

# THE CHARACTERIZATION OF RADIAL SUBSPACES OF BESOV AND LIZORKIN–TRIEBEL SPACES BY DIFFERENCES

WINFRIED SICKEL

*Institute of Mathematics, Friedrich-Schiller-University Jena  
Ernst-Abbe-Platz 2, 07743 Jena, Germany  
E-mail: winfried.sickel@uni-jena.de*

LESZEK SKRZYPCZAK

*Institute of Mathematics, Adam Mickiewicz University  
Umultowska 87, 61-614 Poznań, Poland  
E-mail: lskrzyp@amu.edu.pl*

JAN VYBÍRAL

*Technical University Berlin, Department of Mathematics, Secretary office MA 4-1  
Street of 17. June 136, 10623 Berlin, Germany  
E-mail: vybiral@math.tu-berlin.de*

**Abstract.** We continue our earlier investigations (see [SSV, SS2]) of radial subspaces of Besov and Lizorkin–Triebel spaces on  $\mathbb{R}^d$ . This time we study characterizations of these subspaces by differences.

**1. Introduction.** The study of radial subspaces of spaces of differentiable functions was initiated at the end of the seventies by papers of Strauss [St1], Coleman, Glazer and Martin [CGM], Berestycki and Lions [BL] and Lions [L]. It became clear that the symmetry of a function in a Sobolev-type space context implies several remarkable properties, which do not extend to arbitrary functions in those spaces. For example, the *Radial Lemma* of Strauss states that every radial function  $f \in H^1(\mathbb{R}^d)$ ,  $d \geq 2$ , is almost everywhere equal

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to a function  $\tilde{f}$ , continuous for  $x \neq 0$ , such that

$$|\tilde{f}(x)| \leq c |x|^{(1-d)/2} \|f\|_{H^1(\mathbb{R}^d)}, \quad x \in \mathbb{R}^d \setminus \{0\}, \quad (1)$$

where  $c$  depends only on  $d$ . The main aim of our papers [SS1, SSV, SS2] has been the study of the decay and regularity properties of radial functions (very much in the spirit of the Radial Lemma of Strauss) in the more general framework of Besov and Lizorkin–Triebel spaces. Let us refer also to the recent paper of Cho and Ozawa [CO] for some extensions of the Radial Lemma. The philosophy used in our earlier papers consists in the fact that all the information about a (locally integrable) radial function  $f : \mathbb{R}^d \rightarrow \mathbb{C}$  is contained in its trace

$$(\operatorname{tr} f)(t) := f(t, 0, \dots, 0), \quad t \in \mathbb{R}. \quad (2)$$

The corresponding extension operator is then given by

$$(\operatorname{ext} g)(x) = g(|x|), \quad x \in \mathbb{R}^d, \quad (3)$$

where  $g$  is an even function on  $\mathbb{R}$ . In [SSV] and [SS2] we have investigated properties of the operators  $\operatorname{tr}$  and  $\operatorname{ext}$  in the frame of both homogeneous and inhomogeneous Besov and Lizorkin–Triebel spaces. Here we have extensively used adapted atomic decompositions characterizing radial subspaces. Afterwards this has been applied to describe the interesting interplay between regularity and decay properties of radial functions (or distributions) from these classes. In case of Besov and Lizorkin–Triebel spaces on  $\mathbb{R}^d$  probably the most transparent characterizations are given in terms of differences. One could ask whether the proofs, given in [SSV, SS2], would become more transparent by working with differences. For that reason we shall derive a characterization by differences for the traces of elements of  $RF_{p,q}^s(\mathbb{R}^d)$  and  $RB_{p,q}^s(\mathbb{R}^d)$ . Not only for technical reasons we shall restrict ourselves to spaces with values of  $s$  strictly less than 1, see the detailed comments in Subsection 3.2. It turns out that even in this simplified situation (because of  $s < 1$  it will be enough to work with first order differences) the characterizations we have found do not seem to be a convenient tool to derive those properties of radial functions as stated in the Lemma of Strauss. However, under certain extra conditions on the parameters the spaces  $\operatorname{tr}(RF_{p,q}^s(\mathbb{R}^d))$  and  $\operatorname{tr}(RB_{p,q}^s(\mathbb{R}^d))$  allow an interpretation as weighted Besov and Lizorkin–Triebel spaces, where the weight is given by  $w(t) := |t|^{d-1}$ ,  $t \in \mathbb{R}$ , see [SSV]. Hence, our characterizations of  $\operatorname{tr}(RF_{p,q}^s(\mathbb{R}^d))$  and  $\operatorname{tr}(RB_{p,q}^s(\mathbb{R}^d))$  in terms of differences may be understood as a first hint how complicated those characterizations of weighted Besov and Lizorkin–Triebel spaces may look like in case of weights with singularities.

The paper is organized as follows. In Section 2 we derive the announced characterization by differences of the spaces  $\operatorname{tr}(RF_{p,q}^s(\mathbb{R}^d))$ . The next section is used for establishing characterization by differences of  $\operatorname{tr}(RB_{p,q}^s(\mathbb{R}^d))$ . Section 4 is devoted to a comparison of  $\operatorname{tr}(RF_{p,q}^s(\mathbb{R}^d))$  and  $\operatorname{tr}(RB_{p,q}^s(\mathbb{R}^d))$  with some weighted spaces on the real line. Finally, in the Appendix at the end of the paper we collect some definitions and properties of the function spaces under consideration.

**Notation.** As usual,  $\mathbb{N}$  denotes the natural numbers,  $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ ,  $\mathbb{Z}$  denotes the integers and  $\mathbb{R}$  the real numbers. For the complex numbers we use the symbol  $\mathbb{C}$ , for the Euclidean  $d$ -space we use  $\mathbb{R}^d$ , and  $\mathbb{Z}^d$  denotes the collection of all elements in  $\mathbb{R}^d$  having

integer components. Many times we shall use the abbreviation

$$\sigma_{p,q}(d) := d \max\left(0, \frac{1}{p} - 1, \frac{1}{q} - 1\right). \tag{4}$$

The symbol  $\sigma_{d-1}$  stands for the usual  $(d - 1)$ -dimensional Hausdorff measure in  $\mathbb{R}^d$ .

At very few places we shall need the Fourier transform  $\mathcal{F}$  as well as its inverse transformation  $\mathcal{F}^{-1}$ , always defined on the Schwartz space  $\mathcal{S}'(\mathbb{R}^d)$  of tempered distributions.

If  $X$  and  $Y$  are two quasi-Banach spaces, then the symbol  $X \hookrightarrow Y$  indicates that the embedding is continuous. The set of all linear and bounded operators  $T : X \rightarrow Y$ , denoted by  $\mathcal{L}(X, Y)$ , is equipped with the standard quasi-norm. As usual, the symbol  $c$  denotes positive constants which depend only on the fixed parameters  $s, p, q$  and probably on auxiliary functions, unless otherwise stated; its value may vary from line to line. Sometimes we will use the symbols “ $\lesssim$ ” and “ $\gtrsim$ ” instead of “ $\leq$ ” and “ $\geq$ ”, respectively. The meaning of  $A \lesssim B$  is given by: there exists a constant  $c > 0$  such that  $A \leq cB$ . Similarly  $\gtrsim$  is defined. The symbol  $A \asymp B$  will be used as an abbreviation of  $A \lesssim B \lesssim A$ .

If  $E$  denotes a space of functions on  $\mathbb{R}^d$  then by  $RE$  we mean the subset of radial functions in  $E$  and we endow this subset with the same quasi-norm as the original space. Inhomogeneous Besov and Lizorkin–Triebel spaces are denoted by  $B_{p,q}^s$  and  $F_{p,q}^s$ , respectively. Definitions as well as some references are given in the Appendix.

