ALGEBRAIC TOPOLOGY—OLD AND NEW
BANACH CENTER PUBLICATIONS, VOLUME 85
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2009

ON RESIDUE FORMULAS FOR CHARACTERISTIC NUMBERS

FRANCISCO GÓMEZ RUIZ

Departamento de Álgebra, Geometría y Topología, Universidad de Málaga
Ap. 59, 29080 Málaga, Spain
E-mail: fgomez@agt.cie.uma.es

Abstract. We show that coefficients of residue formulas for characteristic numbers associated to a smooth toral action on a manifold can be taken in a quotient field $\mathbf{Q}(X_1, \dots, X_r)$. This yields canonical identities over the integers and, reducing modulo two, residue formulas for Stiefel Whitney numbers.

1. Introduction. The classical formulas of Baum and Cheeger, [2], and Bott, [5], give the Pontrjagin or Chern numbers as a sum of residues at the zeros of a Killing or holomorphic vector field. In this paper we substitute these residues by elements of the quotient field $\mathbf{Q}(X_1, \dots, X_r)$ of the polynomial ring $\mathbf{Z}[X_1, \dots, X_r]$, r being the dimension of the torus acting on the manifold M. My motivation for doing this work was actually trying to get rational numbers as residues. In case M is a compact Riemannian manifold and v is a Killing vector field on M, we substitute v by the corresponding action of the associated torus G: if φ_t are the isometries of M induced by v, then G is the closure of the one parameter subgroup φ_t in the compact group of all isometries of M. We observe that as v changes in the Lie algebra of G the Baum-Cheeger residues of v factorize through a unique residue in $\mathbf{Q}(X_1, \dots, X_r)$.

Residue formulas for Killing vector fields or toral actions and Pontrjagin classes have been given by N. Alamo and F. Gómez [1], P. Baum and J. Cheeger [2], R. Bott [5], F. Gómez [7], D. Lehmann [9]; for holomorphic vector fields and Chern classes by P. Baum and R. Bott [3], R. Bott [4]; for toral actions and Stiefel Whitney classes by J. Daccah and A. Wassermann [6].

We consider the following situation: G is a torus of dimension r acting smoothly on a compact connected oriented smooth manifold M of dimension $2m$, F^G is the fixed point

2000 *Mathematics Subject Classification*: Primary 57R20, 57S15; Secondary 55R25.
Key words and phrases: toral action, characteristic class, fixed point set.
The paper is in final form and no version of it will be published elsewhere.

DOI: 10.4064/bc85-0-13

set of the action of G on M. By a theorem of Kobayashi [8], each connected component F of F^G is a submanifold.

We distinguish two cases:

(a) τ_M has a complex structure which is preserved by the action of G, and

(b) the dimension of M is $4k$.

In case (a), suppose that $I = (i_1, \ldots, i_m)$ with $i_1 + 2i_2 + \cdots + mi_m = m$ and denote by $c_I(M) = \int_M c_1^{i_1} \ldots c_m^{i_m}$ the Chern number corresponding to I, where c_i is the ith Chern class of τ_M.

In case (b), suppose that $I = (i_1, \ldots, i_k)$ with $i_1 + 2i_2 + \cdots + ki_k = k$ and denote as usual by $p_I(M) = \int_M p_1^{i_1} \ldots p_k^{i_k}$ the Pontrjagin number corresponding to I, where p_i is the ith Pontrjagin class of τ_M.

