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COHOMOLOGY C_∞-ALGEBRA AND RATIONAL HOMOTOPY TYPE

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Abstract. In the rational cohomology of a 1-connected space a structure of C_∞-algebra is constructed and it is shown that this object determines the rational homotopy type.

1. Introduction. Usually invariants of algebraic topology are not *complete*: the isomorphism of invariants does not guarantee the equivalence of spaces. The invariants which carry richer algebraic structure contain more information about the space. For example "cohomology algebra" allows one to distinguish spaces which cannot be distinguished by "cohomology groups".

Let us assume that all R-modules $H^*(X,R)$ are free. In [11, 12] we obtain an A_∞-algebra structure on $H^*(X,R)$. This structure consists of a collection of operations

$$\{m_i : H^*(X,R) \otimes \overset{i}{\cdots} \otimes H^*(X,R) \to H^*(X,R),\ i = 2,3,\dots\}.$$

In fact this structure extends the usual structure of cohomology algebra: the first operation $m_2 : H^*(X,R) \otimes H^*(X,R) \to H^*(X,R)$ coincides with the cohomology multiplication.

The cohomology algebra equipped with this additional structure, which we call cohomology A_∞-algebra, carries more information about the space than the cohomology algebra. For example just the cohomology algebra $H^*(X,R)$ does not determine the cohomology of the loop space $H^*(\Omega X,R)$, but the cohomology A_∞-algebra $(H^*(X,R),\{m_i\})$ does. Dually, the Pontriagin ring $H_*(G)$ does not determine the homology $H_*(B_G)$ of the classifying space, but the homology A_∞-algebra $(H_*(G),\{m_i\})$ does.

These A_∞-algebras have several applications in the cohomology theory of fibre bundles too, see [12].

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But this invariant is not complete either. One cannot expect the existence of more or less simply complete algebraic invariant in general case but for the rational homotopy category there are various complete homotopy invariants (algebraic models):

(i) The model of Quillen [22] L_X, which is a differential graded Lie algebra;

(ii) The minimal model of Sullivan [2] M_X, which is a commutative graded differential algebra;

(iii) The filtered model of Halperin and Stasheff [9] ΛX, which is a filtered commutative graded differential algebra.

The rational cohomology algebra $H^*(X, Q)$ is not a complete invariant even for rational spaces: two spaces might have isomorphic cohomology algebras, but different rational homotopy types.

The main result of this paper is the construction of a complete rational homotopy invariant: *the cohomology C_∞-algebra.*

This notion of C_∞-algebra is the commutative version of Stasheff's notion of A_∞-algebra. It was mentioned in [23]; in [13] it was called commutative A_∞-algebra and denoted by CA_∞; in [21] it was called *balanced* A_∞-algebra; the term C_∞-algebra was introduced in [7].

We show that in the rational case on the cohomology $H^*(X, Q)$ there arises a structure of C_∞-algebra $(H^*(X, Q), \{m_i\})$. The main application of this structure is the following: it completely determines the rational homotopy type, that is, 1-connected spaces X and X' have the same rational homotopy type if and only if their cohomology C_∞-algebras $(H^*(X, Q), \{m_i\})$ and $(H^*(X', Q), \{m'_i\})$ are isomorphic.

We present also several applications of this complete rational homotopy invariant to some problems of rational homotopy theory.

The C_∞-algebra structure on the homology of a *commutative* dg algebra and the applications of this structure in rational homotopy theory were actually presented in the hardly available small book [15] (see also the preprint [14]).

Applications of cohomology C_∞-algebra in rational homotopy theory are inspired by the existence of Sullivan's commutative cochains $A(X)$ in this case. The cohomology C_∞-algebra $(H^*(X, Q), \{m_i\})$ carries the same amount of information as $A(X)$ does. Actually these two objects are equivalent in the category of C_∞-algebras.

Outside of the rational category generally we do not have commutative cochains, so some additional structures, such as Steenrod $\smile_i$ products, and much more, must be involved. For example as the first step one should add the operations which form a so called homotopy G-algebra structure (in fact the little square operad) ([6], [18]). These in fact are cochain operations which control interaction between $\smile$ and $\smile_1$ products. Next, some new operations which control interaction between $\smile$ and $\smile_i$, $i = 1, 2, 3, \dots$ products show up ([16]). Next there must be operations which control interaction between $\smile_i$ and $\smile_j$ products, etc.

We presume that finally we obtain some specific E_∞ algebra structure on singular cochains, see [10], [19], [1].

The final achievement in this direction is Mandell's result: the E_∞-algebra structure on cochain algebra determines (in some cases) the homotopy type [20].

In the rational case the E_∞ operad can be replaced by the commutative operad $\mathcal{C}$ acting on appropriate cochains. And in order to step from cochains to cohomology we replace $\mathcal{C}$ by the operad C_∞.

2. A_∞-algebras. The notion of A_∞-algebra was introduced by J. Stasheff [25]. This notion generalizes the notion of differential graded algebra (dga).

DEFINITION 2.1. An A_∞-algebra is a graded module $M = \{M^k\}_{k\in Z}$ equipped with a sequence of operations

$$\{m_i : M \otimes \overset{i}{\cdots} \otimes M \to M, i = 1, 2, 3, \dots\}$$

satisfying the conditions $m_i((\otimes^i M)^q) \subset M^{q-i+2}$, that is *deg* $m_i = 2 - i$, and

$$(1)\qquad \sum_{k=0}^{i-1}\sum_{j=1}^{i-k} \pm m_{i-j+1}(a_1 \otimes \cdots \otimes a_k \otimes m_j(a_{k+1} \otimes \cdots \otimes a_{k+j}) \otimes \cdots \otimes a_i) = 0.$$

In fact for an A_∞-algebra $(M, \{m_i\})$ the first two operations form a *nonassociative* dga (M, m_1, m_2) with differential m_1 and multiplication m_2 which is associative just up to homotopy and the suitable homotopy is the operation m_3.