**2. The characterization of radial Lizorkin–Triebel spaces by differences.** In

case of first order Sobolev spaces  $RW_p^1(\mathbb{R}^d)$  there is a simple characterization of the trace spaces, see [SSV]. Let  $d \geq 2$  and  $1 \leq p < \infty$ . The mapping  $\text{tr}$  is a linear isomorphism (with inverse  $\text{ext}$ ) of  $RW_p^1(\mathbb{R}^d)$  onto the closure of  $RC_0^\infty(\mathbb{R})$  with respect to the norm

$$\|g\|_{L_p(\mathbb{R}, |t|^{d-1})} + \|g'\|_{L_p(\mathbb{R}, |t|^{d-1})}. \tag{5}$$

As usual,

$$\|g\|_{L_p(\mathbb{R}, |t|^{d-1})} := \left( \int_{-\infty}^{\infty} |g(t)|^p |t|^{d-1} dt \right)^{1/p}. \tag{6}$$

The characterization of  $\text{tr}(RW_p^1(\mathbb{R}^d))$  makes clear what type of result can be expected in the general situation of Lizorkin–Triebel spaces: we shall look for a characterization as a weighted Lizorkin–Triebel space on  $\mathbb{R}$ , where the weight is given by  $w(t) := |t|^{d-1}$ ,  $t \in \mathbb{R}$ . However, the outcome will be a bit more technical.

**2.1. Some preliminaries.** Now we look for a counterpart of (5) in case of the radial Lizorkin–Triebel spaces  $RF_{p,q}^s(\mathbb{R}^d)$ . Recall in this context that  $W_p^1(\mathbb{R}^d) = F_{p,2}^1(\mathbb{R}^d)$ ,  $1 < p < \infty$ , in the sense of equivalent norms. Our point of departure is the following characterization of  $F_{p,q}^s(\mathbb{R}^d)$ , for which we refer to [T2, Thm. 3.5.3]. We shall use the abbreviations

$$M_{t,u}f(x) := \left( t^{-d} \int_{|h|<t} |f(x+h) - f(x)|^u dh \right)^{1/u}, \quad t > 0, \quad 0 < u < \infty,$$

and

$$M_{t,\infty}f(x) := \sup_{|h|<t} |f(x+h) - f(x)|, \quad t > 0.$$

Suppose  $0 < p < \infty$ ,  $0 < q \leq \infty$ ,  $1 \leq v \leq \infty$  and

$$s > d \max\left(0, \frac{1}{p} - \frac{1}{v}, \frac{1}{q} - \frac{1}{v}\right). \tag{7}$$

Let  $0 < u \leq v$ ,  $s < 1$  and  $T > 0$ . Then  $F_{p,q}^s(\mathbb{R}^d)$  is the collection of all  $f \in L_{\max(p,v)}(\mathbb{R}^d)$  such that

$$\|f\|_{F_{p,q}^s(\mathbb{R}^d)}^* := \|f\|_{L_p(\mathbb{R}^d)} + \left\| \left( \int_0^T t^{-sq} (M_{t,u} f(\cdot))^q \frac{dt}{t} \right)^{1/q} \Big|_{L_p(\mathbb{R}^d)} \right\|. \tag{8}$$

Moreover,  $\|\cdot\|_{F_{p,q}^s(\mathbb{R}^d)}^*$  is equivalent to  $\|\cdot\|_{F_{p,q}^s(\mathbb{R}^d)}$  on  $L_{\max(p,v)}(\mathbb{R}^d)$ .

REMARK 2.1.

- (i) By taking  $v = 1$  it becomes obvious that we have a characterization with first order differences as long as  $\sigma_{p,q}(d) < s < 1$ , see (4).
- (ii) There are many references dealing with the characterization of Lizorkin–Triebel spaces by differences (partly in a different form than here), we refer in particular to Strichartz [Str] ( $1 < p < \infty$ ,  $q = 2$ ), Seeger [S] and Triebel [T1, 2.5].

We need a modification of the above characterization. Let  $0 < A < B < \infty$  be fixed real numbers. Let  $\Omega_t(x)$ ,  $t > 0$ ,  $x \in \mathbb{R}^d$ , be a family of open sets in  $\mathbb{R}^d$  such that

$$\{h \in \mathbb{R}^d : |h| < At\} \subset \Omega_t(x) \subset \{h \in \mathbb{R}^d : |h| < Bt\}. \tag{9}$$

We define

$$M_{t,u}^\Omega f(x) := \left( t^{-d} \int_{\Omega_t(x)} |f(x+h) - f(x)|^u dh \right)^{1/u}, \quad t > 0, \quad 0 < u \leq \infty.$$

Then we have the following.

LEMMA 2.2. *Suppose  $0 < p < \infty$ ,  $0 < q \leq \infty$ ,  $1 \leq v \leq \infty$ ,  $0 < u \leq v$ ,*

$$d \max\left(0, \frac{1}{p} - \frac{1}{v}, \frac{1}{q} - \frac{1}{v}\right) < s < 1 \tag{10}$$

*and  $T > 0$ . Let  $\Omega_t(x)$ ,  $t > 0$ ,  $x \in \mathbb{R}^d$ , be a family of open sets in  $\mathbb{R}^d$  such that (9) is satisfied for some  $0 < A < B < \infty$ . Then  $F_{p,q}^s(\mathbb{R}^d)$  is the collection of all  $f \in L_{\max(p,v)}(\mathbb{R}^d)$  such that*

$$\|f\|_{F_{p,q}^s(\mathbb{R}^d)}^* := \|f\|_{L_p(\mathbb{R}^d)} + \left\| \left( \int_0^T t^{-sq} (M_{t,u}^\Omega f(\cdot))^q \frac{dt}{t} \right)^{1/q} \Big|_{L_p(\mathbb{R}^d)} \right\|. \tag{11}$$

*Moreover,  $\|\cdot\|_{F_{p,q}^s(\mathbb{R}^d)}^*$  is equivalent to  $\|\cdot\|_{F_{p,q}^s(\mathbb{R}^d)}$  on  $L_{\max(p,v)}(\mathbb{R}^d)$ .*

*Proof.* As a first step one can prove the lemma in case  $\Omega_t(x) = \{h \in \mathbb{R}^d : |h| \leq Ct\}$  for a fixed  $C > 0$  by following the original proof in [T2, 3.5.3]. Afterwards the claim follows by an obvious monotonicity argument. ■

**2.2. Differences and radial Lizorkin–Triebel spaces.** We start with an elementary observation. Let  $f$  be a radial function in  $L_p(\mathbb{R}^d)$ . Then the mapping  $\text{tr}$ , see (2), is an isomorphism onto the space  $RL_p(\mathbb{R}, |t|^{d-1})$  with inverse  $\text{ext}$ . Hence,  $\text{tr} f$  is well-defined for any  $f \in RF_{p,q}^s(\mathbb{R}^d)$  such that  $s > \sigma_{p,p}(d)$ . This is a consequence of the Sobolev type

embedding

$$F_{p,q}^s(\mathbb{R}^d) \hookrightarrow L_u(\mathbb{R}^d), \quad p \leq u \leq \begin{cases} \frac{d}{d/p-s} & \text{if } s < d/p; \\ \infty & \text{if } s > d/p, \end{cases}$$

see, e.g., [T1, 2.7.1]. The main aim of this section is to establish characterization by differences of radial subspaces of Lizorkin–Triebel spaces by using  $\text{tr } f$  instead of  $f$  itself. To begin with we concentrate on the use of the means  $M_{t,\infty}^\Omega$ .

