Big[\frac{1}{\delta}\int_0^{\delta} |\varphi_x(t) - \varphi_x(t \pm \gamma)|^p \, dt\Big]^{1/p} \ll w_x(\gamma) \\ \text{and } \; w_x f(\delta)_p \ll w_x(\delta), \quad \text{where } \; \gamma, \delta > 0 \Big\}$$

is nonempty. It is clear that $\Omega_{\alpha,p}(w_x) \subseteq \Omega_{\alpha,p'}(w_x)$, for $p' \leq p < \infty$.

Our main results are the following:

THEOREM 1. *Let* (1.1) *and* (1.6) *hold. Suppose w_x is such that*

$$\Big\{ u^{p/q} \int_u^{\pi} \frac{(w_x(t))^p}{t^{1+p/q}} \, dt \Big\}^{1/p} = O(u H_x(u)) \quad \text{as } \; u \to 0^+, \tag{2.1}$$

where $H_x(u) \geq 0$, $1 < p \leq q$ and

$$\int_0^t H_x(u) \, du = O(t H_x(t)) \quad \text{as } \; t \to 0^+. \tag{2.2}$$

If $f \in \Omega_{\alpha,p}(w_x)$, then

$$H^q_{n,A,\gamma} f(x) = O\Big(a_{n,n} H_x(a_{n,n}) + \Big\{ \sum_{k=0}^{n} a_{n,k} (E_{\alpha k/2}(f)_{S^p})^q \Big\}^{1/q} \Big), \tag{2.3}$$

where q is such that $1 < q(q-1)^{-1} \leq p \leq q$.

THEOREM 2. *Let* (1.1), (1.5), (2.1) *and* (2.2) *hold. If $f \in \Omega_{\alpha,p}(w_x)$, then*

$$H^q_{n,A,\gamma} f(x) = O\Big(a_{n,0} H_x(a_{n,0}) + \Big\{ \sum_{k=0}^{n} a_{n,k} (E_{\alpha k/2}(f)_{S^p})^q \Big\}^{1/q} \Big), \tag{2.4}$$

where q is such that $1 < q(q-1)^{-1} \leq p \leq q$.

Consequently, we can immediately derive the results on norm approximation.

THEOREM 3. *Let* (1.1) *and* (1.6) *hold. Suppose* $\omega f(\cdot)_{S^{\widetilde{p}}}$ *is such that*

$$\left\{u^{p/q}\int_u^{\pi}\frac{(\omega f(t)_{S^{\widetilde{p}}})^p}{t^{1+p/q}}\,dt\right\}^{1/p}=O(uH(u))\qquad as\quad u\to 0^+ \tag{2.5}$$

with $1<p\le q\le \widetilde{p}$, *where additionally* H (≥ 0) *instead of* H_x *satisfies the condition* (2.2). *If* $f\in\Omega_{\alpha,\widetilde{p}}$, *then*

$$\left\|H^{q'}_{n,A,\gamma}f(\cdot)\right\|_{S^{\widetilde{p}}}=O(a_{n,n}H(a_{n,n})),$$

with $q'\in(0,q]$, *where* q *is such that* $1<q(q-1)^{-1}\le p\le q$.

THEOREM 4. *Let* (1.1) *and* (1.5) *hold. Suppose* $\omega f(\cdot)_{S^{\widetilde{p}}}$ *is such that* (2.5) *holds, with* $1<p\le q\le\widetilde{p}$, *where additionally* H (≥ 0) *instead of* H_x *satisfies the condition* (2.2). *If* $f\in\Omega_{\alpha,\widetilde{p}}$, *then*

$$\left\|H^{q'}_{n,A,\gamma}f(\cdot)\right\|_{S^{\widetilde{p}}}=O(a_{n,0}H(a_{n,0})),$$

with $q'\in(0,q]$, *where* q *is such that* $1<q(q-1)^{-1}\le p\le q$.

REMARK 1. Analyzing our proofs and dividing the integral in the formula

$$\left\{\sum_{k=0}^{n}a_{n,k}\left|\int_0^{\infty}\varphi_x(t)\Psi_{k+\kappa}(t)\,dt\right|^q\right\}^{1/q}$$

into parts with $\frac{\pi}{n+1}$ instead of $a_{n,n}$ or $a_{n,0}$ we can obtain the next series of theorems analogously to [8].