DEFINITION 2.2. A morphism of A_∞-algebras

$$\{f_i\} : (M, \{m_i\}) \to (M', \{m'_i\})$$

is a sequence $\{f_i : \otimes^i M \to M', i = 1, 2, \dots,\ deg\ f_1 = 1 - i\}$ such that

$$(2)\qquad \sum_{k=0}^{i-1}\sum_{j=1}^{i-k} \pm f_{i-j+1}(a_1 \otimes \cdots \otimes a_k \otimes m_j(a_{k+1} \otimes \cdots \otimes a_{k+j}) \otimes \cdots \otimes a_i)$$
$$= \sum_{t=1}^{i}\ \sum_{k_1+\cdots+k_t=i} \pm m'_t(f_{k_1}(a_1 \otimes \cdots \otimes a_{k_1}) \otimes \cdots \otimes f_{k_t}(a_{i-k_t+1} \otimes \cdots \otimes a_i)).$$

The composition of A_∞ morphisms

$$\{h_i\} : (M, \{m_i\}) \xrightarrow{\{f_i\}} (M', \{m'_i\}) \xrightarrow{\{g_i\}} (M'', \{m''_i\})$$

is defined as

$$(3)\qquad h_n(a_1 \otimes \cdots \otimes a_n)$$
$$= \sum_{t=1}^{n}\ \sum_{k_1+\cdots+k_t=n} g_n(f_{k_1}(a_1 \otimes \cdots \otimes a_{k_1}) \otimes \cdots \otimes f_{k_t}(a_{n-k_t+1} \otimes \cdots \otimes a_n)).$$

The bar construction argument (see (4.1) below) allows one to show that this composition satisfies the condition (2).

For a morphism $\{f_i\} : (M, \{m_i\}) \to (M', \{m'_i\})$ the first component $f_1 : (M, m_1) \to (M', m'_1)$ is a chain map which is *multiplicative* just up to homotopy and the suitable homotopy is the map f_2.

An A_∞ algebra of type $(M, \{m_1, m_2, 0, 0, \dots\})$ is a dga with the differential m_1 and strictly associative multiplication m_2. Furthermore, a morphism of such A_∞-algebras of type $\{f_1, 0, 0, \dots\}$ is a strictly multiplicative chain map. Thus the category of dg algebras is a subcategory of the category of A_∞-algebras.

3. C_∞-algebras. The *shuffle* product $\mu_{sh} : M^{\otimes m} \otimes M^{\otimes n} \to M^{\otimes(m+n)}$ is defined as

$$\mu((a_1 \otimes \dots \otimes a_n) \otimes (a_{n+1} \otimes \dots \otimes a_{n+m})) = \sum \pm a_{\sigma(1)} \otimes \dots \otimes a_{\sigma(n+m)}, \tag{4}$$

where summation is taken over all (m,n)-shuffles, that is, over all permutations of the set $(1, 2, \dots, n+m)$ which satisfy the condition: $i < j$ if $1 \le \sigma(i) < \sigma(j) \le n$ or $n+1 \le \sigma(i) < \sigma(j) \le n+m$.

Definition 3.1 ([23], [13], [21], [7]). A C_∞-algebra is an A_∞-algebra $(M, \{m_i\})$ which additionally satisfies the following condition: each operation m_i vanishes on shuffles, that is, for $a_1, \dots, a_i \in M$ and $k = 1, 2, \dots, i-1$

$$m_i(\mu_{sh}((a_1 \otimes \dots \otimes a_k) \otimes (a_{k+1} \otimes \dots \otimes a_i))) = 0. \tag{5}$$

Definition 3.2. A morphism of C_∞-algebras is defined as a morphism of A_∞-algebras $\{f_i\} : (M, \{m_i\}) \to (M', \{m'_i\})$ whose components f_i vanish on shuffles, that is,

$$f_i((\mu_{sh}(a_1 \otimes \dots \otimes a_k) \otimes (a_{k+1} \otimes \dots \otimes a_i))) = 0. \tag{6}$$

The composition is defined as in the A_∞ case and the bar construction argument (see (4.1) below) allows one to show that the composition is a C_∞ morphism.

In particular for the operation m_2 we have $m_2(a \otimes b \pm b \otimes a) = 0$, so a C_∞-algebra of type $(M, \{m_1, m_2, 0, 0, \dots\})$ is a commutative dg algebra (cdga) with the differential m_1 and strictly associative and commutative multiplication m_2. Thus the category of cdg algebras is a subcategory of the category of C_∞-algebras.

4. Tensor coalgebra. The notions of A_∞ and C_∞ algebras can be interpreted in terms of *differentials* on the tensor coalgebra.

The *tensor coalgebra* of a graded module V is defined as

$$T^c(V) = R \oplus V \oplus V \otimes V \oplus V \otimes V \otimes V \oplus \dots = \sum_{i=0}^{\infty} V^{\otimes i}$$

with the comultiplication $\Delta : T^c(V) \to T^c(V) \otimes T^c(V)$ given by

$$\Delta(a_1 \otimes \dots \otimes a_n) = \sum_{i=0}^{n} (a_1 \otimes ... \otimes a_i) \otimes (a_{i+1} \otimes \dots \otimes a_n).$$

The tensor coalgebra is a cofree object in the category of graded coalgebras: for a map of graded modules $\alpha : C \to V$ there exists a unique morphism of graded coalgebras $f_\alpha : C \to T^c(V)$ such that $p_1 f_\alpha = \alpha$, where $p_n : T^c(V) \to V^{\otimes n}$ is the clear projection. The coalgebra map f_α is defined as $f_\alpha = \sum_k (\alpha \otimes \dots \alpha)\Delta^k$, where $\Delta^k : C \to C^{\otimes k}$ is the k-th iteration of the comultiplication $\Delta : C \to C \otimes C$, i.e. $\Delta^1 = id$, $\Delta^2 = \Delta$, $\Delta^k = (\Delta^{k-1} \otimes id)\Delta$.