**THEOREM 2.3.** *Let  $d \geq 2$ ,  $0 < p < \infty$ ,  $0 < q \leq \infty$  and*

$$d \max\left(\frac{1}{p}, \frac{1}{q}\right) < s < 1. \tag{12}$$

*Then the radial function  $f \in L_p(\mathbb{R}^d)$  belongs to  $F_{p,q}^s(\mathbb{R}^d)$  if and only if  $g := \text{tr } f$  satisfies*

$$\begin{aligned} \|g\|^\# &:= \|g\|_{L_p(\mathbb{R}, |t|^{d-1})} \\ &+ \left( \int_{-\infty}^\infty |r|^{d-1} \left[ \int_0^1 t^{-sq} \sup_{-t \leq w \leq t} |g(r+w) - g(r)|^q \frac{dt}{t} \right]^{p/q} dr \right)^{1/p} < \infty. \end{aligned}$$

*Moreover,  $\|g\|^\#$  is equivalent to  $\|f\|_{RF_{p,q}^s(\mathbb{R}^d)}$ .*

*Proof. Step 1. Preparations.* By  $\langle x, y \rangle$  we denote the scalar product of  $x, y \in \mathbb{R}^d$ . We are going to use Lemma 2.2 with a particular family  $\Omega_t(x)$ . In case  $t = 1$  we put

$$\Omega_1(x) := \{h \in \mathbb{R}^d : |x + h| \leq 2\} \quad \text{if } |x| \leq 1 \tag{13}$$

and

$$\begin{aligned} \Omega_1(x) &:= \left\{ h \in \mathbb{R}^d : \exists \tau > 0 \exists y \in \mathbb{R}^d \text{ such that } x + h = \tau y, \right. \\ &\left. |y| = |x|, \quad \langle x, y \rangle > |x|^2 - \frac{1}{2} \text{ and } |x| - \frac{1}{2} < \tau|x| < |x| + \frac{1}{2} \right\} \tag{14} \end{aligned}$$

for  $|x| > 1$ .

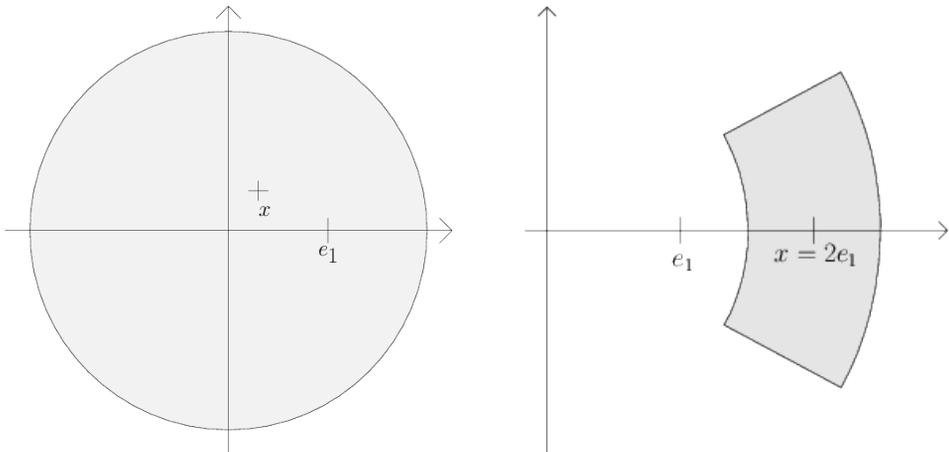


Fig. 1. The set  $x + \Omega_1(x)$  for  $|x| \leq 1$  (left) and  $|x| > 1$  (right).

For the general definition we make use of scaling, i.e., we define

$$h \in \Omega_1(x) \iff th \in \Omega_t(x), \quad t > 0. \quad (15)$$

A scaling argument yields that the family  $(\Omega_t(x))_{t,x}$  satisfies (9) if and only if the subfamily  $(\Omega_1(x))_{1,x}$  satisfies (9) (with  $t = 1$ ). Now we investigate the subfamily with  $t = 1$ . The condition (9) is obviously satisfied in case  $|x| \leq 1$ . To prove it also for  $|x| > 1$ , we argue as follows. Let  $h \in \Omega_1(x)$ . Clearly,  $1/2 < \tau < 3/2$  and

$$|h| = |\tau y - x| = |\tau x - x + \tau y - \tau x| \leq |x| \cdot |\tau - 1| + \tau|x - y| \leq \frac{1}{2} + \frac{3}{2} \cdot |x - y| \leq 2,$$

where the last step in the estimate follows from

$$|x - y|^2 = 2|x|^2 - 2\langle x, y \rangle \leq 2|x|^2 - 2|x|^2 + 2 \cdot 1/2 = 1.$$

This proves  $\Omega_1(x) \subset \{h \in \mathbb{R}^d : |h| \leq 2\}$ . Next we shall prove the relation  $\{h \in \mathbb{R}^d : |h| \leq 1/4\} \subset \Omega_1(x)$ . Let  $|h| \leq 1/4$ . We define

$$\tau := \frac{|x+h|}{|x|} \quad \text{and} \quad y := \frac{|x|}{|x+h|} \cdot (x+h).$$

Then  $|y| = |x|$  and

$$|x| - 1/2 < \tau|x| < |x| + 1/2$$

follows easily. Finally, we have to show that the conditions  $|x| \geq 1$  and  $|h| \leq 1/4$  imply

$$\left\langle x, \frac{x+h}{|x+h|} \right\rangle \cdot |x| \geq |x|^2 - \frac{1}{2}, \quad (16)$$

see (14). First we claim

$$\langle x, x+h \rangle \geq |x+h| \cdot \sqrt{|x|^2 - |h|^2}. \quad (17)$$

But

$$\begin{aligned} (\langle x, x+h \rangle)^2 &= |x|^4 + 2\langle x, h \rangle |x|^2 + \langle x, h \rangle^2 \\ &\geq |x|^4 + 2\langle x, h \rangle |x|^2 + |h|^2|x|^2 - |h|^2|x|^2 - |h|^4 - 2\langle x, h \rangle |h|^2 \\ &= \langle x+h, x+h \rangle (|x|^2 - |h|^2). \end{aligned}$$

Since  $\langle x, x+h \rangle > 0$  the claim (17) follows. Hence

$$\langle x, x+h \rangle \cdot \frac{|x|}{|x+h|} \geq |x| \cdot \sqrt{|x|^2 - |h|^2} \geq |x|^2 - \frac{1}{2},$$

since  $|x| \geq 1$ . This proves that our family  $(\Omega_1(x))_x$  and therefore  $(\Omega_t(x))_{t,x}$  satisfies (9) with  $A = 1/4$  and  $B = 3$ .

*Step 2.* Let  $\mathcal{A} : \mathbb{R}^d \rightarrow \mathbb{R}^d$  be a rotation around the origin. Hence,  $\mathcal{A}$  is a  $(d \times d)$  matrix. The mapping  $x \mapsto \mathcal{A}(x)$  is linear, length and angles are invariant under this mapping. Simple calculations show that  $\Omega_t(\mathcal{A}(x)) = \mathcal{A}(\Omega_t(x))$ .

Now, let  $x, z$  be two points in  $\mathbb{R}^d$  such that  $|x| = |z|$  and let  $\mathcal{A}$  be any rotation around the origin satisfying  $\mathcal{A}(x) = z$ . For the moment we shall deal with  $M_{t,u}^\Omega$  for all  $u$ ,

$0 < u \leq \infty$ . Let  $f \in RF_{p,q}^s(\mathbb{R}^d)$ . With  $g := \text{tr } f$  we conclude

$$\begin{aligned} M_{t,u}^\Omega f(z) &= \left( t^{-d} \int_{\Omega_t(z)} |g(|z+h|) - g(|z|)|^u dh \right)^{1/u} \\ &= \left( t^{-d} \int_{\Omega_t(\mathcal{A}(x))} |g(|\mathcal{A}(x)+h|) - g(|\mathcal{A}(x)|)|^u dh \right)^{1/u}. \end{aligned}$$

A transformation of coordinates  $h' := \mathcal{A}^{-1}(h)$  and  $|\det \mathcal{A}^{-1}| = 1$  yield

$$\begin{aligned} M_{t,u}^\Omega f(z) &= \left( t^{-d} \int_{\Omega_t(x)} |g(|\mathcal{A}(x+h')|) - g(|\mathcal{A}(x)|)|^u dh' \right)^{1/u} \\ &= \left( t^{-d} \int_{\Omega_t(x)} |g(|x+h'|) - g(|x|)|^u dh' \right)^{1/u} \\ &= M_{t,u}^\Omega f(x). \end{aligned}$$

Hence,  $M_{t,u}^\Omega f$  is a radial function.

