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NONFIBERED KNOTS AND REPRESENTATION SHIFTS

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Abstract. A conjecture of [12] states that a knot is nonfibered if and only if its infinite cyclic cover has uncountably many finite covers. We prove the conjecture for a class of knots that includes all knots of genus 1, using techniques from symbolic dynamics.

1. Introduction. Let G be a finitely presented group with epimorphism $\chi : G \to \mathbb{Z}$. The kernel K of χ need not be finitely generated. However, K is finitely presented as a $\mathbb{Z}$-operator group [10]. In [11], [12] the authors exploited this structure to show that the representations of K into a fixed finite group Σ form a *shift of finite type*, a simple dynamical system described by a finite directed graph. We call this dynamical system the *representation shift* of K in Σ. When G is a knot or link group, representation shifts inform us about the algebraic topology of finite covering spaces from a purely dynamical perspective.

We review basic definitions of representation shifts and give a partial solution to Conjecture 4.4 of [12]. The complete solution would characterize nonfibered knots as those knots with some representation shift having positive topological entropy.

2. Review of representation shifts. An *augmented group system* [7] is a triple $\mathcal{G} = (G, \chi, x)$ consisting of a finitely presented group G, epimorphism $\chi : G \to \mathbb{Z}$ and distinguished element $x \in G$ such that $\chi(x) = 1$. Two such systems $\mathcal{G}_i = (G_i, \chi_i, x_i),\ i = 1, 2$ are *equivalent* (and regarded as the same) if there exists an isomorphism $f : G_1 \to G_2$ such that $f(x_1) = x_2$ and $\chi_1 = \chi_2 \circ f$.

EXAMPLE 2.1. An augmented group system is associated to an oriented knot $k \subset \mathbb{S}^3$ in a canonical manner. Let $G = \pi_1(\mathbb{S}^3 \setminus k, p)$, where the base point p is contained on the boundary $\partial N(k)$ of a tubular neighborhood $N(k) = \mathbb{S}^1 \times \mathbb{D}^2$ of k. Let x be the homotopy

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class of a meridian $m \subset \partial N(k)$, with orientation acquired from k. Finally, let $\chi : G \to \mathbb{Z}$ be the abelianization homomorphism that sends x to 1. It follows from the uniqueness of tubular neighborhoods that $\mathcal{G} = (G, \chi, x)$ is well defined.

We denote the kernel of χ by K. Given any finite group Σ, we consider the space $\mathrm{Hom}(K, \Sigma)$ of representations $\rho : K \to \Sigma$. The basis for its topology is given by the sets

$$\mathcal{N}_{a_1,\dots,a_s}(\rho) = \{\rho' \mid \rho'(a_i) = \rho(a_i),\ i = 1, \dots, s\},$$

where $a_1, \dots, a_s$ varies over all finite collections of elements of K. The topology is the compact-open topology where K and Σ are discrete spaces. Roughly speaking, representations are close in $\mathrm{Hom}(K, \Sigma)$ if they agree on large finitely generated subgroups of K. The distinguished element x induces a self-map σ_x of $\mathrm{Hom}(K, \Sigma)$ defined by

$$\sigma_x \rho(a) = \rho(x^{-1} a x) \quad \forall a \in K.$$

It is easily seen that σ_x is a homeomorphism.

The *representation shift* associated to $\mathcal{G} = (G, \chi, x)$ and Σ is the pair $(\mathrm{Hom}(K, \Sigma), \sigma_x)$. We denote it by $\Phi_\Sigma(\mathcal{G})$. It is a dynamical system well defined up to topological conjugacy [10]. More precisely, if $\mathcal{G}_i,\ i = 1, 2$, are equivalent augmented group systems, then there exists a homeomorphism F of the underlying spaces of $\Phi_\Sigma(\mathcal{G}_i)$ such that $F \circ \sigma_{x_1} = \sigma_{x_2} \circ F$.

The representation shift $\Phi_\Sigma(\mathcal{G})$ is an example of a *shift of finite type*, a special type of expansive 0-dimensional dynamical system, one that can be described by a finite directed graph. (See [4].) We use combinatorial group theory to construct such a graph for a representation shift.