3. Lemmas. To prove our theorems we need the following lemmas.

LEMMA 1 ([8]). *If* (2.1) *and* (2.2) *hold, then*

$$\int_0^u\frac{w_xf(t)}{t}\,dt=O(uH_x(u))\quad(u\to 0_+). \tag{3.1}$$

LEMMA 2 ([11, Theorem 5.20 II, Ch. XII]). *Suppose that* $1<q(q-1)^{-1}\le p\le q$ *and* $\xi=\frac{1}{p}+\frac{1}{q}-1$. *If* $|t^{-\xi}g(t)|^p$ *is Lebesgue integrable, then*

$$\left\{\frac{|a_0(g)|^q}{2}+\sum_{k=1}^{\infty}(|a_k(g)|^q+|b_k(g)|^q)\right\}^{1/q}\ll\left\{\int_{-\pi}^{\pi}|t^{-\xi}g(t)|^p\,dt\right\}^{1/p}, \tag{3.2}$$

where $a_k(g)$ *and* $b_k(g)$ *denote the Fourier coefficients of* 2π*-periodic function* g.

4. Proofs of the results

Proof of Theorem 1. In the proof we will use the function $\Phi_xf(\delta,\nu)=\frac{1}{\delta}\int_{\nu}^{\nu+\delta}\varphi_x(u)\,du$, with $\delta=\frac{\pi}{n+1}$, and its estimate from [9, Lemma 1, p. 218]

$$|\Phi_xf(\delta_1,\delta_2)|\le w_x(\delta_1)+w_x(\delta_2) \tag{4.1}$$

for $f\in\Omega_{\alpha,p}(w_x)$ and any $\delta_1,\delta_2>0$.

Since for $n=0$ our estimate is evident, we consider $n>0$ only.

Denote by S_k^*f the sums of the form

$$S_{\alpha k/2}f(x)=\sum_{|\lambda_\nu|\le\alpha k/2}A_\nu(f)e^{i\lambda_\nu x}$$

such that the interval $(\frac{\alpha k}{2}, \frac{\alpha(k+1)}{2})$ does not contain any λ_ν. Applying Lemma 1.10.2 of [7] we easily verify that

$$S_k^* f(x) - f(x) = \int_0^\infty \varphi_x(t)\Psi_k(t)\,dt,$$

where $\varphi_x(t) := f(x+t) + f(x-t) - 2f(x)$ and $\Psi_k(t) = \Psi_{\alpha k/2, \alpha(k+1)/2}(t)$, i.e.

$$\Psi_k(t) = \frac{4\sin\frac{\alpha t}{4}\sin\frac{\alpha(2k+1)t}{4}}{\alpha\pi t^2}$$

(see also [3], p. 41 and [10]). Evidently, if the interval $(\frac{\alpha k}{2}, \frac{\alpha(k+1)}{2})$ contains a Fourier exponent λ_ν, then

$$S_{\alpha k/2} f(x) = S_{k+1}^* f(x) - \big(A_\nu(f)e^{i\lambda_\nu x} + A_{-\nu}(f)e^{-i\lambda_\nu x}\big).$$

Since (see [1, p. 78] and [2, p. 7])

$$\Big\{\sum_{\nu=-\infty}^{\infty} |A_\nu(f)|^q\Big\}^{1/q} \le \|f\|_{B^p} \quad \text{and} \quad \|f\|_{B^p} \le \|f\|_{S^p},$$

where $\|\cdot\|_{B^p}$, with $p \ge 1$, is the Besicovitch norm, we have

$$|A_{\pm\nu}(f)| = |A_{\pm\nu}(f - g_{\alpha\mu/2})| \le \|f - g_{\alpha\mu/2}\|_{S^p} = E_{\alpha\mu/2}(f)_{S^p},$$

for some $g_{\alpha\mu/2} \in B_{\alpha\mu/2}$, with $\frac{\alpha k}{2} < \frac{\alpha\mu}{2} < \lambda_\nu$. Therefore, the deviation

$$\Big\{\sum_{k=0}^{n} a_{n,k}\big|S_{\alpha k/2} f(x) - f(x)\big|^q\Big\}^{1/q}$$

can be estimated from above by

$$\Big\{\sum_{k=0}^{n} a_{n,k}\Big|\int_0^\infty \varphi_x(t)\Psi_{k+\kappa}(t)\,dt\Big|^q\Big\}^{1/q} + \Big\{\sum_{k=0}^{n} a_{n,k}(E_{\alpha k/2}(f)_{S^p})^q\Big\}^{1/q},$$

