FUNCTION SPACES X
BANACH CENTER PUBLICATIONS, VOLUME 102
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2014

GROTHENDIECK–LIDSKIĬ THEOREM FOR SUBSPACES OF QUOTIENTS OF L_p-SPACES

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Abstract. Generalizing A. Grothendieck's (1955) and V. B. Lidskiĭ's (1959) trace formulas, we have shown in a recent paper that for $p \in [1,\infty]$ and $s \in (0,1]$ with $1/s = 1 + |1/2 - 1/p|$ and for every s-nuclear operator T in every subspace of any $L_p(\nu)$-space the trace of T is well defined and equals the sum of all eigenvalues of T. Now, we obtain the analogous results for subspaces of quotients (equivalently: for quotients of subspaces) of L_p-spaces.

1. Main result. In the note [13], we have proved that if $p \in [1,\infty]$ and $1/s = 1 + |1/2 - 1/p|$, then for any subspace (or quotient) of an L_p-space and for every s-nuclear operator T in the space the nuclear trace of T is well-defined and equals the sum of all eigenvalues of T. The main fact, we are going to obtain here, is

THEOREM 1. *Let Y be a subspace of a quotient (or a quotient of a subspace) of an L_p-space, $1 \leq p \leq \infty$. If $T \in N_s(Y,Y)$ (s-nuclear), where $1/s = 1 + |1/2 - 1/p|$, then*

1. *the (nuclear) trace of T is well defined,*
2. *$\sum_{n=1}^{\infty} |\lambda_n(T)| < \infty$, where $\{\lambda_n(T)\}$ is the system of all eigenvalues of the operator T (written in according to their algebraic multiplicities) and*

$$\operatorname{trace} T = \sum_{n=1}^{\infty} \lambda_n(T).$$

2010 *Mathematics Subject Classification*: Primary 47B06.
Key words and phrases: approximation properties, s-nuclear operators, eigenvalue distributions.
The paper is in final form and no version of it will be published elsewhere.

DOI: 10.4064/bc102-0-13

Let us mention that in the proof we have to repeat some ideas of proofs from [13] (in particular, of the proof of main lemma there) as well as, simultaneously, to use the main lemma of [13] itself (so, we will get a generalization of the lemma by using a part of its proof and also its statement).

2. Preliminaries. All the terminology and facts we use here can be found in [5]–[8]. Let X, Y be Banach spaces. For $s \in (0,1]$, denote by $X^*\widehat{\otimes}_s Y$ the completion of the tensor product $X^* \otimes Y$ (considered as a linear space of all finite rank operators) with respect to the quasi-norm

$$\|z\|_s := \inf\Big\{ \Big(\sum_{k=1}^{N} \|x'_k\|^s \, \|y_k\|^s \Big)^{1/s} : \, z = \sum_{k=1}^{N} x'_k \otimes y_k \Big\}.$$

Let Ψ_p, for $p \in [1,\infty]$, be the ideal of all operators which can be factored through a subspace of a quotient of an L_p-space. Put $N_s(X,Y) :=$ image of $X^*\widehat{\otimes}_s Y$ in the space $L(X,Y)$ of all bounded linear transformations under the canonical factor map $X^*\widehat{\otimes}_s Y \to N_s(X,Y) \subset L(X,Y)$. We consider the (Grothendieck) space $N_s(X,Y)$ of all s-nuclear operators from X to Y with the natural quasi-norm, induced from $X^*\widehat{\otimes}_s Y$.

Finally, let $\Psi_{p,s}$ be the quasi-normed product $N_s \circ \Psi_p$ of the corresponding ideals (see [6], p. 107, 7.1, about the products of quasi-normed operator ideals) equipped with the natural quasi-norm $\nu_{p,s}$: if $A \in N_s \circ \Psi_p(X,Y)$ then $A = \varphi \circ T$ with $T = \beta\alpha \in \Psi_p$, $\varphi = \delta\Delta\gamma \in N_s$ and

