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ON THE RATE OF SUMMABILITY BY MATRIX MEANS IN THE GENERALIZED HÖLDER METRIC

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Abstract. We will generalize and improve the results of T. Singh [Publ. Math. Debrecen 40 (1992), 261–271] obtaining the L. Leindler type estimates from [Acta Math. Hungar. 104 (2004), 105–113].

1. Introduction. Let f be a continuous and 2π-periodic function and let

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1.1}$$

be its Fourier series. Denote by $S_n(x) = S_n(f;x)$ the n-th partial sum of (1.1) and by $\omega(f,\delta)$ the modulus of continuity of $f \in C_{2\pi}$.

The usual supremum norm will be denoted by $\|\cdot\|_C$.

Let ω be a nondecreasing continuous function on the interval $[0, 2\pi]$ having the properties

$$\omega(0) = 0, \quad \omega(\delta_1 + \delta_2) \le \omega(\delta_1) + \omega(\delta_2).$$

Such a function will be called a *modulus of continuity.*

Denote by H^ω the class of functions

$$H^\omega := \big\{ f \in C_{2\pi} : |f(x) - f(y)| \le C\omega(|x-y|) \big\},$$

where C is a positive constant. For $f \in H^\omega$, we define the norm $\|\cdot\|_\omega$ by the formula

$$\|f\|_\omega := \|f\|_C + \sup_{x,y} |\Delta^\omega f(x,y)|,$$

where

$$\Delta^\omega f(x,y) = \frac{|f(x) - f(y)|}{\omega(|x-y|)}, \quad x \ne y,$$

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and $\Delta^0 f(x,y)=0$. If $\omega(t)=C_1|t|^\alpha$ $(0<\alpha\le 1)$, where C_1 is a positive constant, then

$$H^\alpha=\left\{f\in C_{2\pi}:|f(x)-f(y)|\le C_1|x-y|^\alpha,\ 0<\alpha\le 1\right\}$$

is a Banach space and the metric induced by the norm $\|\cdot\|_\alpha$ on H^α is said to be a Hölder metric.

Let $A:=(a_{nk})$ $(k,n=0,1,\dots)$ be a lower triangular infinite matrix of real numbers satisfying the condition

$$a_{nk}\ge 0\ (k,n=0,1,\dots),\quad a_{nk}=0,\ k>n,\quad\text{and}\ \sum_{k=0}^{n}a_{nk}=1. \tag{1.2}$$

Let the A-transformation of $(S_n(f;x))$ be given by

$$T_n(f):=T_n(f;x):=\sum_{k=0}^{n}a_{nk}S_k(f;x)\quad(n=0,1,\dots). \tag{1.3}$$

Now, we define two classes of sequences (see [3]).

A sequence $c:=(c_n)$ of nonnegative numbers tending to zero is called the *Rest Bounded Variation Sequence*, or briefly $c\in RBVS$, if it has the property

$$\sum_{n=m}^{\infty}|c_n-c_{n+1}|\le K(c)c_m \tag{1.4}$$

for all natural numbers m, where $K(c)$ is a constant depending only on c.

A sequence $c:=(c_n)$ of nonnegative numbers will be called the *Head Bounded Variation Sequence*, or briefly $c\in HBVS$, if it has the property

$$\sum_{n=0}^{m-1}|c_n-c_{n+1}|\le K(c)c_m \tag{1.5}$$

for all natural numbers m, or only for all $m\le N$ if the sequence c has only finite number of nonzero terms and the last nonzero term is c_N.

Therefore we assume that the sequence $(K(\alpha_n))_{n=0}^\infty$ is bounded, that is, that there exists a constant K such that

$$0\le K(\alpha_n)\le K$$

holds for all n, where $K(\alpha_n)$ denotes the sequence of constants appearing in the inequalities (1.4) or (1.5) for the sequence $\alpha_n:=(a_{nk})_{k=0}^\infty$. Now, we can give the conditions to be used later on. We assume that for all n and $0\le m\le n$

$$\sum_{k=m}^{\infty}|a_{nk}-a_{nk+1}|\le Ka_{nm} \tag{1.6}$$

or

$$\sum_{k=0}^{m-1}|a_{nk}-a_{nk+1}|\le Ka_{nm} \tag{1.7}$$

if $\alpha_n:=(a_{nk})_{k=0}^\infty$ belongs to $RBVS$ or $HBVS$, respectively.