THEOREM. *We associate canonically to the G-vector bundle $\tau_{M|F}$ a residue class $Res_I(F)$ in the cohomology of F with coefficients in the quotient field $\mathbf{Q}(X_1, \ldots, X_r)$ such that the Chern number $c_I(M)$, resp. the Pontrjagin number $p_I(M)$, is given by the sum $\sum_F \int_F Res_I(F)$.*

2. Residues. The canonical decomposition for the representations of G on the tangent space of M at the points of F clearly induces a canonical G-vector bundle decomposition

$$\tau_{M|F} = \nu_0^F \oplus \nu_1^F \oplus \cdots \oplus \nu_{s(F)}^F$$

where $\nu_0^F = \tau_F$ is the tangent bundle to F and $\nu^F = \nu_1^F \oplus \cdots \oplus \nu_{s(F)}^F$ is the normal bundle of the inclusion $F \hookrightarrow M$. It is well known that each ν_j^F admits a complex structure, unique up to conjugation, and corresponding integer vectors $n_j^F \in \mathbf{Z}^r$, $j = 1, \ldots, s(F)$, so that the action of G on $\tau_{M|F}$ is given by

$$a.(v_0 \oplus v_1 \oplus \cdots \oplus v_{s(F)}) = v_0 \oplus a^{n_1^F}.v_1 \oplus \cdots \oplus a^{n_{s(F)}^F}.v_{s(F)},$$

where, if $a = (a_1, \ldots, a_r) \in G$ and $n = (n_1, \ldots, n_r) \in \mathbf{Z}^r$, a^n means $a_1^{n_1} \ldots a_r^{n_r} \in S^1$.

Set $X = (X_1, \ldots, X_r)$ and consider the linear polynomials

$$\langle n_j^F, X \rangle = \sum_{k=1}^{r} n_{jk} X_k \in \mathbf{Z}[X_1, \ldots, X_r],$$

$j = 1, \ldots, s(F)$.

Set $rank(\nu^F) = 2m^F$, $rank(\nu_j^F) = 2m_j^F$, $j = 1, \ldots, s(F)$.

In case (a), suppose that $I = (i_1, \ldots, i_m)$ with $i_1 + 2i_2 + \cdots + mi_m = m$ and consider the symmetric polynomial in m variables $c_I(Y_1, \ldots, Y_m) = \sigma_1(Y_1, \ldots, Y_m)^{i_1} \ldots \sigma_m(Y_1, \ldots, Y_m)^{i_m}$.

Define the residue at F associated to I, $Res_I(F)$, by

$$\frac{c_I(c_{01}^F + \langle n_0^F, X \rangle, \ldots, c_{0m_0^F}^F + \langle n_0^F, X \rangle; \ldots; c_{s(F)1}^F + \langle n_{s(F)}^F, X \rangle, \ldots, c_{s(F)m_{s(F)}^F}^F + \langle n_{s(F)}^F, X \rangle)}{\prod_{i=1}^{s(F)} (\prod_{j=1}^{m_i^F} (c_{ij}^F + \langle n_i^F, X \rangle))},$$

where $n_0^F = 0$ and $c_{i1}^F, \ldots, c_{im_i^F}^F$, $i = 0, \ldots, s(F)$, are formal variables of degree two so that $\sigma_\lambda(c_{i1}^F, \ldots, c_{im_i^F}^F)$ is the λth Chern class of ν_i^F.

Explicitly the numerator of $Res_I(F)$ is given by

$$\prod_{\lambda=1}^{m}\Big(\sum_{\substack{\alpha_0+\cdots+\alpha_{s(F)}=\lambda\\ 0\le\alpha_i\le m_i^F}} c_{\alpha_0}(F)\prod_{i=1}^{s(F)}\Big(\sum_{\alpha+\beta=\alpha_i}\binom{m_i^F-\beta}{\alpha}\langle n_i^F,X\rangle^{\alpha}c_\beta(\nu_i^F)\Big)^{i_\lambda}\Big)$$

and the denominator by

$$\prod_{i=1}^{s(F)}\sum_{\lambda+\mu=m_i^F}\langle n_i^F,X\rangle^{\lambda}c_\mu(\nu_i^F),$$

where $c_\mu(\nu_i^F)$ is the μth Chern class of ν_i^F.