The tensor coalgebra has a similar universal property for coderivations, i.e. maps $\partial : C \to C'$ satisfying $\Delta\partial = (\partial \otimes id + id \otimes \partial)\Delta$. Namely, for each homomorphism $\beta : T^c(V) \to V$ there exists a unique coderivation $\partial_\beta : T^c(V) \to T^c(V)$ such that $p_1 \partial_\beta = \beta$. The coderivation ∂_β is defined as $\partial_\beta = \sum_{k,i} (id \otimes \beta \otimes id)\Delta^3$.

The *shuffle* multiplication $\mu_{sh} : T^c(V) \otimes T^c(V) \to T^c(V)$, introduced by Eilenberg and MacLane [3], turns $(T^c(V), \Delta, \mu_{sh})$ into a graded bialgebra.

This multiplication is defined as a graded coalgebra map induced by the universal property of $T^c(V)$ by $\alpha : T^c(V) \otimes T^c(V) \to V$ given by $\alpha(v \otimes 1) = \alpha(1 \otimes v) = v$ and $\alpha = 0$ otherwise. This multiplication is associative and in fact is given by

$$\mu_{sh}([a_1, \dots, a_m] \otimes [a_{i+1}, \dots, a_n]) = \sum \pm [a_{\sigma(1)}, \dots, a_{\sigma(n)}]),$$

where the summation is taken over all (m, n)-shuffles.

4.1. *Bar construction of an A_∞-algebra.* Let $(M, \{m_i\})$ be an A_∞-algebra. We consider the tensor coalgebra $T^c(s^{-1}M)$ where $s^{-1}M$ is the *desuspension* of M, i.e. $(s^{-1}M)^n = M^{n+1}$. We use the standard notation $s^{-1}a_1 \otimes \cdots \otimes s^{-1}a_n = [a_1, \dots, a_n]$. The structure maps m_i define the map $\beta : T^c(s^{-1}M) \to s^{-1}M$ by $\beta[a_1, \dots, a_n] = [s^{-1}m_n(a_1 \otimes \cdots \otimes a_n)]$. Extending this β as a coderivation we obtain $d_\beta : T^c(s^{-1}M) \to T^c(s^{-1}M)$ which in fact looks as

$$d_\beta[a_1, \dots, a_n] = \sum_k \pm [a_1, \dots, a_k, m_j(a_{k+1} \otimes \cdots \otimes a_{k+j}), a_{k+j+1}, \dots a_n].$$

The defining condition (1) of an A_∞-algebra guarantees that $d_\beta d_\beta = 0$. The resulting dg coalgebra $(T^c(s^{-1}M), d_\beta, \Delta)$ is called *bar construction* of A_∞-algebra $(M, \{m_i\})$ and is denoted by $\tilde{B}(M)$.

For an A_∞-algebra of type $(M, \{m_1, m_2, 0, 0, \dots\})$ this bar construction coincides with the ordinary bar construction of this dga.

A morphism of A_∞-algebras $\{f_i\} : (M, \{m_i\}) \to (M', \{m'_i\})$ defines a dg coalgebra map of bar constructions $F = \tilde{B}(\{f_i\})$ as follows: $\{f_i\}$ defines the map $\alpha : T^c(s^{-1}M) \to s^{-1}M$ by $\alpha[a_1, \dots, a_n] = [s^{-1}f_n(a_1 \otimes \cdots \otimes a_n)]$. Extending this α as a coalgebra map we obtain $F : T^c(s^{-1}M) \to T^c(s^{-1}M)$ which in fact looks as

$$F[a_1, \dots, a_n] = \sum \pm [f_{k_1}(a_1 \otimes \cdots \otimes a_{k_1}), \dots, f_{k_t}(a_{n-k_t+1} \otimes \cdots \otimes a_n)].$$

The defining condition (2) of an A_∞ morphism guarantees that F is a chain map.

Now we are able to show that the composition of A_∞ morphisms is correctly defined: to the composition of morphisms (3) corresponds the composition of dg coalgebra maps

$$\tilde{B}((M, \{m_i\})) \xrightarrow{\tilde{B}(\{f_i\})} \tilde{B}((M', \{m'_i\})) \xrightarrow{\tilde{B}(\{g_i\})} \tilde{B}((M'', \{m''_i\}))$$

which is a dg coalgebra map, thus for the projection $p_1\tilde{B}(\{g_i\})\tilde{B}(\{f_i\})$, i.e. for the collection $\{h_i\}$, the condition (2) is satisfied.

4.2. *Bar construction of a C_∞-algebra.* The notion of C_∞-algebra is motivated by the following observation. If a dg algebra (A, d, μ) is graded commutative then the differential of the bar construction BA is not only a coderivation but also a derivation with respect to the shuffle product, so the bar construction $(BA, d_\beta, \Delta, \mu_{sh})$ of a cdga is a dg bialgebra.

By definition the bar construction of an A_∞-algebra $(M, \{m_i\})$ is a dg coalgebra $\tilde{B}(M) = (T^c(s^{-1}M), d_\beta, \Delta)$.

But if $(M,\{m_i\})$ is a C_∞-algebra, then $\tilde{B}(M)$ becomes a dg bialgebra:

Proposition 4.1. *For an A_∞-algebra $(M,\{m_i\})$ the differential of the bar construction d_β is a derivation with respect to the shuffle product if and only if each operation m_i vanishes on shuffles, that is, $(M,\{m_i\})$ is a C_∞-algebra.*

Proof. The map $\Phi : T^c(s^{-1}M)\otimes T^c(s^{-1}M) \to T^c(s^{-1}M)$ defined as $\Phi = d_\beta\mu_{sh} - \mu_{sh}(d_\beta\otimes id + id\otimes d_\beta)$ is a coderivation. Thus, according to the universal property of $T^c(s^{-1}M)$ the map Φ is trivial if and only if $p_1\Phi = 0$ and the last condition means exactly (5).

Proposition 4.2. *Let $\{f_i\} : (M,\{m_i\}) \to (M',\{m'_i\})$ be an A_∞-algebra morphism of C_∞-algebras. Then the induced map of bar constructions $\tilde{B}\{f_i\}$ is a map of dg bialgebras if and only if each f_i vanishes on shuffles, that is, $\{f_i\}$ is a morphism of C_∞-algebras.*

Proof. The map $\Psi = \tilde{B}\{f_i\}\mu_{sh} - \mu_{sh}(\tilde{B}\{f_i\}\otimes\tilde{B}\{f_i\})$ is a coderivation. Thus, according to the universal property of $T^c(s^{-1}M)$ the map Ψ is trivial if and only if $p_1\Psi = 0$ and the last condition means exactly (6).