$$A = \delta\Delta\gamma\beta\alpha : X \to X_p \to Z \to c_0 \to l_1 \to Y,$$

where Z is a Banach space, all maps are continuous and linear, X_p is a subspace of a quotient of an L_p-space, constructed on a measure space, and Δ is a diagonal operator with the diagonal from l_s. Thus, $A = \delta\Delta\gamma\beta\alpha$ and $A \in N_s$. Therefore, if $X = Y$, the spectrum of A, $\mathrm{sp}(A)$, is at most countable with only possible limit point zero. Moreover, A is compact and, therefore, a Riesz operator (see [6], pp. 358–366, for the theory of Riesz operators) with eigenvalues of finite algebraic multiplicities and $\mathrm{sp}(A) \equiv \mathrm{sp}(B)$, where $B := \alpha\delta\Delta\gamma\beta : X_p \to X_p$ is an s-nuclear operator, acting in a subspace of a quotient of an L_p-space.

Definition 2. Let Y be a Banach space and $s \in (0,1]$. We say that Y *possesses the property* AP_s (the approximation property of order s; written down as "$Y \in AP_s$") if for any tensor element $z \in Y^*\widehat{\otimes}_s Y$ the operator $\tilde{z} : Y \to Y$, associated with z, is zero iff the tensor element z is zero as an element of the space $Y^*\widehat{\otimes} Y$.

This is equivalent to the fact that if $z \in Y^*\widehat{\otimes}_s Y$ then it follows from

$$\operatorname{trace} z \circ R = 0, \quad \forall\, R \in Y^* \otimes Y$$

that trace $U \circ z = 0$ for every $U \in L(Y, Y^{**})$. There is a simple characterization of the condition $Y \in AP_s$ in terms of the approximation of the identity id_Y on some sequences of the space Y, but we omit it. We need here only some examples which are crucial for our note. For giving them, we formulate and prove the following statement, which, we hope, is interesting by itself.

Proposition 3. *Let $\alpha \in [0, 1/2]$ and $1/s = 1 + \alpha$. For a Banach space Y, suppose that*

$$(\alpha)\qquad \begin{array}{c}\text{there exists a constant } C > 0 \text{ such that for every } \varepsilon > 0, \text{ for every natural } n\\ \text{and for every } n\text{-dimensional subspace } E \text{ of } Y \text{ there exists}\\ \text{a finite rank operator } R \text{ in } Y \text{ so that } \|R\| \le Cn^\alpha \text{ and } \|R|_E - id_E\|_{L(E,Y)} \le \varepsilon.\end{array}$$

Then $Y \in AP_s$.

Proof. Suppose that there is an element $z \in Y^* \widehat{\otimes}_s Y$ such that $\operatorname{trace} z = b > 0$, but $\tilde z = 0$. Consider a representation of z of the kind

$$z = \sum_{k=1}^{\infty} \mu_k y'_k \otimes y_k,$$

where $\|y'_k\|, \|y_k\| = 1$ and $\mu_k \ge 0$, $\sum_{k=1}^\infty \mu_k^s < \infty$. Without loss of generality, we can (and do) assume that the sequence (μ_k) is decreasing and that $\sum_{k=1}^\infty \mu_k \le 1$. In this situation, $\mu_k^s = o(1/k)$, so, there are $c_k > 0$ with $c_k \to 0$ and $\mu_k \le c_k/k^{1/s}$.

Fix any natural N, large enough, such that for all $m \ge N$

$$\sum_{k=1}^{m} \mu_k \langle y'_k, y_k\rangle \ge b/2.$$

For such an m, put $E := \operatorname{span}\{y_k\}_{k=1}^m$, and apply the condition (α) to find a corresponding operator $R \in Y^* \otimes Y$ for $n = m$ and $\varepsilon = b/4$.

By our assumption, $\operatorname{trace} R \circ z = 0$. From this, we get (for all $m \ge N$)

$$0 = \sum_{k=1}^{m} \mu_k \langle y'_k, Ry_k\rangle + \sum_{k=m+1}^{\infty} \mu_k \langle y'_k, Ry_k\rangle.$$

For the first sum

$$\sum_{k=1}^{m} \mu_k \langle y'_k, Ry_k\rangle \ge \sum_{k=1}^{m} \mu_k \langle y'_k, y_k\rangle - \Big|\sum_{k=1}^{m} \mu_k \langle y'_k, y_k - Ry_k\rangle\Big| \ge b/2 - b/4 = b/4.$$

For the second sum

$$\Big|\sum_{k=m+1}^{\infty} \mu_k \langle y'_k, Ry_k\rangle\Big| \le Cm^\alpha\, \tilde c_m \int_m^\infty x^{-1/s}\, dx \le d_m\, m^\alpha m^{1-1/s} = d_m,$$

where $0 \le \tilde c_m \to 0$, and thus $0 \le d_m \to 0$.