Given an augmented group system $\mathcal{G} = (G, \chi, x)$, we can describe G as an HNN extension $\langle x, B \mid x^{-1} a x = \phi(a),\ \forall a \in U \rangle$, where B is a finitely generated subgroup of K, and U, V are isomorphic finitely generated subgroups of B with isomorphism $\phi : U \to V$ (see [6]). The subgroup B is an *HNN base*. One can choose B so that it contains any prescribed finite subset of K (see [8]).

Example 2.2. Let $\mathcal{G} = (G, \chi, x)$ be an augmented group system associated to a knot, as in Example 2.1. An HNN decomposition for G can be obtained in a natural way. Begin with a π_1-incompressible Seifert surface for k meeting the exterior $E(k) = \mathbb{S}^3 \setminus \mathrm{int}\ N(k)$ in a connected surface S. Split $E(k)$ along S, and denote by $(W; S_0, S_1)$ the resulting cobordism, with boundary comprising two copies S_0, S_1 of S joined by an annulus $\partial S \times I$. Let $B = \pi_1(W, p)$, where the basepoint p lies on the boundary of S_0. Let $U = \pi_1(S_0, p)$. The meridian m appears as a path from $p \in S_0$ to a point $p_1 \in S_1$. Use the path to regard $\pi_1(S_1, p_1)$ as a subgroup V of B. Clearly G is described as $(B; U, V, \phi)$, where ϕ is induced by the gluing of S_0 to S_1 when recovering the exterior $E(k)$.

Conjugation by x induces an automorphism of K. Let $B_j = x^{-j} B x^j$, $U_j = x^{-j} U x^j$ and $V_j = x^{-j} V x^j$, for $j \in \mathbb{Z}$. Then K is described as an infinite amalgamated free product

$$K = \langle B_j \mid V_j = U_{j+1},\ \forall j \in \mathbb{Z} \rangle.$$

A graph Γ describing the representation shift $\Phi_\Sigma(\mathcal{G})$ is constructed as follows. The vertex set consists of all representations $\rho_0 : U \to \Sigma$, a finite set since U is finitely generated. If $\bar{\rho}_0$ is a representation from B to Σ, then we draw a directed edge labeled

$\bar\rho_0$ from the vertex $\rho_0 = \bar\rho_0|_U$ to the vertex $\rho_0' = \bar\rho_0|_V \circ \phi$. ($\Gamma$ may have parallel edges.) Consider a bi-infinite path in Γ given by an edge sequence

$$\cdots\ \bar\rho_{-2}\ \bar\rho_{-1}\ \bar\rho_0\ \bar\rho_1\ \bar\rho_2\ \cdots$$

The representations $B_j \to \Sigma$ given by $a \mapsto \bar\rho_j(x^j a x^{-j})$ have a unique common extension $\rho : K \to \Sigma$. Conversely, any representation $\rho : K \to \Sigma$ arises from such a path, and uniquely. Thus bi-infinite paths of the graph Γ correspond bijectively to elements of $\mathrm{Hom}(K, \Sigma)$. The map σ_x acts as the left coordinate shift on the sequence of edges.

We may "prune" Γ by removing any vertex or edge that is not contained in a bi-infinite path. The resulting graph has finitely many bi-infinite paths iff it consists of a collection of disjoint cycles. It contains uncountably many bi-infinite paths iff it contains two cycles with at least one common vertex.

A representation $\rho \in \Phi_\Sigma(\mathcal{G})$ has *period* r if $\sigma_x^r(\rho) = \rho$. Such representations correspond to closed paths in Γ with length dividing r. The set of representations with period r is denoted by $\mathrm{Fix}(\sigma_x^r)$. If M_r is the r-fold cyclic cover of $\mathbb{S}^3$ branched over a knot k, then $\mathrm{Fix}(\sigma_x^r)$ is in natural bijective correspondence with $\mathrm{Hom}(\pi_1 M_r, \Sigma)$ [12]. This correspondence connects dynamical properties of the representation shift with topological properties of k.