*Step 3.* Let  $e_1 \in \mathbb{R}^d$  be the unit vector in the first direction. We employ Lemma 2.2, Step 1 and Step 2. For the radial function  $f \in RF_{p,q}^s(\mathbb{R}^d)$  and  $g := \text{tr } f$  this yields

$$\begin{aligned} \|f\|_{F_{p,q}^s(\mathbb{R}^d)} &\asymp \|g\|_{L_p(\mathbb{R}, |t|^{d-1})} \\ &\quad + \left( \int_0^\infty r^{d-1} \left[ \int_0^T t^{-sq} (M_{t,u}^\Omega f(re_1))^q \frac{dt}{t} \right]^{p/q} dr \right)^{1/p}. \end{aligned} \tag{18}$$

From now we concentrate on  $u = \infty$ . This causes the restrictions for  $s$  in (12). Then

$$M_{t,\infty}^\Omega f(re_1) = \sup_{h \in \Omega_t(re_1)} |f(re_1+h) - f(re_1)| \begin{cases} \leq \sup_{|w| \leq 3t} |g(r+w) - g(r)| \\ \geq \sup_{|w| \leq t/4} |g(r+w) - g(r)| \end{cases}$$

by using (9) and Step 1. Consequently,

$$\begin{aligned} &\|g\|_{L_p(\mathbb{R}, |t|^{d-1})} + \left( \int_0^\infty r^{d-1} \left[ \int_0^T t^{-sq} \left( \sup_{|w| \leq t/4} |g(r+w) - g(r)| \right)^q \frac{dt}{t} \right]^{p/q} dr \right)^{1/p} \\ &\lesssim \|f\|_{F_{p,q}^s(\mathbb{R}^d)} \lesssim \|g\|_{L_p(\mathbb{R}, |t|^{d-1})} \\ &\quad + \left( \int_0^\infty r^{d-1} \left[ \int_0^{T'} t^{-sq} \left( \sup_{|w| \leq 3t} |g(r+w) - g(r)| \right)^q \frac{dt}{t} \right]^{p/q} dr \right)^{1/p}, \end{aligned}$$

where we can choose  $T, T' > 0$  as we want. Using Lemma 2.2 with  $T = 1/3$  (estimate from below) and afterwards with  $T = 4$  (estimate from above) we conclude

$$\begin{aligned} \|f\|_{F_{p,q}^s(\mathbb{R}^d)} &\asymp \|g\|_{L_p(\mathbb{R}, |t|^{d-1})} \\ &\quad + \left( \int_0^\infty r^{d-1} \left[ \int_0^1 t^{-sq} \left( \sup_{|w| \leq t} |g(r+w) - g(r)| \right)^q \frac{dt}{t} \right]^{p/q} dr \right)^{1/p}. \end{aligned}$$

Since  $g$  is even we may replace  $\int_0^\infty r^{d-1} \dots dr$  by  $\int_{-\infty}^\infty |r|^{d-1} \dots dr$ . The proof is complete. ■

REMARK 2.4. The scale of Lizorkin–Triebel spaces generalizes three well-known scales of function spaces, namely Sobolev spaces, Bessel potential spaces and Slobodeckij spaces. In fact, we have

- $W_p^m(\mathbb{R}^d) = F_{p,2}^m(\mathbb{R}^d)$ ,  $1 < p < \infty$ ,  $m \in \mathbb{N}_0$ ;
- $H_p^s(\mathbb{R}^d) = F_{p,2}^s(\mathbb{R}^d)$ ,  $1 < p < \infty$ ,  $s \in \mathbb{R}$ ;
- $W_p^s(\mathbb{R}^d) = F_{p,p}^s(\mathbb{R}^d)$ ,  $1 \leq p < \infty$ ,  $s > 0$ ,  $s \notin \mathbb{N}$ ,

see, e.g., [T1, 2.2]. Hence, Theorem 2.3 also gives characterizations of  $RH_p^s(\mathbb{R})$ ,  $1 < p < \infty$ ,  $1/2 < s < 1$ , and of  $RW_p^s(\mathbb{R}^d)$ ,  $1 < p < \infty$ ,  $d/p < s < 1$ .

Of course, the restrictions in  $s$ , given in (12), are rather inconvenient. They are mainly caused by the use of the means  $M_{t,\infty}^\Omega$ . For this reason we turn now to the use of  $M_{t,u}^\Omega$ ,  $0 < u < \infty$ . Employing (9) we find

$$\begin{aligned} M_{t,u}^\Omega f(re_1) &\leq \left( t^{-d} \int_{|h| \leq 3t} |f(re_1 + h) - f(re_1)|^u dh \right)^{1/u} \\ &= \left( t^{-d} \int_{|w - re_1| \leq 3t} |g(|w|) - g(r)|^u dw \right)^{1/u} \\ &= \left( t^{-d} \int_{\max(0, r-3t)}^{r+3t} |g(\lambda) - g(r)|^u \sigma_{d-1}(Q_{\lambda,3t}(r)) d\lambda \right)^{1/u} \end{aligned}$$

where

$$Q_{\lambda,t}(r) := \{w \in \mathbb{R}^d : |w| = \lambda, |w - re_1| \leq t\}. \tag{19}$$

Similarly, also by using (9), we obtain some sort of reverse inequality

$$M_{t,u}^\Omega f(re_1) \geq \left( t^{-d} \int_{\max(0, r-t/4)}^{r+t/4} |g(\lambda) - g(r)|^u \sigma_{d-1}(Q_{\lambda,t/4}(r)) d\lambda \right)^{1/u}.$$

By obvious manipulations with  $T$  this proves the following lemma.

LEMMA 2.5. *Let  $d \geq 2$ ,  $0 < p < \infty$ ,  $0 < q \leq \infty$ ,  $1 \leq v < \infty$ ,  $0 < u \leq v$  and*

$$d \max\left(0, \frac{1}{p} - \frac{1}{v}, \frac{1}{q} - \frac{1}{v}\right) < s < 1. \tag{20}$$

*Let  $T > 0$ . Then the radial function  $f \in L_p(\mathbb{R}^d)$  belongs to  $F_{p,q}^s(\mathbb{R}^d)$  if and only if  $g := \text{tr } f$  satisfies*

$$\begin{aligned} \|g\|^\Delta := \|g|_{L_p(\mathbb{R}, |t|^{d-1})}\| + \left( \int_0^\infty r^{d-1} \left[ \int_0^T t^{-sq} \left( t^{-d} \int_{\max(0, r-t)}^{r+t} |g(\lambda) - g(r)|^u \sigma_{d-1}(Q_{\lambda,t}(r)) d\lambda \right)^{q/u} \frac{dt}{t} \right]^{p/q} dr \right)^{1/p} < \infty. \end{aligned} \tag{21}$$

Moreover,  $\|g\|^\Delta$  is equivalent to  $\|f|_{F_{p,q}^s(\mathbb{R}^d)}\|$ .