where κ equals 0 or 1. Applying the Minkowski inequality we obtain

$$\begin{aligned}
&\Big\{\sum_{k=0}^{n} a_{n,k}\Big|\int_0^\infty \varphi_x(t)\Psi_{k+\kappa}(t)\,dt\Big|^q\Big\}^{1/q} \\
&\quad = \Big\{\sum_{k=0}^{n} a_{n,k}\Big|\Big(\int_0^{2\pi a_{n,n}/\alpha} + \int_{2\pi a_{n,n}/\alpha}^{2\pi/\alpha} + \int_{2\pi/\alpha}^{\infty}\Big)\varphi_x(t)\Psi_{k+\kappa}(t)\,dt\Big|^q\Big\}^{1/q} \\
&\quad \le \Big\{\sum_{k=0}^{n} a_{n,k}|I_1(k)|^q\Big\}^{1/q} + \Big\{\sum_{k=0}^{n} a_{n,k}|I_2(k)|^q\Big\}^{1/q} + \Big\{\sum_{k=0}^{n} a_{n,k}|I_3(k)|^q\Big\}^{1/q}.
\end{aligned}$$

By (1.1), integrating by parts, we obtain

$$
\begin{aligned}
&\Big\{\sum_{k=0}^{n} a_{n,k}|I_1(k)|^q\Big\}^{1/q}\\
&\quad\le \Big\{\sum_{k=0}^{n} a_{n,k}\Big|\frac{4}{\alpha\pi}\int_0^{2\pi a_{n,n}/\alpha}\varphi_x(t)\,\frac{\sin\frac{\alpha t}{4}}{t^2}\,\sin\frac{\alpha t}{4}(2k+2\kappa+1)\,dt\Big|^q\Big\}^{1/q}\\
&\quad\le \frac{1}{\pi}\int_0^{2\pi a_{n,n}/\alpha}\frac{|\varphi_x(t)|}{t}\,dt=\frac{1}{\pi}\int_0^{2\pi a_{n,n}/\alpha}\frac{1}{t}\Big(\frac{d}{dt}\int_0^t|\varphi_x(s)|\,ds\Big)\,dt\\
&\quad=\frac{1}{\pi}\Big[\frac{1}{t}\int_0^t|\varphi_x(s)|\,ds\Big]_{t=0}^{t=2\pi a_{n,n}/\alpha}+\frac{1}{\pi}\int_0^{2\pi a_{n,n}/\alpha}\frac{1}{t^2}\Big(\int_0^t|\varphi_x(s)|\,ds\Big)\,dt\\
&\quad=\frac{1}{\pi}\,w_xf\Big(\frac{2\pi}{\alpha}a_{n,n}\Big)_1+\frac{1}{\pi}\int_0^{2\pi a_{n,n}/\alpha}\frac{1}{t}\,w_xf(t)_1\,dt\\
&\quad\ll w_xf(a_{n,n})_1+\int_0^{2\pi a_{n,n}/\alpha}\frac{1}{t}\,w_xf(t)_1\,dt\\
&\quad=\frac{1}{\pi}\,w_xf\Big(\frac{2\pi}{\alpha}a_{n,n}\Big)_1+\frac{1}{\pi}\int_0^{2\pi a_{n,n}/\alpha}\frac{1}{t}\,w_xf(t)_1\,dt\\
&\quad\ll w_xf(a_{n,n})_1+\int_0^{2\pi a_{n,n}/\alpha}\frac{1}{t}\,w_xf(t)_1\,dt. \qquad (4.2)
\end{aligned}
$$

It is clear that $w_xf(t)_1/t$ is nonincreasing with respect to $t>0$ and $w_xf(t)_1\le w_xf(t)_p$ for $p\ge 1$. Using these properties we have

$$
\begin{aligned}
\Big\{\sum_{k=0}^{n} a_{n,k}|I_1(k)|^q\Big\}^{1/q} &\ll a_{n,n}\int_{a_{n,n}}^{\pi}\frac{w_xf(t)_1}{t^2}+\int_0^{a_{n,n}}\frac{1}{t}\,w_xf\Big(\frac{2\pi}{\alpha}t\Big)_1\,dt\\
&\ll \Big\{a_{n,n}\int_{a_{n,n}}^{\pi}\frac{(w_xf(t)_1)^p}{t^2}\Big\}^{1/p}+\int_0^{a_{n,n}}\frac{1}{t}\,w_xf(t)_1\,dt.
\end{aligned}
$$