Topological entropy is a measure of complexity of a dynamical system. For a shift of finite type, it can be computed as the log of the spectral radius of the adjacency matrix A of any directed graph that describes the shift. (Here $A_{i,j}$ is the number of edges from the ith vertex to the jth.) Consequently, the topological entropy of $\Phi_\Sigma(\mathcal{G})$, which we denote by $h_\Sigma(\mathcal{G})$, is the exponential growth rate of $\mathrm{tr} A^r = |\mathrm{Fix}(\sigma_x^r)| = |\mathrm{Hom}(\pi_1 M_r, \Sigma)|$ (see [12]). This is positive if and only if $\Phi_\Sigma(\mathcal{G})$ is uncountable. Notice that if K is finitely generated, then $\Phi_\Sigma(\mathcal{G})$ is finite for all Σ, and so in this case $h_\Sigma(\mathcal{G})=0$.

Let S_N denote the symmetric group on $\{1, \ldots, N\}$. It is well known that elements $\rho \in \mathrm{Hom}(K, S_N)$ correspond in a finite-to-one manner with subgroups $H \leq K$ with index no greater than N. The correspondence is

$$\rho \mapsto \{g \in K \mid \rho(g)(1) = 1\}.$$

The preimage of a subgroup of index N consists of $(N-1)!$ transitive representations. (A representation ρ is *transitive* if $\rho(K)$ operates transitively on $\{1, \ldots, N\}$.) Note that if $\Phi_{S_N}(\mathcal{G})$ is uncountable, then K contains uncountably many subgroups of some index no greater than N. Hence the infinite cyclic cover of k has uncountably many finite covers.

We summarize the results of this section. Recall that any finite group embeds in a sufficiently large symmetric group.

PROPOSITION 2.3. *Let $k \subset \mathbb{S}^3$ be a knot with associated augmented group system $\mathcal{G}$. Then the following statements are equivalent.*

(i) *The infinite cyclic cover of k has uncountably many finite covers.*
(ii) *The representation shift $\Phi_\Sigma(\mathcal{G})$ is uncountable for some finite group Σ.*
(iii) *The topological entropy $h_\Sigma(\mathcal{G})$ is positive for some finite group Σ.*
(iv) $\lim_{r\to\infty} \frac{1}{r} \log |\mathrm{Hom}(\pi_1 M_r, \Sigma)|$ *is positive for some finite group Σ.*

3. Nonfibered knots. We recall that a knot $k \subset \mathbb{S}^3$ is fibered if its exterior $E(k) = \mathbb{S}^3 \setminus \mathrm{int}\, N(k)$ fibers over the circle. It is no loss of generality to assume that the fibration restricts to the standard projection $\partial N(k) \simeq k \times \mathbb{S}^1 \to \mathbb{S}^1$. Hence $E(k)$ is seen to be homeomorphic to a mapping torus $S \times I/F$, where $F : S \to S$ is a homeomorphism of a minimal-genus Seifert surface S of k.

If k is fibered, then the commutator subgroup G' of its group is finitely generated and free, isomorphic to $\pi_1 S$. Conversely, a theorem of J. Stallings [13] implies that if k is a knot such that G' is finitely generated, then in fact G' is free and k is fibered.

If k is fibered and $\mathcal{G}$ is its associated augmented group system, then for any finite group Σ, the representation shift $\Phi_\Sigma(\mathcal{G})$ is finite. Its order is $|\Sigma|^{2g}$, where g is the genus of k (equal to the genus of its fiber). The trefoil and figure-eight knots are the only fibered knots of genus 1.

Conjecture 4.4 of [12] proposes a characterization of nonfibered knots. It states that k is nonfibered iff the entropy $h_\Sigma(\mathcal{G})$ is positive for some finite group Σ.