Thus the bar functor maps the subcategory of C_∞-algebras to the category of dg bialgebras.

4.3. *Adjunctions.* The bar and cobar functors

$$B : DGAlg \to DGCoalg,\ \Omega : DGCoalg \to DGAlg$$

are adjoint and there exist standard weak equivalences $\Omega B(A) \to A$, $\quad C \to B\Omega C$. So $\Omega B(A) \to A$ is a *free resolution* of a dga A.

If A is commutative, then the cobar-bar resolution gets out of the category: $\Omega B(A)$ is not commutative.

In this case instead of the cobar-bar functors we must use the adjoint functors Γ, $\mathcal{A}$, see [26], which we describe now.

For a commutative dg algebra the bar construction is a dg bialgebra, so the restriction of the bar construction is the functor $B : CDGAlg \to DGBialg$. Furthermore, the functor of indecomposables $Q : DGBialg \to DGLieCoalg$ maps the category of dg bialgebras to the category of dg Lie calgebras. Let Γ be the composition

$$\Gamma : CDGAlg \xrightarrow{B} DGBialg \xrightarrow{Q} DGLieCoalg.$$

There is the adjoint of Γ $\mathcal{A} : DGLieCoalg \to CDGAlg$, which is dual to the Chevalley-Eilenberg functor. There is the standard weak equivalence $\mathcal{A}\Gamma A \to A$.

5. Minimality. Let $\{f_i\} : (M,\{m_i\}) \to (M',\{m'_i\})$ be a morphism of A_∞-algebras. It follows from (2) that the first component $f_1 : (M,m_1) \to (M',m'_1)$ is a chain map.

A weak equivalence of A_∞-algebras is defined as a morphism $\{f_i\}$ for which $B(\{f_i\})$ is a weak equivalence of dg coalgebras. A standard spectral sequence argument allows to prove the following

Proposition 5.1. *A morphism of A_∞-algebras is a weak equivalence if and only if its first component $f_1 : (M,m_1) \to (M',m'_1)$ is a weak equivalence of chain complexes.*

PROPOSITION 5.2. *A morphism of A_∞-algebras is an isomorphism if and only if its first component $f_1 : (M, m_1) \to (M', m'_1)$ is an isomorphism.*

Proof. The components of the opposite morphism $\{g_i\} : (M', \{m'_i\}) \to (M, \{m_i\})$ can be solved inductively from the equation $\{g_i\}\{f_i\} = \{id_M, 0, 0, \dots\}$.

DEFINITION 5.3. We call an A_∞-algebra $(M, \{m_i\})$ minimal if $m_1 = 0$.

In this case (M, m_2) is a *strictly* associative graded algebra.

From the above propositions easily follows

PROPOSITION 5.4. *Each weak equivalence of minimal A_∞-algebras is an isomorphism.*

It is clear that all the above is true for C_∞-algebras, thus

PROPOSITION 5.5. *Each weak equivalence of minimal C_∞-algebras is an isomorphism.*

DEFINITION 5.6. We call a minimal A_∞-algebra (C_∞-algebra) $(M, \{m_i\})$ degenerate if it is isomorphic in the category of A_∞ (C_∞) algebras to the graded (commutative) algebra (M, m_2).

6. Minimal A_∞ and C_∞ algebras and Hochschild and Harrison cohomology. Here we present the connection of the notion of minimal A_∞ (resp. C_∞)-algebra with Hochschild (resp. Harrison) cochain complexes, studied in [13], see also [18].

Let H be a graded algebra. Consider the Hochshild cochain complex $C^{*,*}(H, H)$ which is bigraded in this case:

$$C^{n,m}(H, H) = Hom^m(H^{\otimes n}, H),$$

where Hom^m means homomorphisms of degree m.

This bigraded complex carries a structure of homotopy Gerstenhaber algebra, see [13], [7], [6], [18], which consists of the following structure maps:

(i) The Hochschild differential $\delta : C^{n-1,m}(H, H) \to C^{n,m}(H, H)$ given by

$$\begin{aligned}\delta f(a_1 \otimes \cdots \otimes a_n) = a_1 \cdot f(a_2 \otimes \cdots \otimes a_n)& \\ + \sum_k \pm f(a_1 \otimes \cdots \otimes a_{k-1} \otimes a_k \cdot a_{k+1} \otimes \cdots \otimes a_n)& \\ \pm f(a_1 \otimes \cdots \otimes a_{n-1}) \cdot a_n.&\end{aligned}$$

(ii) The $\smile$ product defined by

$$f \smile g(a_1 \otimes \cdots \otimes a_{n+m}) = f(a_1 \otimes \cdots \otimes a_n) \cdot g(a_{n+1} \otimes \cdots \otimes a_{n+m}).$$

(iii) The *brace* operations $f\{g_1, \dots, g_i\}$ which we write as

$$f\{g_1, \dots, g_i\} = E_{1,i}(f; g_1, \dots, g_i),$$
$$E_{1,i} : C^{n,m} \otimes C^{n_1,m_1} \otimes \cdots \otimes C^{n_i,m_i} \to C^{n+\sum n_t - i, m+\sum m_t},$$

given by

$$E_{1,i}(f;g_1,\dots,g_i)(a_1\otimes\dots\otimes a_{n+n_1+\dots+n_i-i}) \tag{7}$$
$$= \sum_{k_1,\dots,k_i} \pm f(a_1\otimes\dots\otimes a_{k_1}\otimes g_1(a_{k_1+1}\otimes\dots\otimes a_{k_1+n_1})\otimes$$
$$\dots\otimes a_{k_2}\otimes g_2(a_{k_2+1}\otimes\dots\otimes a_{k_2+n_2})\otimes a_{k_2+n_2+1}\otimes\dots$$
$$\otimes a_{k_i}\otimes g_i(a_{k_i+1}\otimes\dots\otimes a_{k_i+n_i})\otimes\dots\otimes a_{n+n_1+\dots+n_i-i}).$$

The first brace operation $E_{1,1}$ has the properties of Steenrod's $\smile_1$ product, so we use the notation $E_{1,1}(f,g)=f\smile_1 g$. In fact this is Gerstenhaber's $f\circ g$ product [4], [5].