Observe that, since all the n_i^F are nonzero, for $i\neq 0$, it makes sense to consider the inverse

$$\frac{1}{\sum_{\lambda+\mu=m_i^F}\langle n_i^F,X\rangle^{\lambda}c_\mu(\nu_i^F)}=\frac{1}{\langle n_i^F,X\rangle^{m_i^F}}\frac{1}{1+\tilde{c}_j},\qquad i=1,\ldots,s(F)$$

where

$$\tilde{c}_i=1+\frac{c_1(\nu_i^F)}{\langle n_i^F,X\rangle}+\frac{c_2(\nu_i^F)}{\langle n_i^F,X\rangle^2}+\cdots+\frac{c_{m_i^F}(\nu_i^F)}{\langle n_i^F,X\rangle^{m_i^F}}$$

and

$$\frac{1}{1+\tilde{c}_i}=1-\tilde{c}_i+\tilde{c}_i^2-\tilde{c}_i^3+\cdots.$$

Therefore

$$\frac{1}{\prod_{i=1}^{s(F)}(\sum_{\lambda+\mu=m_i^F}\langle n_i^F,X\rangle^{\lambda}c_\mu(\nu_i^F))}$$
$$=\frac{1}{\langle n_1^F,X\rangle^{m_1^F}\cdots\langle n_{s(F)}^F,X\rangle^{m_{t(F)}^F}}\frac{1}{(1+\tilde{c}_1)\cdots(1+\tilde{c}_{s(F)})}.$$

In case (b), suppose that $I=(i_1,\ldots,i_k)$ with $i_1+2i_2+\cdots+ki_k=k$ and consider the symmetric polynomial in m variables $p_I(Y_1,\ldots,Y_m)=\sigma_1(Y_1^2,\ldots,Y_m^2)^{i_1}\ldots\sigma_k(Y_1^2,\ldots,Y_m^2)^{i_k}$.

Define the residue at F associated to I, $Res_I(F)$, by

$$\frac{p_I(c_{01}^F+\langle n_0^F,X\rangle,\ldots,c_{0m_0^F}^F+\langle n_0^F,X\rangle;\cdots;c_{s(F)1}^F+\langle n_{s(F)}^F,X\rangle,\ldots,c_{s(F)m_{s(F)}^F}^F+\langle n_{s(F)}^F,X\rangle)}{\prod_{i=1}^{s(F)}(\prod_{j=1}^{m_i^F}(c_{ij}^F+\langle n_i^F,X\rangle))},$$

where $n_0^F=0$ and $c_{i1}^F,\ldots,c_{im_i^F}^F$, $i=0,\ldots,s(F)$, are formal variables of degree two so that $\sigma_\lambda((c_{01}^F)^2,\ldots,(c_{0m_0^F}^F)^2)$ is the λth Pontrjagin class of ν_0^F, and $\sigma_\lambda(c_{i1}^F,\ldots,c_{im_i^F}^F)$ is the λth Chern class of ν_i^F, for $1\le i\le s(F)$.

Explicitly the numerator of $Res_I(F)$ is given by

$$\prod_{\lambda=1}^{k}\Big(\sum_{\substack{\alpha_0+\cdots+\alpha_{s(F)}=\lambda\\ 0\le\alpha_i\le m_i^F}} p_{\alpha_0}(F)\prod_{i=1}^{s(F)}\tilde{\Phi}_{\alpha_i}\Big)^{i_\lambda},$$

with

$$\tilde{\Phi}_{\alpha_i} = \Phi_{\alpha_i}\Big(\sum_{\alpha+\beta=1}\binom{m_i^F-\beta}{\alpha}\langle n_i^F, X\rangle^\alpha c_\beta(\nu_i^F), \dots, \sum_{\alpha+\beta=m_i^F}\binom{m_i^F-\beta}{\alpha}\langle n_i^F, X\rangle^\alpha c_\beta(\nu_i^F)\Big)$$

where Φ_t is given by the formula

$$\sigma_t(Y_1^2, \dots, Y_m^2) = \Phi_t(\sigma_1(Y_1, \dots, Y_m), \dots, \sigma_m(Y_1, \dots, Y_m)).$$

The denominator is given, as in case (a), by

$$\prod_{i=1}^{s(F)}\Big(\sum_{\lambda+\mu=m_i^F}\langle n_i^F, X\rangle^\lambda c_\mu(\nu_i^F)\Big),$$

where $c_\mu(\nu_i^F)$ is the μth Chern class of ν_i^F.