Now, from the last three relations, we obtain: $0 \ge b/4 - d_m$. ∎

Let us consider some consequences of the proposition.

Corollary 4. *Let $\alpha \in [0, 1/2]$ and $1/s = 1 + \alpha$. For a Banach space Y, suppose that there exists a constant $C > 0$ such that for every natural n and for every n-dimensional subspace E of Y there exists a finite rank operator R in Y so that $\|R\| \le Cn^\alpha$ and $R|_E = id_E$. Then $Y \in AP_s$.*

Corollary 5. *Let $\alpha \in [0, 1/2]$ and $1/s = 1 + \alpha$. For a Banach space Y, suppose that there exists a constant $C > 0$ such that for every natural n and for every n-dimensional subspace E of Y there exists a finite-dimensional subspace F of Y, containing E and Cn^α-complemented in Y. Then $Y \in AP_s$.*

Corollary 6. *Let $\alpha \in [0, 1/2]$ and $1/s = 1 + \alpha$. For a Banach space Y, suppose that there exists a constant $C > 0$ such that for every natural n and every n-dimensional subspace E of Y is Cn^α-complemented in Y. Then $Y \in AP_s$. Moreover, every subspace of the space Y has the AP_s.*

It can be shown (but we do not need this in the note) that $Y \in AP_s$ iff for every Banach space X the natural mapping $X^*\widehat{\otimes}_s Y \to L(X, Y)$ is one-to-one (for other related results see, e.g., [11], [12]). Thus, taking this into account, we get:

Corollary 7. *In all above four assertions, in the case of Y with mentioned properties, we have the quasi-Banach equality $X^*\widehat{\otimes}_s Y = N_s(X, Y)$, whichever the space X was. In particular, $Y^*\widehat{\otimes}_s Y = N_s(Y, Y)$.*

Before giving more concrete applications of Proposition 3, let us mention the simplest example.

Example 8. Let $s \in (0, 1]$, $p \in [1, \infty]$ and $1/s = 1 + |1/p - 1/2|$. Any subspace as well as any factor space of any L_p-space have the property AP_s.

We used this example in [13]. The statement of Example 8 follows from Corollary 6 and the results of D. R. Lewis (see [3], Corollary 4).

As a matter of fact, one can get from the work [3] more general facts on complementability concerning L_p-situation. However, we prefer to consider abstract situations and to deal with spaces of nontrivial types and cotypes (partially, for using the results to be obtained in other considerations).

We will apply mainly the results that can be found, e.g., in [1], [6], [8] and [9]. For the definitions of the notions of type and cotype, see any of these references (Rademacher type p = Gauss type p and Rademacher cotype q = Gauss cotype q; so, we can apply results from G. Pisier's lecture [9], assuming that we are working with Rademacher notions).

Let us collect the facts we need. Recall that a subspace E of a Banach space X is b-complemented ($b > 0$) in X, if there exists a linear continuous projection P from X onto E such that $\|P\| \le b$.

Proposition 9. *Let X be a Banach space and $1 < p \le 2$, $2 \le q < \infty$.*

1) *If X is of type p (cotype p) then every subspace is of type p (cotype p).*
2) [1, Proposition 11.11] *If X is of type p then any quotient of X is of type p.*
3) [1, Proposition 11.10] *If X is of type p then X^* is of cotype p'.*
4) *If X^* is of type p then X is of cotype p'.*
5) *If X is of type p then any subspace of any quotient (and any quotient of any subspace) of X is of type p.*
6) [1, Corollary 11.9] *A Banach space has the same type or cotype as its bidual.*
7) [1, Corollary 11.7] *Each L_r-space ($1 \le r < \infty$) has type* $\min\{r, 2\}$ *and cotype* $\max\{r, 2\}$.
8) [9, see Theorem 4.1 and its Corollaries] *If X is of type p and of cotype q then there is a constant $D_{p,q} > 0$ such that every finite-dimensional subspace E of X is $D_{p,q}\,(\dim E)^{1/p-1/q}$-complemented in X.*

Recall also the well known general fact: in any Banach space every n-dimensional subspace is $n^{1/2}$-complemented.