**The three-dimensional case.** To make use of (21) we need to know  $\sigma_{d-1}(Q_{\lambda,t}(r))$ . For  $d = 3$  this quantity is explicitly known. We have

$$\sigma_2(Q_{\lambda,t}(r)) = \begin{cases} \frac{\pi\lambda}{r} (t^2 - (\lambda - r)^2) & \text{if } \max(0, r - t) < \lambda < r + t, t \leq r + \lambda; \\ 4\pi\lambda^2 & \text{if } t > r + \lambda; \\ 0 & \text{otherwise.} \end{cases} \tag{22}$$

We need some more notation. Depending on the behaviour of  $\sigma_2(Q_{\lambda,t}(r))$  we introduce the following splitting of the area of integration.

$$I_0(r, t) := \{\lambda \geq 0 : \lambda < t - r\}; \tag{23}$$

$$I_1(r, t) := \{\lambda \geq 0 : |\lambda - r| < \frac{3}{4}t, t - r \leq \lambda\}; \tag{24}$$

$$I_2(r, t) := \{\lambda \geq 0 : r + \frac{3}{4}t \leq \lambda < r + t, t - r \leq \lambda\}; \tag{25}$$

$$I_3(r, t) := \{\lambda \geq 0 : |r - t| \leq \lambda \leq r - \frac{3}{4}t\}. \tag{26}$$

Obviously it follows

$$(\max(0, r - t), r + t) \cap [t - r, \infty) = I_1(r, t) \cup I_2(r, t) \cup I_3(r, t).$$

On  $I_1$  we can use

$$\frac{7}{16}t^2 \leq (t^2 - (\lambda - r)^2) \leq t^2. \tag{27}$$

Because of

$$\frac{7}{4}t \leq t + \lambda - r \leq 2t, \quad \lambda \in I_2,$$

and

$$\frac{7}{4}t \leq t - \lambda + r \leq 2t, \quad \lambda \in I_3,$$

we obtain

$$\frac{7}{4}t(t - \lambda + r) \leq (t^2 - (\lambda - r)^2) \leq 2t(t - \lambda + r), \quad \lambda \in I_2, \tag{28}$$

and

$$\frac{7}{4}t(t + \lambda - r) \leq (t^2 - (\lambda - r)^2) \leq 2t(t + \lambda - r), \quad \lambda \in I_3. \tag{29}$$

Inserting (27), (28) and the value of  $\sigma_2$  in case  $\lambda < t - r$  into (21), we get the following characterization of  $RF_{p,q}^s(\mathbb{R}^3)$  in terms of differences.

**THEOREM 2.6.** *Let  $0 < p < \infty, 0 < q \leq \infty, 1 \leq v < \infty, 0 < u \leq v$  and*

$$3 \max\left(0, \frac{1}{p} - \frac{1}{v}, \frac{1}{q} - \frac{1}{v}\right) < s < 1.$$

*Then the radial function  $f \in L_p(\mathbb{R}^3)$  belongs to  $F_{p,q}^s(\mathbb{R}^3)$  if and only if  $g := \text{tr } f$  satisfies*

$$\begin{aligned} \|g\|^\Delta &:= \left(\int_0^\infty r^2 |g(r)|^p dr\right)^{1/p} \\ &+ \left(\int_0^1 r^2 \left[\int_r^1 t^{-sq} \left(t^{-3} \int_{I_0(r,t)} \lambda^2 |g(\lambda) - g(r)|^u d\lambda\right)^{q/u} \frac{dt}{t} \right]^{p/q} dr\right)^{1/p} \\ &+ \left(\int_0^\infty r^{2-p/u} \left[\int_0^1 t^{-sq} \left(t^{-1} \int_{I_1(r,t)} \lambda |g(\lambda) - g(r)|^u d\lambda\right)^{q/u} \frac{dt}{t} \right]^{p/q} dr\right)^{1/p} \\ &+ \left(\int_0^\infty r^{2-p/u} \left[\int_0^1 t^{-sq} \left(t^{-2} \int_{I_2(r,t)} \lambda(t - \lambda + r) |g(\lambda) - g(r)|^u d\lambda\right)^{q/u} \frac{dt}{t} \right]^{p/q} dr\right)^{1/p} \\ &+ \left(\int_0^\infty r^{2-p/u} \left[\int_0^1 t^{-sq} \left(t^{-2} \int_{I_3(r,t)} \lambda(t + \lambda - r) |g(\lambda) - g(r)|^u d\lambda\right)^{q/u} \frac{dt}{t} \right]^{p/q} dr\right)^{\frac{1}{p}} < \infty. \end{aligned}$$

Moreover,  $\|g\|^\Delta$  is equivalent to  $\|f\|_{F_{p,q}^s(\mathbb{R}^3)}$ .

REMARK 2.7.

- (i) The characterization of  $RF_{p,q}^s(\mathbb{R}^3)$  as given above seems to be of limited use because of its complexity. From our point of view the value of Theorem 2.6 consists in exactly this negative observation. However, (21) can be used to derive an embedding into weighted spaces, see Subsection 4.2 below.
- (ii) The formula (22) for  $\sigma_2(Q_{\lambda,t}(r))$  plays a role when calculating the solution of the following wave equation in three dimensions

$$\begin{aligned} u_{tt}(x, t) &= c^2 \Delta u(x, t), & x \in \mathbb{R}^3, \quad t > 0, \\ u(x, 0) &= 0, & x \in \mathbb{R}^3, \\ u_t(x, 0) &= \begin{cases} 1 & \text{if } |x| < \varrho, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

We refer to the textbooks of Strauss [St2, 9.2, Exercise 6] and of Drábek, Holubová [DH, 13.7, Exercise 6] in this connection.

**The two-dimensional case.** Elementary calculations, based on the law of Cosines, lead to the following formula in case  $d = 2$ :

$$\sigma_1(Q_{\lambda,t}(r)) = \begin{cases} 2\lambda \arccos\left(\frac{r^2 + \lambda^2 - t^2}{2r\lambda}\right) & \text{if } \max(0, r - t) < \lambda < r + t, \quad t \leq r + \lambda; \\ 2\pi\lambda & \text{if } t > r + \lambda; \\ 0 & \text{otherwise.} \end{cases} \tag{30}$$

According to the behaviour of  $2\lambda \arccos\left(\frac{r^2 + \lambda^2 - t^2}{2r\lambda}\right)$  one has to split the area of integration in order to simplify (21). We omit details.

**3. The characterization of radial Besov spaces by differences.** Our strategy is the same as in case of Lizorkin–Triebel spaces.

**3.1. Differences and radial Besov spaces.** The point of departure is the following counterpart of Lemma 2.2.

LEMMA 3.1. *Suppose  $0 < p, q \leq \infty$ ,  $1 \leq v \leq \infty$ ,  $0 < u \leq v$ ,*

$$d \max\left(0, \frac{1}{p} - \frac{1}{v}\right) < s < 1 \tag{31}$$

*and  $T > 0$ . Let  $\Omega_t(x)$ ,  $t > 0$ ,  $x \in \mathbb{R}^d$ , be a family of open sets in  $\mathbb{R}^d$  such that (9) is satisfied for some  $0 < A < B < \infty$ . Then  $B_{p,q}^s(\mathbb{R}^d)$  is the collection of all  $f \in L_{\max(p,v)}(\mathbb{R}^d)$  such that*

$$\|f\|_{B_{p,q}^s(\mathbb{R}^d)}^* := \|f\|_{L_p(\mathbb{R}^d)} + \left(\int_0^T t^{-sq} \|(M_{t,u}^\Omega f(\cdot))\|_{L_p(\mathbb{R}^d)}^q \frac{dt}{t}\right)^{1/q}. \tag{32}$$

*Moreover,  $\|\cdot\|_{B_{p,q}^s(\mathbb{R}^d)}^*$  is equivalent to  $\|\cdot\|_{B_{p,q}^s(\mathbb{R}^d)}$  on  $L_{\max(p,v)}(\mathbb{R}^d)$ .*

*Proof.* As in case of Lemma 2.2 the last assertion represents an easy modification of Theorem 3.5.3(ii) in [T2, 3.5.3]. ■

Again a Sobolev-type embedding

$$B_{p,q}^s(\mathbb{R}^d) \hookrightarrow L_u(\mathbb{R}^d), \quad p \leq u < \begin{cases} \frac{d}{d/p-s} & \text{if } s < d/p; \\ \infty & \text{if } s > d/p, \end{cases}$$

see, e.g., [T1, 2.7.1], guarantees that  $\text{tr}$  is well-defined on  $RB_{p,q}^s(\mathbb{R}^d)$  if  $s > \sigma_{p,p}(d)$ . Now we are making use of the same arguments as in case of Lizorkin–Triebel spaces. The radiality of  $M_{t,u}^\Omega f$  and the obvious counterpart of Step 3 of the proof of Theorem 2.3 yield the following characterization of radial Besov spaces.