Since $f\in\Omega_{\alpha,p}(w_x)$ and (2.2) holds, Lemma 1 and (2.1) give

$$\Big\{\sum_{k=0}^{n} a_{n,k}|I_1(k)|^q\Big\}^{1/q}=O(a_{n,n}H_x(a_{n,n})).$$

If (1.6) holds, then

$$a_{n,\mu}-a_{n,m}\le|a_{n,\mu}-a_{n,m}|\le\sum_{k=\mu}^{m-1}|a_{n,k}-a_{n,k+1}|\le Ka_{n,m}$$

for any $n\ge m\ge\mu\ge 0$. Hence we have

$$a_{n,\mu}\le(K+1)a_{n,m}. \qquad (4.3)$$

From this, we get

$$\begin{aligned}
&\Big\{\sum_{k=0}^{n} a_{n,k}|I_2(k)|^q\Big\}^{1/q}\\
&\quad\le \{(K+1)a_{n,n}\}^{1/q}\Big\{\sum_{k=0}^{n}\Big|\frac{4}{\alpha\pi}\int_{2\pi a_{n,n}/\alpha}^{2\pi/\alpha}\frac{\varphi_x(t)\sin\frac{\alpha t}{4}}{t^2}\sin\frac{\alpha t}{4}\,(2k+2\kappa+1)\,dt\Big|^q\Big\}^{1/q}\\
&\quad\ll \frac{8}{\alpha^2}\,(a_{n,n})^{1/q}\Big\{\sum_{k=0}^{n}\Big|\frac{\alpha}{2\pi}\int_{2\pi a_{n,n}/\alpha}^{2\pi/\alpha}\frac{\varphi_x(t)\sin\frac{\alpha t}{4}}{t^2}\sin\frac{\alpha t}{4}\,(2\kappa+1)\cos\frac{\alpha k t}{2}\,dt\Big|^q\Big\}^{1/q}\\
&\quad+\frac{8}{\alpha^2}(a_{n,n})^{1/q}\Big\{\sum_{k=0}^{n}\Big|\frac{\alpha}{2\pi}\int_{2\pi a_{n,n}/\alpha}^{2\pi/\alpha}\frac{\varphi_x(t)\sin\frac{\alpha t}{4}}{t^2}\cos\frac{\alpha t}{4}\,(2\kappa+1)\sin\frac{\alpha k t}{2}\,dt\Big|^q\Big\}^{1/q}.
\end{aligned}$$

Using inequality (3.2), we have

$$\Big\{\sum_{k=0}^{n} a_{n,k}|I_2(k)|^q\Big\}^{1/q}\ll (a_{n,n})^{1/q}\Big\{\int_{2\pi a_{n,n}/\alpha}^{2\pi/\alpha}\frac{|\varphi_x(t)|^p}{t^{1+p/q}}\,dt\Big\}^{1/p}.$$

Integrating by parts, we obtain

$$\begin{aligned}
&\Big\{\sum_{k=0}^{n} a_{n,k}|I_2(k)|^q\Big\}^{1/q}\\
&\ll (a_{n,n})^{1/q}\Big\{\Big[\frac{1}{t^{1+p/q}}\int_0^t|\varphi_x(s)|^p\,ds\Big]_{t=2\pi a_{n,n}/\alpha}^{t=2\pi/\alpha}\\
&\qquad\qquad+\Big(1+\frac{p}{q}\Big)\int_{2\pi a_{n,n}/\alpha}^{2\pi/\alpha}\frac{1}{t^{2+p/q}}\Big(\int_0^t|\varphi_x(s)|^p\,ds\Big)\,dt\Big\}^{1/p}\\
&=(a_{n,n})^{1/q}\Big\{\Big[\frac{1}{t^{p/q}}(w_xf(t)_p)^p\Big]_{t=2\pi a_{n,n}/\alpha}^{t=2\pi/\alpha}+\Big(1+\frac{p}{q}\Big)\int_{2\pi a_{n,n}/\alpha}^{2\pi/\alpha}\frac{1}{t^{1+p/q}}(w_xf(t)_p)^p\,dt\Big\}^{1/p}\\
&\ll (a_{n,n})^{1/q}\Big\{\Big(w_xf\Big(\frac{2\pi}{\alpha}\Big)_p\Big)^p+\int_{2\pi a_{n,n}/\alpha}^{2\pi/\alpha}\frac{1}{t^{1+p/q}}(w_xf(t)_p)^p\,dt\Big\}^{1/p}.
\end{aligned}\tag{4.4}$$

Since $f\in\Omega_{\alpha,p}(w_x)$, (2.1) gives

$$\begin{aligned}
\Big\{\sum_{k=0}^{n} a_{n,k}|I_2(k)|^q\Big\}^{1/q}&\ll (a_{n,n})^{1/q}\Big\{(w_x(\pi))^p+\int_{a_{n,n}}^{\pi}\frac{1}{t^{1+p/q}}(w_x(t))^p\,dt\Big\}^{1/p}\\
&\ll\Big\{(a_{n,n})^{p/q}\int_{a_{n,n}}^{\pi}\frac{(w_x(t))^p}{t^{1+p/q}}\,dt\Big\}^{1/p}=O(a_{n,n}H_x(a_{n,n})).
\end{aligned}$$