Let ω and ω^* be two given moduli of continuity satisfying the following condition (for $0 \le p < q \le 1$):

$$\frac{(\omega(t))^{p/q}}{\omega^*(t)} = O(1) \quad (t \to 0_+). \tag{1.8}$$

In [5] R. Mohapatra and P. Chandra obtained some results on degree of approximation by the means (1.3) in the Hölder metric. Recently, T. Singh in [6] established the following two theorems generalizing some results of P. Chandra [1] with a mediate function H such that

$$\int_u^\pi \frac{\omega(f;t)}{t^2}\,dt = O(H(u))\ (u \to 0_+), \quad H(t) \ge 0 \tag{1.9}$$

and

$$\int_0^t H(u)\,du = O(tH(t)) \quad (t \to 0_+). \tag{1.10}$$

THEOREM 1.1 ([6]). *Assume that* $A = (a_{nk})$ *satisfies condition* (1.2) *and* $a_{nk} \le a_{nk+1}$ *for* $k = 0, 1, \dots, n-1$; $n = 0, 1, \dots$. *Then for* $f \in H^\omega$, $0 \le p < q \le 1$

$$\begin{aligned}\|T_n(f) - f\|_{\omega^*} = O\Big[&\big\{\omega(|x-y|)\big\}^{p/q}\big\{\omega^*(|x-y|)\big\}^{-1}\\ &\times \Big\{\Big(H\Big(\frac{\pi}{n}\Big)\Big)^{1-p/q} a_{nn}(n^{p/q} + a_{nn}^{-p/q})\Big\}\Big] + O\Big(a_{nn}H\Big(\frac{\pi}{n}\Big)\Big),\end{aligned} \tag{1.11}$$

if $\omega(t)$ *satisfies* (1.9) *and* (1.10), *and*

$$\begin{aligned}\|T_n(f) - f\|_{\omega^*} = O\Big[&\big\{\omega(|x-y|)\big\}^{p/q}\big\{\omega^*(|x-y|)\big\}^{-1}\\ &\times \Big\{\Big(\omega\Big(\frac{\pi}{n}\Big)\Big)^{1-p/q} + a_{nn}n^{p/q}\Big(H\Big(\frac{\pi}{n}\Big)\Big)^{1-p/q}\Big\}\Big] + O\Big\{\omega\Big(\frac{\pi}{n}\Big) + a_{nn}H\Big(\frac{\pi}{n}\Big)\Big\},\end{aligned} \tag{1.12}$$

if $\omega(t)$ *satisfies* (1.9).

THEOREM 1.2 ([6]). *Assume that* $A = (a_{nk})$ *satisfies condition* (1.2) *and* $a_{nk} \ge a_{nk+1}$ *for* $k = 0, 1, \dots, n-1$; $n = 0, 1, \dots$ *and* $\omega(f;t)$ *satisfies* (1.9) *and* (1.10). *Then for* $f \in H^\omega$, $0 \le p < q \le 1$

$$\begin{aligned}\|T_n(f) - f\|_{\omega^*} = O\Big[&\{\omega(|x-y|)\}^{p/q}\{\omega^*(|x-y|)\}^{-1}\\ &\times \{(H(a_{n0}))^{1-p/q} a_{n0}(n^{p/q} + a_{n0}^{-p/q})\}\Big] + O(a_{n0}H(a_{n0})).\end{aligned} \tag{1.13}$$

Another generalization of the results of Chandra [2] was obtained by L. Leindler in [3]. Namely, he proved the following theorems.