**THEOREM 3.2.** *Let  $d \geq 2$ ,  $0 < p, q \leq \infty$  and  $d/p < s < 1$ . Then the radial function  $f \in L_p(\mathbb{R}^d)$  belongs to  $B_{p,q}^s(\mathbb{R}^d)$  if and only if  $g := \text{tr } f$  satisfies*

$$\|g\|^\# := \|g|L_p(\mathbb{R}, |t|^{d-1})\| + \left( \int_0^1 t^{-sq} \left[ \int_{-\infty}^\infty |r|^{d-1} (t^{-1} \sup_{|w| \leq t} |g(r+w) - g(r)|)^p dr \right]^{q/p} \frac{dt}{t} \right)^{1/q} < \infty.$$

Moreover,  $\|g\|^\#$  is equivalent to  $\|f|B_{p,q}^s(\mathbb{R}^d)\|$ .

The same arguments, leading to Lemma 2.5, may be used to derive its counterpart for Besov spaces.

**LEMMA 3.3.** *Let  $d \geq 2$ ,  $0 < p, q \leq \infty$ ,  $1 \leq v < \infty$ ,  $0 < u \leq v$  and*

$$d \max\left(0, \frac{1}{p} - \frac{1}{v}\right) < s < 1.$$

*Let  $T > 0$ . Then the radial function  $f \in L_p(\mathbb{R}^d)$  belongs to  $B_{p,q}^s(\mathbb{R}^d)$  if and only if  $g := \text{tr } f$  satisfies*

$$\|g\|^\Delta := \|g|L_p(\mathbb{R}, |t|^{d-1})\| + \left( \int_0^T t^{-sq} \left[ \int_0^\infty r^{d-1} \left( t^{-d} \int_{\max(0, r-t)}^{r+t} |g(\lambda) - g(r)|^u \sigma_{d-1}(Q_{\lambda,t}(r)) d\lambda \right)^{p/u} dr \right]^{q/p} \frac{dt}{t} \right)^{1/q} < \infty. \quad (33)$$

Moreover,  $\|g\|^\Delta$  is equivalent to  $\|f|B_{p,q}^s(\mathbb{R}^d)\|$ .

By restricting to  $d = 3$  and by employing the splitting from (23)–(26) this results in the following.

**THEOREM 3.4.** *Let  $0 < p, q \leq \infty$ ,  $1 \leq v < \infty$ ,  $0 < u \leq v$  and*

$$3 \max\left(0, \frac{1}{p} - \frac{1}{v}\right) < s < 1.$$

*Then the radial function  $f \in L_p(\mathbb{R}^3)$  belongs to  $B_{p,q}^s(\mathbb{R}^3)$  if and only if  $g := \text{tr } f$  satisfies*

$$\|g\|^\Delta := \left( \int_0^\infty r^2 |g(r)|^p dr \right)^{1/p} + \left( \int_0^1 t^{-sq} \left[ \int_0^t r^2 \left( t^{-3} \int_{I_0(r,t)} \lambda^2 |g(\lambda) - g(r)|^u d\lambda \right)^{p/u} dr \right]^{q/p} \frac{dt}{t} \right)^{1/q} + \left( \int_0^1 t^{-sq} \left[ \int_0^\infty r^{2-p/u} \left( t^{-1} \int_{I_1(r,t)} \lambda |g(\lambda) - g(r)|^u d\lambda \right)^{p/u} dr \right]^{q/p} \frac{dt}{t} \right)^{1/q}$$

$$\begin{aligned}
 &+ \left( \int_0^1 t^{-sq} \left[ \int_0^\infty r^{2-p/u} \left( t^{-2} \int_{I_2(r,t)} \lambda(t-\lambda+r) |g(\lambda) - g(r)|^u d\lambda \right)^{p/u} dr \right]^{q/p} \frac{dt}{t} \right)^{1/q} \\
 &+ \left( \int_0^1 t^{-sq} \left[ \int_0^\infty r^{2-p/u} \left( t^{-2} \int_{I_3(r,t)} \lambda(t+\lambda-r) |g(\lambda) - g(r)|^u d\lambda \right)^{p/u} dr \right]^{q/p} \frac{dt}{t} \right)^{\frac{1}{q}} < \infty.
 \end{aligned}$$

Moreover,  $\|g\|^\Delta$  is equivalent to  $\|f\|_{B_{p,q}^s(\mathbb{R}^3)}$ .

**3.2. Some concluding remarks.** Here we shall indicate the main problems connected with a desirable extension of Theorem 2.3 and Theorem 3.2, respectively.

The smoothness  $s > 1$  requires the use of higher order differences. Let  $N \in \mathbb{N}$  and  $x \in \mathbb{R}^d$ . Then the  $N$ -th order difference of  $f$  in  $x$  is defined to be

$$\Delta_h^N f(x) := \sum_{j=0}^N \binom{N}{j} (-1)^{N-j} f(x + jh)$$

By using these higher order differences the following extension of Lemma 2.2 can be proved.

LEMMA 3.5. *Suppose  $0 < p < \infty$ ,  $0 < q \leq \infty$ ,  $1 \leq v \leq \infty$ ,  $0 < u \leq v$ ,  $N \in \mathbb{N}$ ,*

$$d \max\left(0, \frac{1}{p} - \frac{1}{v}, \frac{1}{q} - \frac{1}{v}\right) < s < N,$$

and  $T > 0$ . Let  $\Omega_t(x)$ ,  $t > 0$ ,  $x \in \mathbb{R}^d$ , be a family of open sets in  $\mathbb{R}^d$  such that (9) is satisfied for some  $0 < A < B < \infty$ . Then  $F_{p,q}^s(\mathbb{R}^d)$  is the collection of all  $f \in L_{\max(p,v)}(\mathbb{R}^d)$  such that

$$\|f\|_{F_{p,q}^s(\mathbb{R}^d)}^* := \|f\|_{L_p(\mathbb{R}^d)} + \left\| \left( \int_0^T t^{-sq} (M_{t,u}^{N,\Omega} f(\cdot))^q \frac{dt}{t} \right)^{1/q} \Big|_{L_p(\mathbb{R}^d)} \right\|,$$

where

$$M_{t,u}^{N,\Omega} f(x) := \left( t^{-d} \int_{\Omega_t(x)} |\Delta_h^N f(x)|^u dh \right)^{1/u}.$$

Moreover,  $\|\cdot\|_{F_{p,q}^s(\mathbb{R}^d)}^*$  is equivalent to  $\|\cdot\|_{F_{p,q}^s(\mathbb{R}^d)}$  on  $L_{\max(p,v)}(\mathbb{R}^d)$ .

Of course, there is a counterpart for Besov spaces as well.

LEMMA 3.6. *Suppose  $N \in \mathbb{N}$ ,  $0 < p, q \leq \infty$ ,  $1 \leq v \leq \infty$ ,  $0 < u \leq v$ ,*

$$d \max\left(0, \frac{1}{p} - \frac{1}{v}\right) < s < N,$$

and  $T > 0$ . Let  $\Omega_t(x)$ ,  $t > 0$ ,  $x \in \mathbb{R}^d$ , be a family of open sets in  $\mathbb{R}^d$  such that (9) is satisfied for some  $0 < A < B < \infty$ . Then  $B_{p,q}^s(\mathbb{R}^d)$  is the collection of all  $f \in L_{\max(p,v)}(\mathbb{R}^d)$  such that

$$\|f\|_{B_{p,q}^s(\mathbb{R}^d)}^* := \|f\|_{L_p(\mathbb{R}^d)} + \left( \int_0^T t^{-sq} \|(M_{t,u}^{N,\Omega} f(\cdot))\|_{L_p(\mathbb{R}^d)}^q \frac{dt}{t} \right)^{1/q}.$$

Moreover,  $\|\cdot\|_{B_{p,q}^s(\mathbb{R}^d)}^*$  is equivalent to  $\|\cdot\|_{B_{p,q}^s(\mathbb{R}^d)}$  on  $L_{\max(p,v)}(\mathbb{R}^d)$ .