Now let $(H,\{m_i\})$ be a minimal A_∞-algebra, so (H,m_2) is an associative graded algebra with multiplication $a\cdot b=m_2(a\otimes b)$.

Each operation m_i can be considered as a Hochschild cochain $m_i\in C^{i,2-i}(H,H)$. Let $m=m_3+m_4+\dots\in C^{*,2-*}(H,H)$. The defining condition of an A_∞-algebra (2.1) means exactly $\delta m=m\smile_1 m$. So a minimal A_∞-algebra structure on H in fact is a *twisting cochain* in the Hochschild complex with respect to the $\smile_1$ product.

There is a notion of equivalence of such twisting cochains: $m\sim m'$ if there exists $p=p^{2,-1}+p^{3,-2}+\dots+p^{i,1-i}+\dots,\ ,\ p^{i,1-i}\in C^{i,1-i}(H,H)$ such that

$$m-m'=\delta p+p\smile p+p\smile_1 m+m'\smile_1 p+E_{1,2}(m';p,p)+E_{1,3}(m';p,p,p)+\cdots. \tag{8}$$

PROPOSITION 6.1. *Two twisting cochains $m,m'\in C^{*,2-*}(H,H)$ are equivalent if and only if $(H,\{m_i\})$ and $(H',\{m'_i\})$ are isomorphic A_∞-algebras.*

Proof. Indeed, $\{p_i\}:(H,\{m_i\})\to(H,\{m'_i\})$ with $p_1=id,\ p_i=p^{i,1-i}$ is the needed isomorphism: the condition (8) coincides with the defining condition (2) of a morphism of A_∞-algebras and Proposition 5.2 implies that this morphism is an isomorphism.

This gives the possibility of perturbation of twisting cochains without changing their equivalence class:

PROPOSITION 6.2. *If m is a twisting cochain (i.e. a minimal A_∞-algebra structure on H) and $p\in C^{n,1-n}(H,H)$ is an arbitrary cochain, then there exists a twisting cochain $\bar m$, equivalent to m, such that $m_i=\bar m_i$ for $i\le n$ and $\bar m_{n+1}=m_{n+1}+\delta p$.*

Proof. The twisting cochain $\bar m$ can be solved inductively from the equation (8).

THEOREM 6.3. *Suppose for a graded algebra H that in Hochschild cohomology*

$$Hoch^{n,2-n}(H,H)=0$$

for $n\ge 3$. Then each $m\sim 0$, that is each minimal A_∞-algebra structure on H is degenerate.

Proof. From the equality $\delta m=m\smile_1 m$ in dimension 4 we obtain $\delta m_3=0$ i.e. m_3 is a cocycle. Since $Hoch^{3,-1}(H,H)=0$ there exists $p^{2,-1}$ such that $m_3=\delta p^{2,-1}$. Perturbing our twisting cochain m by $p^{2,-1}$ we obtain a new twisting cochain $\bar m=\bar m_3+\bar m_4+\dots$ equivalent to m with $\bar m_3=0$. Now the component $\bar m_4$ becomes a cocycle, which can be killed using $Hoch^{4,-2}(H,H)=0$ etc.

Suppose now that (H,μ) is a commutative graded algebra. The Harrison cochain complex $\bar C^*(H,H)$ is defined as a subcomplex of the Hochschild complex consisting of cochains which vanish on shuffles. If $(H,\{m_i\})$ is a C_∞-algebra then the twisting element $m = m_3 + m_4 + \dots$ belongs to the Harrison subcomplex $\bar{C}^*(H,H) \subset C^*(H,H)$ and we have:

THEOREM 6.4. *Suppose that for a graded commutative algebra H in Harrison cohomology*

$$Harr^{n,2-n}(H,H) = 0$$

for $n \geq 3$. Then each $m \sim 0$, that is each minimal C_∞-algebra structure on H is degenerate.

7. The A_∞-algebra structure in homology. Let (A,d,μ) be a dg algebra and let $(H(A),\mu^*)$ be its homology algebra. Although the product in $H(A)$ is associative, there appears a structure of a (generally nondegenerate) minimal A_∞-algebra, which can be considered as an A_∞ *deformation* of $(H(A),\mu^*)$, [18] . Namely, in [11], [12] the following result was proved (see also [23], [8]):

THEOREM 7.1. *Suppose that all homology modules $H^i(A)$ of a dg algebra A are free. Then there exist: a structure of minimal A_∞-algebra $(H(A),\{m_i\})$ on $H(A)$ and a weak equivalence of A_∞-algebras*

$$\{f_i\} : (H(A),\{m_i\}) \to (A,\{d,\mu,0,0,\dots\})$$

such that $m_1 = 0$, $m_2 = \mu^*$, $f_1^* = id_{H(A)}$.

Furthermore, for a dga map $f : A \to A'$ there exists a morphism of A_∞-algebras $\{f_i\} : (H(A)\{m_i\}) \to (H(A')\{m_i'\})$ with $f_1 = f^*$.

Such a structure is unique up to isomorphism in the category of A_∞-algebras: if $(H(A),\{m_i\})$ and $(H(A),\{m_i'\})$ are two such A_∞-algebra structures on $H(A)$ then for $id : A \to A$ there exists $\{f_i\} : (H(A)\{m_i\}) \to (H(A)\{m_i'\})$ with $f_1 = id$, so by Proposition 5.2 $\{f_i\}$ is an isomorphism.