THEOREM 1.3 ([3]). *Let* (1.2), (1.7) *and* (1.9) *hold. Then for* $f \in C_{2\pi}$

$$\|T_n(f) - f\|_C = O\Big(\omega\Big(\frac{\pi}{n}\Big)\Big) + O\Big(a_{nn}H\Big(\frac{\pi}{n}\Big)\Big). \tag{1.14}$$

If, in addition, $\omega(f;t)$ *satisfies condition* (1.10) *then*

$$\|T_n(f) - f\|_C = O(a_{nn}H(a_{nn})). \tag{1.15}$$

THEOREM 1.4 ([3]). *Let* (1.2), (1.6), (1.9) *and* (1.10) *hold. Then for* $f \in C_{2\pi}$

$$\|T_n(f) - f\|_C = O(a_{n0}H(a_{n0})). \tag{1.16}$$

In the present paper we will generalize and improve the results of T. Singh [6] obtaining the L. Leindler type estimates from [3] in the generalized Hölder metric instead of the supremum norm.

Throughout the paper we shall use the following notation:

$$\phi_x(t) = f(x+t) + f(x-t) - 2f(x),$$

$$A_{nk} = \sum_{r=n-k+1}^{n} a_{nr} \quad (k = 1, 2, \ldots, n+1),$$

$$A_n(u) = \sum_{k=0}^{n} a_{nk} \frac{\sin(k+\frac{1}{2})u}{\sin(u/2)} .$$

By K_1, $K_2, \ldots$ we shall designate either an absolute constant or a constant depending on the indicated parameters, not necessarily the same at each occurrence.

2. Main results.

Our main results are the following.

THEOREM 2.1. *Let* (1.2), (1.7) *and* (1.8) *hold. Suppose* $\omega(f;t)$ *satisfies* (1.9), *then for* $f \in H^\omega$

$$\|T_n(f) - f\|_{\omega^*} = O\Big(\Big\{\sum_{k=1}^{n+1} \frac{A_{nk}}{k}\Big\}^{p/q} \Big\{a_{nn} H\Big(\frac{\pi}{n}\Big)\Big\}^{1-p/q}\Big). \tag{2.1}$$

If, in addition, $\omega(f;t)$ *satisfies condition* (1.10), *then*

$$\|T_n(f) - f\|_{\omega^*} = O\Big(\Big\{\sum_{k=1}^{n+1} \frac{A_{nk}}{k}\Big\}^{p/q} \Big\{a_{nn} H(a_{nn})\Big\}^{1-p/q}\Big). \tag{2.2}$$

THEOREM 2.2. *Let* (1.2), (1.8), (1.6) *and* (1.9) *hold. Then, for* $f \in H^\omega$

$$\|T_n(f) - f\|_{\omega^*} = O\Big(\Big\{a_{n0} H\Big(\frac{\pi}{n}\Big)\Big\}^{1-p/q}\Big). \tag{2.3}$$

If, in addition, $\omega(f;t)$ *satisfies* (1.10), *then*

$$\|T_n(f) - f\|_{\omega^*} = O(\{a_{n0} H(a_{n0})\}^{1-p/q}). \tag{2.4}$$

REMARK 2.1. We can observe, that under the condition (1.8), Theorems 1.1 and 1.2 are the corollaries of Theorems 2.1 and 2.2, respectively. The assumption $a_{nk} \le a_{nk+1}$ $(k = 0, 1, \ldots, n-1;\ n = 0, 1, \ldots)$ of Theorem 1.1 implies the inequality

$$\sum_{k=1}^{n+1} \frac{A_{nk}}{k} \le (n+1) a_{nn},$$

whence by the Theorem 2.1, we obtain the relation of the (1.11) type. The estimate (1.13) from Theorem 1.2 is also a consequence of the estimate of Theorem 2.2 and sometimes is better since $(n^{p/q} a_{n0})$ can be unbounded.

REMARK 2.2. If in the assumptions of Theorems 2.1 or 2.2 we take $\omega(|t|) = O(|t|^q)$, $\omega^*(|t|) = O(|t|^p)$ with $p = 0$, then from (2.1), (2.2) and (2.3), we have the estimates (1.14), (1.15) and (1.16), respectively.