For the third term we obtain

$$\begin{aligned}
\Big\{\sum_{k=0}^{n} a_{n,k}|I_3(k)|^q\Big\}^{1/q}&\le\Big\{\sum_{k=0}^{n}a_{n,k}\Big|\sum_{\mu=1}^{\infty}\int_{2\pi\mu/\alpha}^{2\pi(\mu+1)/\alpha}[\varphi_x(t)-\Phi_xf(\delta_k,t)]\Psi_{k+\kappa}(t)\,dt\Big|^q\Big\}^{1/q}\\
&\quad+\Big\{\sum_{k=0}^{n}a_{n,k}\Big|\sum_{\mu=1}^{\infty}\int_{2\pi\mu/\alpha}^{2\pi(\mu+1)/\alpha}\Phi_xf(\delta_k,t)\Psi_{k+\kappa}(t)\,dt\Big|^q\Big\}^{1/q}\\
&=\Big\{\sum_{k=0}^{n}a_{n,k}|I_{31}(k)|^q\Big\}^{1/q}+\Big\{\sum_{k=0}^{n}a_{n,k}|I_{32}(k)|^q\Big\}^{1/q}
\end{aligned}$$

and

$$
\begin{aligned}
|I_{31}(k)| &\le \frac{4}{\alpha\pi}\sum_{\mu=1}^{\infty}\int_{2\pi\mu/\alpha}^{2\pi(\mu+1)/\alpha}|\varphi_x(t)-\Phi_x f(\delta_k,t)|t^{-2}\,dt\\
&\le \frac{4}{\alpha\pi}\sum_{\mu=1}^{\infty}\int_{2\pi\mu/\alpha}^{2\pi(\mu+1)/\alpha}\Big[\frac{1}{\delta_k t^2}\int_0^{\delta_k}|\varphi_x(t)-\varphi_x(t+u)|\,du\Big]\,dt\\
&= \frac{4}{\alpha\pi}\frac{1}{\delta_k}\int_0^{\delta_k}\sum_{\mu=1}^{\infty}\Big\{\int_{2\pi\mu/\alpha}^{2\pi(\mu+1)/\alpha}\frac{1}{t^2}|\varphi_x(t)-\varphi_x(t+u)|\,dt\Big\}\,du\\
&= \frac{4}{\alpha\pi}\,\frac{1}{\delta_k}\int_0^{\delta_k}\sum_{\mu=1}^{\infty}\Big\{\Big[\frac{1}{t^2}\int_0^t|\varphi_x(s)-\varphi_x(s+u)|\,ds\Big]_{t=2\pi\mu/\alpha}^{t=2\pi(\mu+1)/\alpha}\\
&\qquad+2\int_{2\pi\mu/\alpha}^{2\pi(\mu+1)/\alpha}\Big[\frac{1}{t^3}\int_0^t|\varphi_x(s)-\varphi_x(s+u)|\,ds\Big]\,dt\Big\}\,du\\
&\ll \Big|\frac{1}{\delta_k}\int_0^{\delta_k}\sum_{\mu=1}^{\infty}\Big\{\frac{1}{[2\pi(\mu+1)/\alpha]^2}\int_0^{2\pi(\mu+1)/\alpha}|\varphi_x(s)-\varphi_x(s+u)|\,ds\\
&\qquad-\frac{1}{[2\pi\mu/\alpha]^2}\int_0^{2\pi\mu/\alpha}|\varphi_x(s)-\varphi_x(s+u)|\,ds\Big\}\,du\Big|\\
&\qquad+\frac{1}{\delta_k}\int_0^{\delta_k}\sum_{\mu=1}^{\infty}\Big\{\int_{2\pi\mu/\alpha}^{2\pi(\mu+1)/\alpha}\Big[\frac{1}{t^3}\int_0^t|\varphi_x(s)-\varphi_x(s+u)|\,ds\Big]\,dt\Big\}\,du.
\end{aligned}
$$