We need in this note only the following immediate consequence of Proposition 9 and Corollary 6:

COROLLARY 10. *Let $s \in (0,1]$, $p \in [1,\infty]$ and $1/s = 1+|1/p-1/2|$. If a Banach space Y is isomorphic to a subspace of a quotient (or to a quotient of a subspace) of an L_p-space then it has the property AP_s.*

In particular, we get again (cf. [10] and see [2]):

COROLLARY 11. *Every Banach space has the property $AP_{2/3}$.*

3. Main lemma. We are going to formulate and to prove now the main lemma in this paper. It is interesting to note that in the proof we will use a part of the proof of Lemma from [13] as well as the statement of that Lemma itself. Let us recall the formulation of Lemma of [13].

LEMMA 12. *Let $s \in (0,1]$, $p \in [1,\infty]$ and $1/s = 1+|1/2-1/p|$. Then the system of all eigenvalues (with their algebraic multiplicities) of any operator $T \in N_s(Y,Y)$, acting in any subspace Y of any L_p-space, belongs to the space l_1. The same is true for the factor spaces of L_p-spaces.*

The next assertion contains this lemma as a particular case.

LEMMA 13. *Let $s \in (0,1]$, $p \in [1,\infty]$ and $1/s = 1+|1/2-1/p|$. Then the system of all eigenvalues (with their algebraic multiplicities) of any operator $T \in N_s(Y,Y)$, acting in any subspace Y of any quotient of any L_p-space (equivalently: in any quotient Y of any subspace of any L_p-space), belongs to the space l_1.*

Proof. Let $p \in [1,\infty]$. Let Y be a subspace of a quotient W $(= L_p/V$ for some $V \subset L_p)$ of an L_p-space and $T \in N_s(Y,Y)$ with an s-nuclear representation

$$T = \sum_{k=1}^{\infty} \mu_k y'_k \otimes y_k,$$

where $\|y'_k\|, \|y_k\| = 1$ and $\mu_k \geq 0$, $\sum_{k=1}^{\infty} \mu_k^s < \infty$. The operator T can be factored in the following way:

$$T = B\Delta_s j \Delta_{1-s} A : Y \longrightarrow l_\infty \longrightarrow l_r \hookrightarrow c_0 \longrightarrow l_1 \longrightarrow Y,$$

where A and B are linear bounded, j is the natural injection, $\Delta_s \sim (\mu_k^s)_k$ and $\Delta_{1-s} \sim (\mu_k^{1-s})$ are the natural diagonal operators from c_0 into l_1 and from l_∞ into l_r, respectively. Here, r is defined via the conditions $1/s = 1+|1/p-1/2|$ and $\sum_k \mu_k^s < \infty$: we need the convergence of the series $\sum_k \mu_k^{(1-s)r}$. Therefore, we take r such that $1/r = 1/s - 1$, or $1/r = |1/p-1/2|$.

Let $\Phi : L_p \to W$ be a factor map, so that $Y \subset W$. Denote by Y_0, $Y_0 \subset L_p$, the preimage of Y under the map Φ, $Y_0 := \Phi^{-1}(Y)$. Consider the operator $\Phi|_{Y_0} : Y_0 \to Y$ (it is a factor map) and the following diagram:

$$B\Delta_s j \Delta_{1-s} A \Phi|_{Y_0} : Y_0 \longrightarrow Y \longrightarrow l_\infty \longrightarrow l_r \hookrightarrow c_0 \longrightarrow l_1 \longrightarrow Y.$$

$\Phi|_{Y_0}$ is a factor map, so we can find a lifting $Q : l_1 \to Y$ with $B = \Phi|_{Y_0} Q : l_1 \to Y_0 \to Y$. Now, we get that the operator T can be factored as follows:

$$T = \Phi|_{Y_0} Q \Delta_s j \Delta_{1-s} A : Y \longrightarrow l_\infty \longrightarrow l_r \hookrightarrow c_0 \longrightarrow l_1 \longrightarrow Y_0 \longrightarrow Y.$$