3. Lemmas. To prove our theorems we need the following lemmas.

LEMMA 3.1 ([2]). *If* (1.9) *and* (1.10) *hold then*

$$\int_0^r \frac{\omega(f;t)}{t}\,dt = O(rH(r)) \quad (r \to 0_+). \tag{3.1}$$

LEMMA 3.2 ([4]). *If* (1.7) *holds, then for* $\frac{1}{n} \le u \le \pi$

$$|A_n(u)| \le \frac{\pi^2(K+1)^2+\pi}{u}\,A_{n,\overline{u}^{-1}}, \tag{3.2}$$

where $\overline{u}^{\,-1} := \max\{1,[u^{-1}]\}$.

LEMMA 3.3 ([4]). *If* (1.6) *holds, then for* $f \in C_{2\pi}$

$$\|T_n(f)-f\|_C \le 8(K+1)(2K+1)\sum_{k=0}^{n} a_{nk}E_k(f), \tag{3.3}$$

where $E_n(f)$ *denotes the best approximation of function* f *by trigonometric polynomials of order at most* n.

LEMMA 3.4 ([4]). *If* (1.6) *holds, then*

$$\int_0^\pi |A_n(t)|\,dt \le 4K(K+1). \tag{3.4}$$

LEMMA 3.5. *If* (1.2), (1.6) *hold and* $\omega(f;t)$ *satisfies* (1.9) *then*

$$\sum_{k=0}^{n} a_{nk}\omega\Big(f;\frac{\pi}{k+1}\Big) = O\Big(a_{n0}H\Big(\frac{\pi}{n}\Big)\Big). \tag{3.5}$$

If, in addition, $\omega(f;t)$ *satisfies* (1.10) *then*

$$\sum_{k=0}^{n} a_{nk}\omega\Big(f;\frac{\pi}{k+1}\Big) = O(a_{n0}H(a_{n0})). \tag{3.6}$$

Proof. First we prove (3.5). If (1.6) holds, then

$$a_{nn}-a_{nm} \le |a_{nm}-a_{nn}| \le \sum_{k=m}^{n-1}|a_{nk}-a_{nk+1}| \le \sum_{k=m}^{\infty}|a_{nk}-a_{nk+1}| \le Ka_{nm}$$

for any $n \ge m \ge 0$, whence

$$a_{nn} \le (K+1)a_{nm}. \tag{3.7}$$

From this, using (1.9), we get

$$\sum_{k=0}^{n} a_{nk}\omega\Big(f;\frac{\pi}{k+1}\Big) \le (K+1)a_{n0}\sum_{k=0}^{n}\omega\Big(f;\frac{\pi}{k+1}\Big)$$
$$\le K_1a_{n0}\int_1^{n+1}\omega\Big(f;\frac{\pi}{t}\Big)\,dt = K_1a_{n0}\int_{\pi/(n+1)}^{\pi}\frac{\omega(f;u)}{u^2}\,du = O\Big(a_{n0}H\Big(\frac{\pi}{n}\Big)\Big).$$

Now, we prove (3.6). Since

$$(K+1)(n+1)a_{n0} \ge \sum_{k=0}^{n} a_{nk} = 1,$$

we can see that

$$\sum_{k=0}^{n} a_{nk}\omega\Big(f;\frac{\pi}{k+1}\Big) \le \sum_{k=0}^{k^*} a_{nk}\omega\Big(f;\frac{\pi}{k+1}\Big) + \sum_{k=k^*}^{n} a_{nk}\omega\Big(f;\frac{\pi}{k+1}\Big),$$

where $k^* = \frac{1}{(K+1)a_{n0}} - 1$. Using again (3.7), (1.2) and the monotonicity of the modulus of continuity, we obtain

$$\begin{aligned}\sum_{k=0}^{n} a_{nk}\omega\Big(f;\frac{\pi}{k+1}\Big) &\le (K+1)a_{n0}\sum_{k=0}^{k^*}\omega\Big(f;\frac{\pi}{k+1}\Big) + \omega(f;\pi(K+1)a_{n0})\sum_{k=k^*}^{n} a_{nk}\\ &\le K_1 a_{n0}\int_1^{k^*+1}\omega\Big(f;\frac{\pi}{t}\Big)\,dt + \omega(f;\pi(K+1)a_{n0})\\ &\le K_1 a_{n0}\int_{a_{n0}}^{\pi}\frac{\omega(f;u)}{u^2}\,du + 2\pi(K+1)\omega(f;a_{n0}).\end{aligned} \tag{3.8}$$