Since $f\in\Omega_{\alpha,p}(w_x)$, we have for any x

$$
\lim_{\zeta\to\infty}\frac{1}{\zeta^2}\int_0^{\zeta}|\varphi_x(s)-\varphi_x(s+u)|\,ds\le\lim_{\zeta\to\infty}\frac{1}{\zeta}\,w_x(u)\le\lim_{\zeta\to\infty}\frac{1}{\zeta}\,w_x(\delta_k)\le\lim_{\zeta\to\infty}\frac{1}{\zeta}\,w_x(\pi)=0,
$$

and therefore

$$
\begin{aligned}
|I_{31}(k)| &\le \frac{1}{\delta_k}\int_0^{\delta_k}\frac{\alpha}{2\pi}\Big[\frac{\alpha}{2\pi}\int_0^{2\pi/\alpha}|\varphi_x(s)-\varphi_x(s+u)|\,ds\Big]\,du\\
&\qquad+\frac{1}{\delta_k}\int_0^{\delta_k}w_x(u)\,du\sum_{\mu=1}^{\infty}\Big\{\int_{2\pi\mu/\alpha}^{2\pi(\mu+1)/\alpha}\frac{1}{t^2}\,dt\Big\}\\
&\ll \frac{1}{\delta_k}\int_0^{\delta_k}w_x(u)\,du+w_x(\delta_k)\sum_{\mu=1}^{\infty}\frac{1}{(2\pi/\alpha)\mu^2}\ll w_x(\delta_k).
\end{aligned}
$$

Next, we will estimate the term $|I_{32}(k)|$. So,

$$
\begin{aligned}
I_{32}(k) &= \frac{2}{\alpha\pi}\sum_{\mu=1}^{\infty}\int_{2\pi\mu/\alpha}^{2\pi(\mu+1)/\alpha}\frac{\Phi_x f(\delta_k,t)}{t^2}\,\frac{d}{dt}\Big(-\frac{\cos\frac{\alpha t(k+\kappa)}{2}}{\frac{\alpha(k+\kappa)}{2}}+\frac{\cos\frac{\alpha t(k+\kappa+1)}{2}}{\frac{\alpha(k+\kappa+1)}{2}}\Big)\,dt\\
&= \frac{2}{\alpha\pi}\sum_{\mu=1}^{\infty}\Big[\frac{\Phi_x f(\delta_k,t)}{t^2}\Big(-\frac{\cos\frac{\alpha t(k+\kappa)}{2}}{\frac{\alpha(k+\kappa)}{2}}+\frac{\cos\frac{\alpha t(k+\kappa+1)}{2}}{\frac{\alpha(k+\kappa+1)}{2}}\Big)\Big]_{t=2\pi\mu/\alpha}^{t=2\pi(\mu+1)/\alpha}\\
&\quad+\frac{2}{\alpha\pi}\sum_{\mu=1}^{\infty}\int_{\frac{2\pi\mu}{\alpha}}^{\frac{2\pi(\mu+1)}{\alpha}}\frac{d}{dt}\Big(\frac{\Phi_x f(\delta_k,t)}{t^2}\Big)\Big(\frac{\cos\frac{\alpha t(k+\kappa)}{2}}{\frac{\alpha(k+\kappa)}{2}}-\frac{\cos\frac{\alpha t(k+\kappa+1)}{2}}{\frac{\alpha(k+\kappa+1)}{2}}\Big)\,dt=I_{321}(k)+I_{322}(k).
\end{aligned}
$$

Since $f \in \Omega_{\alpha,p}(w_x)$, we obtain for any x (using (4.1))

$$\lim_{\zeta\to\infty}\left|\frac{\Phi_x f(\delta_k,2\pi\zeta/\alpha)}{[2\pi\zeta/\alpha]^2}\Big(-\frac{\cos[\pi\zeta(k+\kappa)]}{\alpha(k+\kappa)/2}+\frac{\cos[\pi\zeta(k+\kappa+1)]}{\alpha(k+\kappa+1)/2}\Big)\right|$$
$$\le \lim_{\zeta\to\infty}\frac{w_x(\delta_k)+w_x(2\pi\zeta/\alpha)}{2\pi^2\zeta^2 k} \ll \lim_{\zeta\to\infty}\frac{w_x(\delta_k)+\zeta w_x(2\pi/\alpha)}{\zeta^2 k} \ll w_x(\pi)\lim_{\zeta\to\infty}\frac{1+\zeta}{\zeta^2}=0,$$