Remark 3.1. (1) In terms of the HNN base B described above, the condition that k is not fibered is equivalent to the condition that U is a proper subgroup of B. Lemma 2.3 (Substitution Lemma) of [11] provides a strategy for showing that some $\Phi_\Sigma(\mathcal{G})$ is uncountable: Find a periodic element of $\Phi_{S_N}(\mathcal{G})$ such that some symbol, say N, is fixed by every permutation in the image of U but moved by some element of $\rho(K)$. Recall that periodic representations correspond to cycles in the graph Γ. Enlarging S_N to S_{N+1}, we can construct another periodic representation ρ' by replacing N by $N+1$ in each permutation in the image of ρ. In the graph of $\Phi_{S_{N+1}}(\mathcal{G})$, ρ and ρ' correspond to cycles with a common vertex, and hence $\Phi_{S_{N+1}}(\mathcal{G})$ is uncountable.

(2) For our strategy, it suffices to find any representation $\tilde\rho : G \to \Sigma$ such that $\rho(U)$ is a proper subgroup of $\rho(K)$. For given such a representation, and letting $\rho : K \to \Sigma$ be the restriction, we enumerate the cosets of $\rho(U)$ in $\rho(K)$, say $1, \ldots, N$ ($N > 1$). In a natural way, ρ determines an element of $\Phi_{S_N}(\mathcal{G})$: $a \in K$ is sent to the transitive permutation of cosets given by right multiplication by $\rho(a)$. Note that if $a \in U$, then such a permutation fixes the symbol corresponding to $\rho(U)$. Finally, we note that if r is the order of $\tilde\rho(x)$ in Σ, then $\sigma_x^r \rho = \rho$, since $(\sigma_x^r\rho)(a) = \rho(x^{-r} a x^r) = \tilde\rho(x^{-1})^r \rho(a) \tilde\rho(x)^r = \rho(a)$, for all $a \in K$.

The representation $\tilde\rho$ in the Remark 3.1 (2) "separates" the subgroup U from some element $a \in K$.

In general, a subgroup U of a group G is *separable* if for any element $a \in G \setminus U$, there exists a finite-index subgroup of G that contains U but not a. Equivalently, there exists a finite representation $\tilde\rho : G \to \Sigma$ such that $\tilde\rho(a) \notin \tilde\rho(U)$. The strategy outlined in Remark 3.1(2) requires only that U can be separated from *some* element of $K \setminus U$.

Definition 3.2. An element $a \in G \setminus U$ is *separable from* U if there exists a subgroup H of finite index in G containing U but not a.

Question 15 of [14] asks if any finitely generated subgroup of a finitely-generated Kleinian group is separable. An affirmative answer would establish Theorem 3.4 for all hyperbolic knots. Although Thurston's question remains open, a result of D. Long and

G. Niblo [5] enables us to apply our strategy in the case of genus-1 knots (see also remarks that follow).

The theorem of Long and Niblo has been used by S. Friedl and S. Vidussi in [1] to show that twisted Alexander polynomials corresponding to finite representations decide if a genus-1 knot is fibered.

Theorem 3.3 (D. Long and G. Niblo [5]). *Let M be an orientable Haken 3-manifold. If $i : T \hookrightarrow M$ is an incompressibly embedded torus, then $i_*(\pi_1 T)$ is separable in $\pi_1 M$.*

Theorem 3.4. *Let k be a knot of genus 1. Then k is nonfibered iff the conclusions of Proposition 2.3 hold.*

Proof. One implication of the theorem is clear: if the conclusion of Proposition 2.3 holds, then k is nonfibered.

Assume that k is nonfibered. Consider the 3-manifold M obtained by 0-framed surgery on k; that is, by removing and replacing a tubular neighborhood $N(k) \equiv k \times \mathbb{D}^2$ in such a way that each disk $* \times \mathbb{D}^2$ bounds a longitude of k . By results of [3], M is irreducible. We denote the fundamental group of M by $\hat{G}$.

The addition of a meridianal disk converts a genus-1 Seifert surface S for k to a torus $\hat{S}$ in M. Since $\hat{S}$ is dual to a nontrivial cohomology class and M is irreducible, we see that $\hat{S}$ is incompressible. Note in particular that M is Haken.