Let $U_0 := Q\Delta_s j \Delta_{1-s} A : Y \to Y_0$. Then $U_0 \in N_s(Y, Y_0)$, $U := U_0 \Phi|_{Y_0} \in N_s(Y_0, Y_0)$ and $T = \Phi|_{Y_0} U \in N_s(Y, Y)$. By the principle of related operators (see [8], 6.4.3.2), U and T have the same eigenvalues with the same algebraic multiplicities. But U acts in a subspace Y_0 of an L_p-space, so the main Lemma of [13] can be applied. Therefore, by Lemma 12, Lemma 13 is proved. ∎

Corollary 14. *If $s \in (0, 1]$, $p \in [1, \infty]$ with $1/s = 1 + |1/2 - 1/p|$ then the quasi-normed ideal $\Psi_{p,s}$ is of (spectral) type l_1.*

4. Proof of Theorem 1. We prefer to give here a complete proof although we could just refer to the proof of the corresponding theorem in [13] with giving some remarks.

Let Y be a subspace of a quotient of an L_p-space and $T \in N_s(Y, Y)$. By Corollary 10, we may (and do) identify the space $N_s(Y, Y)$ with the corresponding tensor product $Y^* \widehat{\otimes}_s Y$, which, in turn, is a subspace of the projective tensor product $Y^* \widehat{\otimes} Y$. Thus, the nuclear trace of T is well defined, and we have to show that this trace of T is just the spectral trace (= spectral sum) $\sum_{n=1}^\infty \lambda_n(T)$.

By Lemma 13, the sequence $\{\lambda_n(T)\}_{n=1}^\infty$ of all eigenvalues of T, counting by multiplicities, is in l_1. Since the quasi-normed ideal $\Psi_{p,s}$ is of spectral (= eigenvalue) type l_1 (see Corollary 14), we can apply the main result from the paper [14] of M. C. White, which asserts:

(∗∗) *If J is a quasi-Banach operator ideal with eigenvalue type l_1, then the spectral sum is a trace on that ideal J.*

For the sake of completeness and to simplify the understanding, we (as in the paper [13]) give here some information about "trace" on an operator ideal. Namely, recall (see [8], 6.5.1.1, or Definition 2.1 in [14]) that a *trace* on an operator ideal J is a class of complex-valued functions, all of which one writes as τ, one for each component $J(E, E)$, where E is a Banach space, so that

(i) $\tau(e' \otimes e) = \langle e', e\rangle$ for all $e' \in E^*$, $e \in E$;
(ii) $\tau(AU) = \tau(UA)$ for all Banach spaces F and operators $U \in J(E, F)$ and $A \in L(F, E)$;
(iii) $\tau(S + U) = \tau(S) + \tau(U)$ for all $S, U \in J(E, E)$;
(iv) $\tau(\lambda U) = \lambda\tau(U)$ for all $\lambda \in C$ and $U \in J(E, E)$.

Our operator T, evidently, belongs to the space $\Psi_{p,s}(Y, Y)$ and, as was said, $\Psi_{p,s}$ is of eigenvalue type l_1. Thus, the assertion (∗∗) implies that the spectral sum λ, defined by $\lambda(U) := \sum_{n=1}^\infty \lambda_n(U)$ for $U \in \Psi_{p,s}(E, E)$, is a trace on $\Psi_{p,s}$.

By the principle of uniform boundedness (see [7], 3.4.6, page 152, or [5]), there exists a constant $C > 0$ with the property that

$$|\lambda(U)| \leq \|\{\lambda_n(U)\}\|_{l_1} \leq C\nu_{p,s}(U)$$

for all Banach spaces E and operators $U \in \Psi_{p,s}(E, E)$.

Now, remembering that all operators in $\Psi_{p,s}$ can be approximated by finite rank operators and taking into account the conditions (iii)–(iv) for $\tau = \lambda$, we deduce that the nuclear trace of our operator T coincides with $\lambda(T)$ (recall that the continuous trace is uniquely defined in such a situation; see [8], 6.5.1.2).

Acknowledgments. The research was supported by the Higher Education Commission of Pakistan.

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