and therefore

$$I_{321}(k)=\frac{2}{\alpha\pi}\sum_{\mu=1}^{\infty}\Big[\frac{\Phi_x f(\delta_k,2\pi(\mu+1)/\alpha)}{[2\pi(\mu+1)/\alpha]^2}\Big(-\frac{\cos[\pi(\mu+1)(k+\kappa)]}{\alpha(k+\kappa)/2}$$
$$+\frac{\cos[\pi(\mu+1)(k+\kappa+1)]}{\alpha(k+\kappa+1)/2}\Big)-\frac{\Phi_x f(\delta_k,2\pi\mu/\alpha)}{[2\pi\mu/\alpha]^2}\Big(-\frac{\cos[\pi\mu(k+\kappa)]}{\alpha(k+\kappa)/2}+\frac{\cos[\pi\mu(k+\kappa+1)]}{\alpha(k+\kappa+1)/2}\Big)\Big]$$
$$=-\frac{2}{\alpha\pi}\,\frac{\Phi_x f(\delta_k,2\pi/\alpha)}{[2\pi/\alpha]^2}\Big(-\frac{(-1)^{(k+\kappa)}}{\alpha(k+\kappa)/2}+\frac{(-1)^{(k+\kappa+1)}}{\alpha(k+\kappa+1)/2}\Big)$$
$$=-\frac{1}{\pi^3}\,\Phi_x f(\delta_k,2\pi/\alpha)(-1)^{(k+\kappa+1)}\Big(\frac{1}{k+\kappa+1}+\frac{1}{k+\kappa}\Big).$$

Using (4.1), we get

$$|I_{321}(k)| \ll \frac{1}{\pi^3}\frac{2}{k+1}|\Phi_x f(\delta_k,2\pi/\alpha)| \le \frac{2}{\pi^3(k+1)}(w_x(\delta_k)+w_x(2\pi/\alpha)).$$

Similarly

$$I_{322}(k)=\frac{2}{\alpha\pi}\sum_{\mu=1}^{\infty}\int_{2\pi\mu/\alpha}^{2\pi(\mu+1)/\alpha}\Big(\frac{\frac{d}{dt}\Phi_x f(\delta_k,t)}{t^2}-\frac{2\Phi_x f(\delta_k,t)}{t^3}\Big)$$
$$\cdot\Big(\frac{\cos\frac{\alpha t(k+\kappa)}{2}}{\frac{\alpha(k+\kappa)}{2}}-\frac{\cos\frac{\alpha t(k+\kappa+1)}{2}}{\frac{\alpha(k+\kappa+1)}{2}}\Big)\,dt$$

and

$$|I_{322}(k)|$$
$$\ll \frac{8}{\alpha^2(k+1)\pi}\sum_{\mu=1}^{\infty}\Big[\int_{2\pi\mu/\alpha}^{2\pi(\mu+1)/\alpha}\frac{|\varphi_x(t+\delta_k)-\varphi_x(t)|}{\delta_k t^2}\,dt+2\int_{2\pi\mu/\alpha}^{2\pi(\mu+1)/\alpha}\frac{|\Phi_x f(\delta_k,t)|}{t^3}\,dt\Big]$$
$$\le \frac{8}{\alpha^2(k+1)\pi\delta_k}\sum_{\mu=1}^{\infty}\int_{2\pi\mu/\alpha}^{2\pi(\mu+1)/\alpha}\frac{|\varphi_x(t+\delta_k)-\varphi_x(t)|}{t^2}\,dt$$
$$+\frac{16}{\alpha^2(k+1)\pi}\sum_{\mu=1}^{\infty}\int_{2\pi\mu/\alpha}^{2\pi(\mu+1)/\alpha}\frac{w_x(\delta_k)+w_x(t)}{t^3}\,dt$$
$$\ll \frac{1}{(k+1)\delta_k}\,w_x(\delta_k)+\frac{1}{k+1}\sum_{\mu=1}^{\infty}\Big[\Big(w_x(\delta_k)+w_x\Big(\frac{2\pi(\mu+1)}{\alpha}\Big)\Big)\frac{\alpha^2}{4\pi^2\mu^3}\Big]$$
$$\ll w_x(\delta_k)+\frac{1}{k+1}\Big[w_x(\delta_k)\sum_{\mu=1}^{\infty}\frac{1}{\mu^3}+\sum_{\mu=1}^{\infty}\frac{w_x(2\pi(\mu+1)/\alpha)}{\mu^3}\Big]$$

$$\ll w_x(\delta_k) + \frac{1}{k+1}\Big(w_x(\delta_k) + w_x\Big(\frac{4\pi}{\alpha}\Big)\sum_{\mu=1}^{\infty}\frac{\mu+1}{\mu^3}\Big)$$
$$\ll w_x(\delta_k) + \frac{1}{k+1}\Big(w_x(\delta_k) + w_x\Big(\frac{4\pi}{\alpha}\Big)\Big).$$