Now we are in position to describe the difficulties with the extension of Theorem 2.3 and Theorem 3.2, respectively. Let  $f$  be a radial function. Then, as above in Steps 1, 2

of the proof of Theorem 2.3, one can show that  $M_{t,u}^{N,\Omega} f(x)$  is a radial function. However, as it is easy to see, for  $N \geq 2$  we do not have an identity of the form

$$\Delta_h^N f(x) = \Delta_w^N g(|x|)$$

in general (here  $f, h$  and  $x$  are given and we may choose  $w$ ). For this reason we do not have a counterpart of Step 3 (proof of Theorem 2.3) in case of higher order differences.

**4. A comparison with weighted Besov–Lizorkin–Triebel spaces.** This section is devoted to the comparison of  $RF_{p,q}^s(\mathbb{R}^d)$  and  $RB_{p,q}^s(\mathbb{R}^d)$  with certain weighted spaces.

**4.1. Besov–Lizorkin–Triebel spaces and Muckenhoupt weights.** For the Muckenhoupt class  $\mathcal{A}_\infty$  of weights there is a well-established theory for Besov and Lizorkin–Triebel spaces (for definitions we refer to the Appendix), we refer to Bui et al. [B1, B2, BPT1, BPT2], Kokilashvili [K1, K2], Rychkov [R], Bownik et al. [BH, B], Haroske, Piotrowska [HP], Haroske, Skrzypczak [HS] and Izuki, Sawano [IS]. Concerning Muckenhoupt weights we refer to the monograph [Ste] of Stein. Recall, the function  $w(t) := t^{d-1}$ ,  $t \in \mathbb{R}$ , is a Muckenhoupt weight for any  $d \in \mathbb{N}$ . As a consequence of Theorems 8 and 9 in [SSV] one knows the following.

PROPOSITION 4.1. *Let  $d \geq 2$ ,  $0 < p < \infty$  and  $0 < q \leq \infty$ .*

(i) *Let  $s > \sigma_{1,q}(d)$  and either*

$$s > d\left(\frac{1}{p} - \frac{1}{d}\right) \quad \text{or} \quad s = d\left(\frac{1}{p} - \frac{1}{d}\right) \quad \text{and} \quad 0 < p \leq 1.$$

*Then we have the coincidence*

$$RF_{p,q}^s(\mathbb{R}, |t|^{d-1}) = \text{tr}(RF_{p,q}^s(\mathbb{R}^d)) \quad (\text{in the sense of equivalent quasi-norms}).$$

(ii) *Let either*

$$s > d\left(\frac{1}{p} - \frac{1}{d}\right) \quad \text{or} \quad s = d\left(\frac{1}{p} - \frac{1}{d}\right) \quad \text{and} \quad 0 < q \leq 1.$$

*Then we have the coincidence*

$$RB_{p,q}^s(\mathbb{R}, |t|^{d-1}) = \text{tr}(RB_{p,q}^s(\mathbb{R}^d)) \quad (\text{in the sense of equivalent quasi-norms}).$$

Hence, we may turn our characterizations of  $\text{tr}(RF_{p,q}^s(\mathbb{R}^d))$  into characterizations of  $RF_{p,q}^s(\mathbb{R}, |t|^{d-1})$ . As an example we reformulate Theorem 2.6.

COROLLARY 4.2. *Let  $3/2 < p < \infty$ ,  $0 < q \leq \infty$ ,  $1 \leq v < \infty$ ,  $0 < u \leq v$  and*

$$3 \max\left(0, \frac{1}{p} - \frac{1}{3}, \frac{1}{p} - \frac{1}{v}, \frac{1}{q} - \frac{1}{v}\right) < s < 1.$$

*Then the even function  $g \in L_p(\mathbb{R}, |t|^2)$  belongs to  $F_{p,q}^s(\mathbb{R}, |t|^2)$  if and only if  $g$  satisfies*

$$\begin{aligned} \|g\|^\Delta &:= \left( \int_0^\infty r^2 |g(r)|^p dr \right)^{1/p} \\ &+ \left( \int_0^1 r^2 \left[ \int_r^1 t^{-sq} \left( t^{-3} \int_{I_0(r,t)} \lambda^2 |g(\lambda) - g(r)|^u d\lambda \right)^{q/u} \frac{dt}{t} \right]^{p/q} dr \right)^{1/p} \\ &+ \left( \int_0^\infty r^{2-p/u} \left[ \int_0^1 t^{-sq} \left( t^{-1} \int_{I_1(r,t)} \lambda |g(\lambda) - g(r)|^u d\lambda \right)^{q/u} \frac{dt}{t} \right]^{p/q} dr \right)^{1/p} \end{aligned}$$

$$\begin{aligned}
 &+ \left( \int_0^\infty r^{2-p/u} \left[ \int_0^1 t^{-sq} \left( t^{-2} \int_{I_2(r,t)} \lambda(t-\lambda+r) |g(\lambda) - g(r)|^u d\lambda \right)^{q/u} \frac{dt}{t} \right]^{p/q} dr \right)^{1/p} \\
 &+ \left( \int_0^\infty r^{2-p/u} \left[ \int_0^1 t^{-sq} \left( t^{-2} \int_{I_3(r,t)} \lambda(t+\lambda-r) |g(\lambda) - g(r)|^u d\lambda \right)^{q/u} \frac{dt}{t} \right]^{p/q} dr \right)^{\frac{1}{p}} < \infty.
 \end{aligned}$$

Moreover,  $\|g\|^\Delta$  is equivalent to  $\|g\|_{F_{p,q}^s(\mathbb{R}, |t|^2)}$ .

REMARK 4.3. Corollary 4.2 may be understood as a first hint how complicated characterizations of weighted spaces  $F_{p,q}^s(\mathbb{R}, w)$  with  $w$  being a Muckenhoupt weight may look like. This is in certain contrast to the characterization of  $F_{p,q}^s(\mathbb{R}, w)$  with smooth weights  $w$ , see [ST, 5.1] and the next subsection.

**4.2. Weighted Besov–Lizorkin–Triebel spaces and differences.** In the monograph [ST] Schmeißer and Triebel developed the theory of weighted Besov–Lizorkin–Triebel spaces for a certain class of smooth weights. This class of weights covers the following examples:

$$\rho_\alpha(x) := (1 + |x|^2)^{\alpha/2}, \quad x \in \mathbb{R}^d, \quad \alpha > 0,$$

see Remark 2 in [ST, 1.4.1]. We concentrate on  $\alpha = d - 1$ ,  $d \in \mathbb{N}$ . In this situation Theorem 5.1.4 in [ST] reads as follows.