Let us look at the first new operation $m_3 : H(A) \otimes H(A) \otimes H(A) \to H(C)$. Let $f_1 : H(A) \to A$ be a cycle-choosing homomorphism: $f_1(a) \in a \in H(A)$. This map is not multiplicative but $f_1(a \cdot b) - f_1(a) \cdot f(b) \sim 0 \in C$ so there exists $f_2 : H(A) \otimes H(A) \to A$ s.t. $f_1(a \cdot b) - f_1(a) \cdot f(b) = \partial f_2(a \otimes b)$. We define $m_3(a \otimes b \otimes c) \in H(A)$ as the homology class of the cycle

$$f_1(a) \cdot f_2(b \otimes c) \pm f_2(a \cdot b \otimes c) \pm f_2(a \otimes b \cdot c) \pm f_2(a \otimes b) \cdot f_1(c).$$

From this description immediately follows the connection of m_3 with Massey products: If $a,b,c \in H(A)$ is a Massey triple, i.e. if $a \cdot b = b \cdot c = 0$, then $m_3(a \otimes b \otimes c)$ belongs to the Massey product $\langle a,b,c\rangle$. This gives examples of dg algebras with essentially nontrivial homology A_∞-algebras.

7.1. *Main examples and applications.* Taking $A = C^*(X)$, the cochain dg algebra of a 1-connected space X, we obtain an A_∞-algebra structure $(H^*(X),\{m_i\})$ on the cohomology algebra $H^*(X)$.

The cohomology algebra equipped with this additional structure carries more information than just the cohomology algebra. Some applications of this structure are given in [12], [15]. For example the cohomology A_∞-algebra $(H^*(X),\{m_i\})$ determines the cohomology of the loop space $H^*(\Omega X)$ whereas just the algebra $(H^*(X),m_2)$ does not:

THEOREM 7.2. *$H(\tilde{B}(H^*(X),\{m_i\}))=H^*(\Omega X)$.*

Taking $A=C_*(G)$, the chain dg algebra of a topological group G, we obtain an A_∞-algebra structure $(H_*(G),\{m_i\})$ on the Pontriagin algebra $H_*(G)$. The homology A_∞-algebra $(H_*(G),\{m_i\})$ determines the homology of the classifying space $H_*(B_G)$ whereas just the Pontriagin algebra $(H_*(G),m_2)$ does not:

THEOREM 7.3. *$H(B(\tilde{H_*}(G),\{m_i\}))=H_*(B_G)$.*

8. C_∞-algebra structure in homology of a commutative dg algebra. There is a commutative version of the above main theorem, see [14], [15], [21]:

THEOREM 8.1. *Suppose that all homology R-modules $H^i(A)$ of a commutative dg algebra A are free. Then there exist: a structure of minimal C_∞-algebra $(H(A),\{m_i\})$ on $H(A)$ and a weak equivalence of C_∞-algebras*

$$\{f_i\}:(H(A),\{m_i\})\to(A,\{d,\mu,0,0,\dots\})$$

such that $m_1=0$, $m_2=\mu^*$, $f_1^*=id_{H(A)}$.

Furthermore, for a cdga map $f:A\to A'$ there exists a morphism of C_∞-algebras $\{f_i\}:(H(A)\{m_i\})\to(H(A')\{m_i'\})$ with $f_1=f^*$.

Such a structure is unique up to isomorphism in the category of C_∞-algebras.

Below we present some applications of this C_∞-algebra structure in rational homotopy theory.

9. Applications in rational homotopy theory

9.1. *Classification of rational homotopy types.* Let X be a 1-connected space. In the case of rational coefficients there exists Sullivan's *commutative* cochain complex $A(X)$ of X. It is well known that the weak equivalence type of the cdg algebra $A(X)$ determines the rational homotopy type of X: two 1-connected spaces X and Y are *rationally homotopy equivalent* if and only if $A(X)$ and $A(Y)$ are weakly homotopy equivalent cdg algebras. Indeed, in this case $A(X)$ and $A(Y)$ have *isomorphic* minimal models $M_X\approx M_Y$, and this implies that X and Y are rationally homotopy equivalent. This is the key geometrical result of Sullivan which we are going to exploit below.

Now we take $A=A(X)$ and apply Theorem 8.1. Then we obtain on $H(A)=H^*(X,Q)$ a structure of minimal C_∞ algebra $(H^*(X,Q),\{m_i\})$ which we call *rational cohomology C_∞-algebra of X*.

Generally an isomorphism of rational cohomology algebras $H^*(X,Q)$ and $H^*(Y,Q)$ does not imply the existence of a homotopy equivalence $X\sim Y$ even rationally. We claim that $(H^*(X,Q),\{m_i\})$ is a *complete* rational homotopy invariant:

THEOREM 9.1. *Two 1-connected spaces X and X' are rationally homotopy equivalent if and only if $(H^*(X,Q),\{m_i\})$ and $(H^*(X',Q),\{m_i'\})$ are isomorphic as C_∞-algebras.*

Proof. If $X \sim X'$, then $A(X)$ and $A(X')$ are weakly equivalent, that is, there exists a cgda A and two weak equivalences $A(X) \leftarrow A \to A(X')$. This implies the existence of two weak equivalences of the corresponding homology C_∞-algebras

$$(H^*(X,Q),\{m_i\}) \leftarrow (H^*(A),\{m_i\}) \to (H^*(X',Q),\{m_i'\}),$$

which by minimality are both isomorphisms.

Conversely, suppose that $(H^*(X,Q),\{m_i\}) \approx (H^*(X',Q),\{m_i'\})$. Then

$$\mathcal{A}QB(H^*(X,Q),\{m_i\}) \approx \mathcal{A}QB(H^*(X',Q),\{m_i'\}).$$

Denote this cdga by A. Then we have weak equivalences of CGD algebras

$$A(X) \leftarrow \mathcal{A}\Gamma A(X) \leftarrow A \to \mathcal{A}\Gamma A(X') \to A(X').$$

This theorem in fact classifies rational homotopy types with given cohomology algebra H as all possible minimal C_∞-algebra structures on H modulo C_∞ isomorphisms.

Example. Here we describe an example which we will use to illustrate the results of this and the forthcoming sections.