Observe that actually, in both cases,

$$\langle n_1^F, X\rangle^{2m_1^F}\cdots\langle n_{s(F)}^F, X\rangle^{2m_{s(F)}^F} Res_I(F) \in \mathbf{Z}[X_1, \dots, X_r]\otimes_{\mathbf{Z}} H^*(F;\mathbf{Z}).$$

If F is not reduced to a single point, endow F with the orientation so that the given orientation on $\tau_{M|F}$ is the direct sum of the orientations on τ_F and ν^F. It is obvious that $\int_F Re_I(F) \in H_*(F;\mathbf{Q}(X_1, \dots, X_r))$ is independent of the choice of the complex structure and corresponding orientation on ν^F.

In case F is one point, $Res_I(F) \in \mathbf{Q}(X_1, \dots, X_r)$ and we define then $\int_F Res_I(F) = \epsilon_F.Res_I(F)$, where $\epsilon_F = 1$ or -1 according to whether the complex orientation on $\nu^F = \tau_{M_{|F}}$ agrees or not with the given orientation on $\tau_{M_{|F}}$.

Again, $\int_F Res_I(F)$ is independent of the choices.

To prove our theorem, replace the variables $X_1, \dots, X_r$ by real numbers linearly independent over $\mathbf{Q}$, choose a G-invariant Riemannian metric on M and consider the Killing vector field whose flow is given by $\varphi_t(x) = (e^{2\pi t X_1}, \dots, e^{2\pi t X_r})x$. Then, we follow the standard procedure of Bott, Baum, Cheeger of choosing G-invariant tubular neighbourhoods of F and convenient Baum-Cheeger connections.

Corollary. *If none of the integer vectors n_i^F is of the form $2\bar{n}_i^F$, with $\bar{n}_i^F \in \mathbf{Z}^r$, we derive from our main theorem a residue formula for the Stiefel Whitney numbers, by simply reducing modulo* 2.

3. Examples and remarks. 1) As an illustration we consider the following action of the 2-dimensional torus $S^1 \times S^1$ on $\mathbf{C}P^2$:

$$(a, b).\langle z_0, z_1, z_2\rangle = \langle z_0, a^{n_{11}}b^{n_{12}}z_1, a^{n_{21}}b^{n_{22}}z_2\rangle$$

where we suppose $|n_{11}.n_{22} - n_{12}.n_{21}| = 1$.

The three fixed points are $\langle 1, 0, 0\rangle$, $\langle 0, 1, 0\rangle$ and $\langle 0, 0, 1\rangle$.

The representation at $\langle 1, 0, 0\rangle$ is given by

$$(a, b).\Big\langle 1, \frac{z_1}{z_0}, \frac{z_2}{z_0}\Big\rangle = \Big\langle 1, a^{n_{11}}b^{n_{12}}\frac{z_1}{z_0}, a^{n_{21}}b^{n_{22}}\frac{z_2}{z_0}\Big\rangle.$$

The representation at $\langle 0, 1, 0\rangle$ is

$$(a, b).\Big\langle \frac{z_0}{z_1}, 1, \frac{z_2}{z_1}\Big\rangle = \Big\langle a^{-n_{11}}b^{-n_{12}}\frac{z_0}{z_1}, 1, a^{n_{21}-n_{11}}b^{n_{22}-n_{12}}\frac{z_2}{z_1}\Big\rangle.$$

The representation at $\langle 0,0,1\rangle$ is

$$(a,b).\left\langle \frac{z_0}{z_2}, \frac{z_1}{z_2}, 1\right\rangle = \left\langle a^{-n_{21}}b^{-n_{22}}\frac{z_0}{z_2}, a^{n_{11}-n_{21}}b^{n_{12}-n_{22}}\frac{z_1}{z_2}, 1\right\rangle$$