PROPOSITION 4.4. *Let  $d \in \mathbb{N}$ ,  $d \geq 2$ .*

(i) *Let  $0 < p, q < \infty$ , let  $N \in \mathbb{N}$  and  $\sigma_{p,q}(1) < s < N$ . Then  $\|f\|_{F_{p,q}^s(\mathbb{R}, \rho_{d-1})}$  and*

$$\begin{aligned}
 \|f\|_{F_{p,q}^s(\mathbb{R}, \rho_{d-1})}^* &:= \|f\|_{L_p(\mathbb{R}, \rho_{d-1})} \\
 &+ \left\| \left( \int_0^1 t^{-sq} \left[ \int_{|h| \leq 1} |\Delta_{ht}^N f(\cdot)| dh \right]^q \frac{dt}{t} \right)^{1/q} \right\|_{L_p(\mathbb{R}, \rho_{d-1})}
 \end{aligned}$$

*are equivalent.*

(ii) *Let  $0 < p < \infty$ ,  $0 < q \leq \infty$ ,  $N \in \mathbb{N}$  and  $\sigma_{p,p}(1) < s < N$ . Then  $\|f\|_{B_{p,q}^s(\mathbb{R}, \rho_{d-1})}$  and*

$$\begin{aligned}
 \|f\|_{B_{p,q}^s(\mathbb{R}, \rho_{d-1})}^* &:= \|f\|_{L_p(\mathbb{R}, \rho_{d-1})} \\
 &+ \left( \int_{|h| \leq 1} |h|^{-sq} \|\Delta_h^N f\|_{L_p(\mathbb{R}, \rho_{d-1})}^q \frac{dh}{|h|^d} \right)^{1/q}
 \end{aligned}$$

*are equivalent.*

Our next step consists in taking the characterization in Proposition 4.4 and comparing this with formula (21) in case  $u = 1$ . Because of

$$\sigma_{d-1}(Q_{\lambda,t}(r)) \leq \sigma_{d-1}(\{x : |x| = t\}) \lesssim t^{d-1}$$

and

$$\begin{aligned}
 &\int_0^\infty x^{d-1} \left[ \int_0^1 t^{-sq} \left( t^{-1} \int_{|h| \leq t} |g(x+h) - g(x)| dh \right)^q \frac{dt}{t} \right]^{p/q} dx \\
 &\lesssim \int_{-\infty}^\infty (1 + |x|^2)^{(d-1)/2} \left[ \int_0^1 t^{-sq} \left( t^{-1} \int_{|h| \leq t} |g(x+h) - g(x)| dh \right)^q \frac{dt}{t} \right]^{p/q} dx
 \end{aligned}$$

we conclude the continuous embedding

$$RF_{p,q}^s(\mathbb{R}, \rho_{d-1}) \hookrightarrow \text{tr}(RF_{p,q}^s(\mathbb{R}^d)),$$

if  $0 < p, q < \infty$  and  $\sigma_{p,q}(d) < s < 1$ . The same type of arguments leads to

$$RB_{p,q}^s(\mathbb{R}, \rho_{d-1}) \hookrightarrow \text{tr}(RB_{p,q}^s(\mathbb{R}^d)),$$

if  $0 < p, q \leq \infty$  and  $\sigma_{p,p}(d) < s < 1$ .

Both assertions supplement Proposition 4.4 in view of the trivial embeddings

$$F_{p,q}^s(\mathbb{R}, \rho_{d-1}) \hookrightarrow F_{p,q}^s(\mathbb{R}, |t|^{d-1}) \quad \text{and} \quad B_{p,q}^s(\mathbb{R}, \rho_{d-1}) \hookrightarrow B_{p,q}^s(\mathbb{R}, |t|^{d-1})$$

(valid for all admissible parameters as a direct consequence of the definition).

**5. Appendix — Muckenhoupt weights and function spaces.** For the convenience of the reader we collect some definitions and properties around Muckenhoupt weights and associated weighted function spaces.

**5.1. Muckenhoupt weights.** A weight function (or simply a weight) is a nonnegative, locally integrable function on  $\mathbb{R}^d$ . We collect a few facts including the definition of Muckenhoupt weights. As usual,  $p'$  is related to  $p$  via the formula  $1/p + 1/p' = 1$ .

DEFINITION 5.1. Let  $1 < p < \infty$ . Let  $w$  be a nonnegative, locally integrable function on  $\mathbb{R}^d$ . Then  $w$  belongs to the Muckenhoupt class  $\mathcal{A}_p$ , if there exists a constant  $A > 0$  such that for all balls  $B$  the following inequality holds:

$$\left(\frac{1}{|B|} \int_B w(x) dx\right)^{1/p} \cdot \left(\frac{1}{|B|} \int_B w(x)^{-p'/p} dx\right)^{1/p'} \leq A.$$

The Muckenhoupt class  $\mathcal{A}_\infty$  is defined as

$$\mathcal{A}_\infty := \bigcup_{p>1} \mathcal{A}_p.$$

REMARK 5.2.

- (i) Good sources for Muckenhoupt weights are Stein’s monograph [Ste] and Duoandikoetxea’s graduate text [D].
- (ii) It is well-known that the functions

$$w(t) := |t|^{d-1}, \quad t \in \mathbb{R}, \quad \text{and} \quad \rho_{d-1}(t) := (1 + |t|^2)^{(d-1)/2}, \quad t \in \mathbb{R},$$

belong to  $\mathcal{A}_\infty$ .

**5.2. Weighted Besov and Lizorkin–Triebel spaces.** To introduce weighted Besov and Lizorkin–Triebel spaces we make use of tools from Fourier analysis. Let  $\psi \in C_0^\infty(\mathbb{R}^d)$  be a function such that  $\psi(x) = 1$ ,  $|x| \leq 1$  and  $\psi(x) = 0$ ,  $|x| \geq 2$ . Then by

$$\varphi_0(x) := \psi(x), \quad \varphi_1(x) := \varphi_0(x/2) - \varphi_0(x), \quad \varphi_j(x) := \varphi_1(2^{-j+1}x), \quad (34)$$

$j \in \mathbb{N}$ , we define a smooth dyadic decomposition of unity. By using such a decomposition of unity we introduce weighted function spaces as follows.

DEFINITION 5.3. Let  $0 < q \leq \infty$ ,  $s \in \mathbb{R}$  and  $w \in \mathcal{A}_\infty$ .

- (i) Let  $0 < p < \infty$ . Then the *weighted Besov space*  $B_{p,q}^s(\mathbb{R}^d, w)$  is the collection of all  $f \in S'(\mathbb{R}^d)$  such that

$$\|f\|_{B_{p,q}^s(\mathbb{R}^d, w)} := \left( \sum_{j=0}^{\infty} 2^{jsq} \|\mathcal{F}^{-1}[\varphi_j(\xi)\mathcal{F}f(\xi)](\cdot)\|_{L_p(\mathbb{R}^d, w)}^q \right)^{1/q} < \infty.$$

- (ii) Let  $0 < p < \infty$ . Then the *weighted Triebel–Lizorkin space*  $F_{p,q}^s(\mathbb{R}^d, w)$  is the collection of all  $f \in S'(\mathbb{R}^d)$  such that

$$\|f\|_{F_{p,q}^s(\mathbb{R}^d, w)} := \left\| \left( \sum_{j=0}^{\infty} 2^{jsq} |\mathcal{F}^{-1}[\varphi_j(\xi)\mathcal{F}f(\xi)](\cdot)|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^d, w)} < \infty.$$

- (iii) Let  $p = \infty$ . Then the *Besov space*  $B_{\infty,q}^s(\mathbb{R}^d)$  is the collection of all  $f \in S'(\mathbb{R}^d)$  such that

$$\|f\|_{B_{\infty,q}^s(\mathbb{R}^d)} := \left( \sum_{j=0}^{\infty} 2^{jsq} \|\mathcal{F}^{-1}[\varphi_j(\xi)\mathcal{F}f(\xi)](\cdot)\|_{L_\infty(\mathbb{R}^d)}^q \right)^{1/q} < \infty.$$

REMARK 5.4.

- (i) For  $w \equiv 1$  we are in the unweighted case. The associated spaces are denoted by  $B_{p,q}^s(\mathbb{R}^d)$  and  $F_{p,q}^s(\mathbb{R}^d)$ .
- (ii) Observe that we did not define weighted spaces with  $p = \infty$ . However, it will be convenient for us to use the convention  $B_{\infty,q}^s(\mathbb{R}^d, w) := B_{\infty,q}^s(\mathbb{R}^d)$ .
- (iii) Weighted Besov and Lizorkin–Triebel spaces with  $w \in \mathcal{A}_\infty$  have been first studied systematically by Bui [B1, B2], cf. also [BPT1] and [BPT2]. In addition we refer to Kokilashvili [K1, K2], Haroske, Piotrowska [HP] and [HS]. Standard references for unweighted spaces are the monographs [P, T1, T2, T3] as well as [FJ]. More general classes of weights have been treated by Rychkov [R], Bownik and Ho [BH], Bownik [B], Izuki and Sawano [IS], and Wojciechowska [W1, W2].

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