We consider the following commutative graded algebra. Its underlying graded Q-vector space has the generators: e of dimension 0, $x,\ y$ of dimension 2, and z of dimension 5, so

$$H^* = \{H^0 = Q_e,\ 0,\ H^2 = Q_x \oplus Q_y,\ 0,\ 0,\ H^5 = Q_z,\ 0,\ 0,\ \dots\}, \tag{9}$$

and the multiplication is trivial by dimensional reasons, with unit e. In fact

$$H^* = H^*(S^2 \vee S^2 \vee S^5, Q).$$

This example was considered in [9] and it was shown that there are just two rational homotopy types with such cohomology algebra.

The same result can be obtained from our classification.

What minimal C_∞-algebra structures are possible on H^*?

By dimensional reasons only one nontrivial operation $m_3 : H^2 \otimes H^2 \otimes H^2 \to H^5$ is possible.

The defining condition of a C_∞-algebra, namely the vanishing on shuffles implies that

$$m_3(x,x,x) = 0,\ m_3(y,y,y) = 0,\ m_3(x,y,x) = 0,\ m_3(y,x,y) = 0$$

and

$$m_3(x,x,y) = m_3(y,x,x), \quad m_3(x,y,y) = m_3(y,y,x).$$

Thus each C_∞-algebra structure on H^* is characterized by a couple of rational numbers $p,\ q$:

$$m_3(x,x,y) = pz, \quad m_3(x,y,y) = qz.$$

So we write a minimal C_∞-algebra structure on H^* as a column vector $\binom{p}{q}$.

Now let us look at the structure of an isomorphism of C_∞-algebras

$$\{f_i\} : (H^*, m_3) \to (H^*, m_3').$$

Again by dimensional reasons just one non-trivial component $f_1 : H^* \to H^*$ is possible, which in its turn consists of two isomorphisms

$$f_1^2 : H^2 = Q_x \oplus Q_y \to H^2 = Q_x \oplus Q_y, \quad f_1^5 : H^5 = Q_z \to H^5 = Q_z.$$

The first one is represented by a nondegenerate matrix $A = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$,

$$f_1^2(x) = ax \oplus by, \quad f_1^2(y) = cx \oplus dy,$$

and the second one by a nonzero rational number r, $f_2^5(z) = rz$.

A calculation shows that the condition $f_1^5 m_3 = m_3'(f_1^2 \otimes f_1^2 \otimes f_1^2)$, to which the defining condition of an A_∞-algebra morphism (2) reduces, looks as

$$r\begin{pmatrix} p \\ q \end{pmatrix} = det\ A\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p' \\ q' \end{pmatrix}.$$

This condition shows that two minimal C_∞-algebra structures $m_3 = \left(\begin{smallmatrix} p \\ q \end{smallmatrix}\right)$ and $m_3' = \left(\begin{smallmatrix} p' \\ q' \end{smallmatrix}\right)$ are isomorphic if and only if they are related by a nondegenerate linear transformation.

Thus there exist just two isomorphism classes of minimal C_∞-algebra on H^*: the trivial one $(H^*, m_3 = 0)$ and the nontrivial one $(H^*, m_3 \neq 0)$. So we have just two rational homotopy types whose rational cohomology is H^*. We denote them X and Y respectively and we analyze them in the next sections.

Below we give some applications of cohomology C_∞-algebra in various problems of rational homotopy theory.

9.2. *Formality.* Among rational homotopy types with given cohomology algebra, there is one called *formal* which is the "formal consequence of its cohomology algebra" (Sullivan). Explicitly this is the type whose minimal model M_X is isomorphic to the minimal model of the cohomology $H^*(X, Q)$.

Our C_∞ model implies the following criterion of formality:

THEOREM 9.2. *X is formal if and only if its cohomology C_∞-algebra is degenerate, i.e. it is C_∞ isomorphic to one with $m_{\geq 3} = 0$.*

Below we deduce using this criterion some known results about formality.

1. A commutative graded 1-connected algebra H is called *intrinsically formal* if there is only one homotopy type with the cohomology algebra H, that of course is formal.

The above Theorem 6.4 immediately implies the following sufficient condition for formality due to Tanré [27]:

THEOREM 9.3. *If for a 1-connected graded Q-algebra H one has*

$$Harr^{k,k-2}(H, H) = 0, \ \ k = 3, 4, \ldots$$

then H is intrinsically formal, that is there exists only one rational homotopy type with $H^*(X, Q) \approx H$.

2. The following theorem of Halperin and Stasheff from [9] is an immediate consequence of our criterion:

THEOREM 9.4. *A commutative graded Q-algebra of type*

$$H = \{H^0 = Q, 0, 0, \ldots, 0, H^n, H^{n+1}, \ldots, H^{3n-2}, 0, 0, \ldots\}$$

is intrinsically formal.

Proof. Since $deg\ m_i = 2 - i$ there is no room for operations $m_{i>2}$, indeed the shortest range is $m_3 : H^n \otimes H^n \otimes H^n \to H^{3n-1} = 0$.

3. From Theorem 9.2 it easily follows:

THEOREM 9.5. *Any 1-connected commutative graded algebra H with $H^{2k} = 0$ is intrinsically formal.*

Proof. Any A_∞-operation m_i has degree $2-i$, thus

$$m_i : H^{2k_1+1} \otimes \cdots \otimes H^{2k_i+1} \to H^{2(k_1+\cdots+k_i+1)} = 0.$$

Thus any C_∞ operation is trivial too.

From this a result of Baues follows: any space with trivial even dimensional cohomology has the rational homotopy type of a wedge of spheres. Indeed, the cohomology algebra is realized as the cohomology algebra of a wedge of spheres, and by intrinsic formality this is the only homotopy type.

EXAMPLE. The algebra H^* from the example of the previous section is not intrinsically formal since there are two homotopy types, X and Y, with $H^*(X,Q) = H^* = H^*(Y)$. The space X is formal (and actually $X = S^2 \vee S^2 \vee S^5$), since its cohomology C_∞-algebra $(H^*, m_3 = 0)$ is trivial. But the space Y is not: its cohomology C_∞-algebra $(H^*, m_3 \neq 0)$ is not degenerate.