Obtain an HNN decomposition $(\hat{B}; \hat{U}, \hat{V})$ for $\hat{G}$ much as we did for G, by splitting M along $\hat{S}$. Here $\hat{U} = \pi_1 \hat{S}$. Since k is not fibered, neither is M [2]. Hence $\hat{U}$ must be a proper subgroup of $\hat{B}$. Select an element $\hat{a} \in \hat{B} \setminus \hat{U}$. By Theorem 3.3 there exists a finite group Σ and homomorphism $\hat{\rho} : \hat{G} \to \Sigma$ such that $\hat{\rho}(\hat{a}) \notin \hat{U}$.

The group $\hat{G}$ is a quotient of G. Let p be the natural projection. Note that $p(U) = \hat{U}$. Choose $a \in K$ such that $p(a) = \hat{a}$. Define $\rho = \hat{\rho} \circ p : G \to \Sigma$.

Remark 3.1(2) completes the proof. ∎

Genus-1 knots are plentiful, the simplest examples being the twist knots (e.g. the knots $5_2, 6_1$) and doubled knots (obtained from a knot and any push-off by joining with a clasp). We extend the collection of nonfibered knots with uncountable representation shifts by considering also any knot k with group G that maps homomorphically onto the group $\bar{G}$ of a nonfibered genus-1 knot $\bar{k}$. Examples of such knots k include satellite knots with genus-1 pattern knot [9].

Corollary 3.5. *Let k be a knot. Assume that the group of k maps onto the group of a nonfibered knot $\bar{k}$ of genus 1. Then k is nonfibered and the conclusions of Proposition 2.3 hold.*

Proof. Assume that $h : G \twoheadrightarrow \bar{G}$ is an epimorphism, where G, $\bar{G}$ are the groups of k, $\bar{k}$, respectively. Let K, $\bar{K}$ denote the respective commutator subgroups, and x, $\bar{x}$ the meridianal generators of k, $\bar{k}$. Since $h(K) = \bar{K}$ and $\bar{K}$ is not finitely generated, we see at once that K is not finitely generated. Hence k is nonfibered.

For any group Σ, the restricted epimorphism $h : K \twoheadrightarrow \bar{K}$ induces an injection $\mathrm{Hom}(\bar{K}, \Sigma) \hookrightarrow \mathrm{Hom}(K, \Sigma)$. By Theorem 3.4, there exists a finite group Σ such that

$\mathrm{Hom}(\bar{K}, \Sigma)$ is uncountable. Hence $\mathrm{Hom}(K, \Sigma)$, the underlying space of the representation shift $\Phi_\Sigma(\mathcal{G})$, is uncountable. ■

REMARK 3.6. For any finite group Σ, the topological entropy $h_\Sigma(\mathcal{G})$ is at least as great as $h_\Sigma(\bar{\mathcal{G}})$, where $\bar{\mathcal{G}}$ is the augmented group system of $\bar{k}$. The reason is the following.

If $h(x) = \bar{x}$, then for any finite group Σ, the representation shift $\Phi_\Sigma(\bar{\mathcal{G}})$ corresponding to $\bar{k}$ is a subshift of the representation shift $\Phi_\Sigma(\mathcal{G})$ corresponding to k; that is, $\mathrm{Hom}(\bar{K}, \Sigma)$ is a subspace of $\mathrm{Hom}(K, \Sigma)$ with the shift map σ_x restricting to $\sigma_{\bar{x}}$. The epimorphism h induces an embedding: $h^*\rho = \rho \circ h$. It follows that the topological entropy $h_\Sigma(\mathcal{G})$ is at least $h_\Sigma(\bar{\mathcal{G}})$.

If $h(x) \neq \bar{x}$, then there exists $a \in K$ such that $h(ax) = \bar{x}^\epsilon$, where $\epsilon = \pm 1$. We may assume without loss of generality that $\epsilon = 1$. In this case, we replace x by ax. Of course the augmented group system $\mathcal{G}$ and associated representation shifts $\Phi_\Sigma(\mathcal{G})$ change. However, by a result of [10], the topological entropy of the representation shift remains unchanged. Again $h_\Sigma(\mathcal{G}) \geq h_\Sigma(\bar{\mathcal{G}})$.

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