Therefore

$$|I_3(k)| \ll w_x(\delta_k) + \frac{1}{k+1}\Big(w_x(\delta_k) + w_x\Big(\frac{2\pi}{\alpha}\Big) + w_x\Big(\frac{4\pi}{\alpha}\Big)\Big)$$

and thus

$$\Big\{\sum_{k=0}^{n} a_{n,k}|I_3(k)|^q\Big\}^{1/q} \ll \Big\{\sum_{k=0}^{n} a_{n,k}\Big(w_x\Big(\frac{\pi}{k+1}\Big) + \frac{1}{k+1}\,w_x\Big(\frac{\pi}{\alpha}\Big)\Big)^q\Big\}^{1/q}$$
$$\ll \Big\{\sum_{k=0}^{n} a_{n,k}\Big(w_x\Big(\frac{\pi}{k+1}\Big)\Big)^q\Big\}^{1/q}.$$

From (4.3) we obtain

$$\sum_{k=0}^{n} a_{n,k}\Big(w_x\Big(\frac{\pi}{k+1}\Big)\Big)^q \le \sum_{k=0}^{[\frac{1}{(K+1)a_{n,n}}]-1} a_{n,k}\Big(w_x\Big(\frac{\pi}{k+1}\Big)\Big)^q$$
$$+ \sum_{k=[\frac{1}{(K+1)a_{n,n}}]-1}^{n} a_{n,k}\Big(w_x\Big(\frac{\pi}{k+1}\Big)\Big)^q.$$

Using (1.1), (4.3) and the monotonicity of the function w_x, from (2.1) and (3.1), we get

$$\sum_{k=0}^{n} a_{n,k}\Big(w_x\Big(\frac{\pi}{k+1}\Big)\Big)^q \le (K+1)a_{n,n} \sum_{k=0}^{[\frac{1}{(K+1)a_{n,n}}]-1} \Big(w_x\Big(\frac{\pi}{k+1}\Big)\Big)^q$$
$$+ (w_x(\pi(K+1)a_{n,n}))^q \sum_{k=[\frac{1}{(K+1)a_{n,n}}]-1}^{n} a_{n,k}$$
$$\ll a_{n,n}\int_1^{\frac{1}{(K+1)a_{n,n}}} \Big(w_x\Big(\frac{\pi}{t}\Big)\Big)^q\,dt + (w_x(a_{n,n}))^q$$
$$\ll a_{n,n}\int_{a_{n,n}}^{\pi} \frac{(w_x(u))^q}{u^2}\,du + (w_x(a_{n,n}))^q$$
$$\le a_{n,n}\int_{a_{n,n}}^{\pi} \frac{(w_x(u))^q}{u^{1+p/q+1-p/q}}\,du + \Big(4w_x\Big(\frac{a_{n,n}}{2}\Big)\Big)^q$$
$$\le (a_{n,n})^{p/q}\int_{a_{n,n}}^{\pi} \frac{(w_x(u))^q}{u^{1+p/q}}\,du + \Big(8\int_{a_{n,n}/2}^{a_{n,n}} \frac{w_x(u)}{u}\,du\Big)^q$$
$$\ll (a_{n,n})^{p/q}\int_{a_{n,n}}^{\pi} \frac{(w_x(u))^q}{u^{1+p/q}}\,du + \Big(\int_0^{a_{n,n}} \frac{w_x(u)}{u}\,du\Big)^q \ll (a_{n,n}H_x(a_{n,n}))^q.$$

Summing up we prove (2.3) and the proof of the theorem is complete. ∎

Proof of Theorem 2. Under the notation of the previous proof we can write

$$\begin{aligned}
&\Big\{\sum_{k=0}^{n} a_{n,k}\Big|\int_0^{\infty}\varphi_x(t)\Psi_{k+\kappa}(t)\,dt\Big|^q\Big\}^{1/q}\\
&\qquad= \Big\{\sum_{k=0}^{n} a_{n,k}\Big|\Big(\int_0^{2\pi a_{n,0}/\alpha}+\int_{2\pi a_{n,0}/\alpha}^{2\pi/\alpha}+\int_{2\pi/\alpha}^{\infty}\Big)\varphi_x(t)\Psi_{k+\kappa}(t)\,dt\Big|^q\Big\}^{1/q}\\
&\qquad\le \Big\{\sum_{k=0}^{n} a_{n,k}|J_1(k)|^q\Big\}^{1/q}+\Big\{\sum_{k=0}^{n} a_{n,k}|J_2(k)|^q\Big\}^{1/q}+\Big\{\sum_{k=0}^{n} a_{n,k}|J_3(k)|^q\Big\}^{1/q},
\end{aligned}$$

using the Minkowski inequality. If we apply the property of the class $HBVS$ instead of the property of $RBVS$ our proof will be similar to the proof of Theorem 1. ∎

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