We remark here that the formal type is represented by $X = S^2 \vee S^2 \vee S^5$ and it is possible to show that the nonformal one is represented by $Y = S^2 \vee S^2 \cup_{f:S^4 \to S^2 \vee S^2} e^5$, where the attaching map f is a nontrivial element from $\pi_4(S^2 \vee S^2) \otimes Q$.

9.3. *Rational homotopy groups.* Since the cohomology C_∞-algebra $(H^*(X,Q), \{m_i\})$ determines the rational homotopy type it must determine the rational homotopy groups $\pi_i(X) \otimes Q$ too. We present a chain complex whose homology is $\pi_i(X) \otimes Q$. Moreover the Lie algebra structure is determined as well.

For the cohomology C_∞-algebra $(H^*(X,Q), \{m_i\})$ the corresponding bar construction $B(H^*(X,Q), \{m_i\})$ is a dg bialgebra. Acting on this bialgebra by the functor Q of indecomposables we obtain a dg Lie coalgebra.

On the other hand rational homotopy groups $\pi_*(\Omega X) \otimes Q$ form a graded Lie algebra with respect to Whitehead product. Thus the dual cohomotopy groups $\pi^*(\Omega X, Q) = (\pi_*(\Omega X) \otimes Q)^*$ form a graded Lie coalgebra.

THEOREM 9.6. *The homology of the dg Lie coalgebra $QB(H^*(X,Q), \{m_i\})$ is isomorphic to the cohomotopy Lie coalgebra $\pi^*(\Omega X, Q)$.*

Proof. The theorem follows from the sequence of graded Lie coalgebra isomorphisms:

$$\begin{aligned} \pi^*(\Omega X, Q) \approx (\pi_*(\Omega X, Q))^* \approx (PH_*(\Omega x, Q)^* \approx QH^*(\Omega X, Q) \approx \\ QH(B(A(X)) \approx QH(\tilde{B}(H^*(X,Q), \{m_i\}) \approx \\ H(Q\tilde{B}(H^*(X,Q), \{m_i\}). \end{aligned}$$

EXAMPLE. For the algebra H^* from the previous examples the complex $QB(H^*)$ in low dimensions looks as

$$0 \to Q_x \oplus Q_y \xrightarrow{0} Q_{x\otimes x} \oplus Q_{x\otimes y} \oplus Q_{y\otimes y} \xrightarrow{0} Q_{x\otimes x\otimes y} \oplus Q_{x\otimes y\otimes y} \xrightarrow{d=m_3} Q_z \oplus \dots\ .$$

The differential $d = m_3$ is trivial for the formal space X and is nontrivial for Y. Thus for both rational homotopy types we have

$$\pi^2 = H^1(QB(H^*)) = 2Q, \quad \pi^3 = H^2(QB(H^*)) = 3Q,$$

and

$$\pi^4(X) = H^3(QB(H^*), d = 0) = 2Q, \quad \pi^4(Y) = H^3(QB(H^*), d \neq 0) = Ker\ d = Q.$$

9.4. *Realization of homomorphisms.* Let $G : H^*(X, Q) \to H^*(Y, Q)$ be a homomorphism of cohomology algebras. When is this homomorphism realizable as a map of rationalizations $g : Y_Q \to X_Q$, $f^* = F$? In the case when G is an isomorphism this question was considered in [9]. It was considered also in [28]. The following theorem gives a complete answer:

THEOREM 9.7. *A homomorphism G is realizable if and only if it is extendable to a C_∞-map $\{g_1 = G, g_2, g_3, \dots\} : (H^*(X, Q), \{m_i\}) \to (H^*(Y, Q), \{m'_i\})$.*

Proof. One implication is a consequence of the last part of Theorem 8.1.

To show the other implication we use Sullivan's minimal models M_X and M_Y of $A(X)$ and $A(Y)$. It is enough to show that the existence of $\{g_i\}$ implies the existence of a cdg algebra map $g : M_Y \to M_X$.

So we have C_∞-algebra maps

$$M_X \stackrel{\{f_i\}}{\leftarrow} (H^*(X, Q), \{m_i\}) \stackrel{\{g_i\}}{\to} (H^*(y, Q), \{m'_i\}) \stackrel{\{f'_i\}}{\to} M_Y.$$

Recall the following property of a minimal cdg algebra M: for a weak equivalence of cdg algebras $\phi : A \to B$ and a cdg algebra map $f : M \to B$ there exists a cdg algebra map $F : M \to A$ such that ϕF is homotopic to f. Using this property it is easy to show the existence of a cdga map $\beta : M_X \to \mathcal{A}QB(M_X)$, the right inverse of the standard map $\alpha : \mathcal{A}QB(M_X) \to M$. Composing this map with $\mathcal{A}QB(\{f'_i\})\mathcal{A}QB(\{g_i\})$ we obtain a cdga map

$$\mathcal{A}QB(\{f'_i\})\mathcal{A}QB(\{g_i\})\beta : M_X \to M_Y.$$

From this theorem immediately follows the

COROLLARY 9.8. *For formal X and Y spaces each $G : H^*(X, Q) \to H^*(Y, Q)$ is realizable.*

Proof. In this case $\{G, 0, 0, \dots\}$ is a C_∞ extension of G.

EXAMPLE. Consider the homomorphism

$$G : H^*(X) = H^*(Y) \to H^*(S^5)$$

induced by the standard imbedding $g : S^5 \to X = S^2 \vee S^2 \vee S^5$. Of course G is realizable as $g : S^5 \to X$ but not as $S^5 \to Y$. Indeed, for such realizability, according to Theorem 9.7, we need a C_∞-algebra morphism

$$\{g_i\} : (H^*, \{0, 0, m_3, 0, \dots\} \to (H^5(S^5, Q), \{0, 0, 0, \dots\})$$

with $g_1 = G$. By dimensional reasons all the components $g_2, g_3, \dots$ are trivial, so this morphism looks as $\{G, 0, 0, \dots\}$. But this collection is not a morphism of C_∞-algebras since the condition $Gm_3 = 0$, to which the defining condition (2) of an A_∞-algebra morphism degenerates, is not satisfied.

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