The Pontrjagin number $\sigma_1(\mathbf{C}P^2)$, which, of course, we know to be 3, is given by the formula

$$\sigma_1(\mathbf{C}P^2) = \frac{(n_{11}X_1+n_{12}X_2)^2+(n_{21}X_1+n_{22}X_2)^2}{(n_{11}X_1+n_{12}X_2)(n_{21}X_1+n_{22}X_2)}$$
$$+\frac{(-n_{11}X_1-n_{12}X_2)^2+((n_{21}-n_{11})X_1+(n_{22}-n_{12})X_2)^2}{(-n_{11}X_1-n_{12}X_2)((n_{21}-n_{11})X_1+(n_{22}-n_{12})X_2)}$$
$$+\frac{(-n_{21}X_1-n_{22}X_2)^2+((n_{11}-n_{21})X_1+(n_{12}-n_{22})X_2)^2}{(-n_{21}X_1-n_{22}X_2)((n_{11}-n_{21})X_1+(n_{12}-n_{22})X_2)}.$$

Set $\lambda = \frac{n_{11}X_1+n_{12}X_2}{n_{21}X_1+n_{22}X_2}$ and then

$$\sigma_1(\mathbf{C}P^2) = \left(\lambda+\frac{1}{\lambda}\right)+\left(-\frac{\lambda}{1-\lambda}-\frac{1-\lambda}{\lambda}\right)+\left(1-\lambda+\frac{1}{1-\lambda}\right)=3.$$

2) Observe that, in example 1, by giving real values to X_1, X_2 we cannot have that all three residues are integers, or equivalently, we cannot find a Killing vector field with integer residues.

3) The main theorem of this paper makes sense for G being a finite abelian group; is it true in that case?

4) If we consider the Borel bundle $M_G \to BG$, with fibre M, associated to the universal bundle $EG \to BG$ and the G-manifold M; we can extend the action of G on M to an action of G on M_G in the obvious way and the fixed point set is then $BG \times F^G$ with $BG = \mathbf{C}P^\infty \times \overset{r}{\cdots} \times \mathbf{C}P^\infty$. Therefore, the integral cohomology of $BG \times F^G$ is $H^*(F^G) \otimes \mathbf{Z}[X_1,\dots,X_r]$ with degree of X_j equal 2. This explains why it is natural to consider rational residues in the variables $X_1,\dots,X_r$.

Acknowledgments. This research was partially supported by a DGICYT grant (MTM2007-60016) and a Junta de Andalucía grant (FQM0213).

References

[1] N. Alamo and F. Gómez, *Smooth toral actions on principal bundles and characteristic classes*, Lecture Notes in Math. 1410, Springer, Basel, 1989, 1–26.

[2] P. Baum and J. Cheeger, *Infinitesimal isometries and Pontrjagin numbers*, Topology 8 (1969), 173–193.

[3] P. Baum and R. Bott, *Singularities of holomorphic foliations*, J. Diff. Geometry 7 (1972), 279–342.

[4] R. Bott, *A residue formula for holomorphic vector fields*, J. Diff. Geometry 1 (1967), 311–330.

[5] R. Bott, *Vector fields and characteristic numbers*, Mich. Math. J. 14 (1967), 231–244.

[6] J. Daccah and A. Wassermann, *Stiefel-Whitney classes and toral actions*, Topology Appl. 21 (1985), 19–26.

[7] F. Gómez, *A residue formula for characteristic classes*, Topology 21 (1982), 101–124.

[8] S. Kobayashi, *Fixed points of isometries*, Nagoya Math. J. 13 (1958), 63–68.

[9] D. Lehmann, *Résidus des sousvariétés invariantes d'un feuilletage singulier*, Ann. Inst. Fourier 41 (1991), 211–258.

[10] J. W. Milnor and J. D. Stasheff, *Characteristic Classes*, Annals of Math. Studies 76, Princeton U. Press, 1972.