According to

$$\omega(f;a_{n0}) \le 4\omega\Big(f;\frac{a_{n0}}{2}\Big) \le 8\int_{a_{n0}/2}^{a_{n0}}\frac{\omega(f;t)}{t}\,dt \le 8\int_0^{a_{n0}}\frac{\omega(f;t)}{t}\,dt,$$

in view of (3.8), (1.9) and (1.10), relation (3.6) holds. ■

4. Proofs of the theorems. In this section we shall prove Theorems 2.1 and 2.2.

Proof of Theorem 2.1. First we prove (2.1). Setting

$$R_n(x) = T_n(f;x) - f(x) = \frac{1}{2\pi}\int_0^{\pi}\phi_x(t)A_n(t)\,dt$$

and

$$R_n(x,y) = R_n(x) - R_n(y) = \frac{1}{2\pi}\int_0^{\pi}(\phi_x(t)-\phi_y(t))A_n(t)\,dt$$

we get

$$|R_n(x,y)| \le \frac{1}{2\pi}\int_0^{\pi}|\phi_x(t)-\phi_y(t)|\,|A_n(t)|\,dt.$$

It is clear that

$$|\phi_x(t)-\phi_y(t)| \le 4C\omega(|x-y|) \tag{4.1}$$

and

$$|\phi_x(t)-\phi_y(t)| \le 4\omega(f;|t|). \tag{4.2}$$

Then, using (4.1), we have

$$|R_n(x,y)| \le \frac{2C}{\pi}\,\omega(|x-y|)\Big(\int_0^{\pi/n} + \int_{\pi/n}^{\pi}\Big)|A_n(t)|\,dt = \frac{2C}{\pi}\,\omega(|x-y|)(I_1+I_2). \tag{4.3}$$

It is obvious that

$$I_1 \le \int_0^{\pi/n}\frac{1}{\sin(t/2)}\sum_{k=0}^{n} a_{nk}\Big|\sin\Big(k+\frac12\Big)t\Big|\,dt \le \pi\int_0^{\pi/n}\sum_{k=0}^{n} a_{nk}\Big(k+\frac12\Big)\,dt \le \frac32\,\pi^2. \tag{4.4}$$

Using (3.2), we obtain

$$\begin{aligned} I_2 &\le K_1 \int_{\pi/n}^{\pi} \frac{A_{n,t^{-1}}}{t}\,dt = K_1 \int_{1/\pi}^{n/\pi} \frac{A_{n,t}}{t}\,dt = K_1 \sum_{k=1}^{n-1} \int_{k/\pi}^{(k+1)/\pi} \frac{A_{n,t}}{t}\,dt \\ &\le K_1 \sum_{k=1}^{n-1} \frac{A_{n,k+1}}{k} \le 2K_1 \sum_{k=2}^{n} \frac{A_{n,k}}{k} \le 2K_1 \sum_{k=1}^{n+1} \frac{A_{n,k}}{k}\,. \end{aligned} \tag{4.5}$$

If (1.7) holds then

$$a_{n\mu} - a_{nm} \le |a_{n\mu} - a_{nm}| \le \sum_{k=\mu}^{m-1} |a_{nk} - a_{nk+1}| \le K a_{nm}$$

for any $m \ge \mu \ge 0$, whence

$$a_{n\mu} \le (K+1) a_{nm}. \tag{4.6}$$

From this and (1.2) we can observe that

$$\sum_{k=1}^{n+1} \frac{A_{n,k}}{k} = \sum_{k=1}^{n+1} \frac{1}{k} \sum_{r=n-k+1}^{n} a_{nr} \ge \frac{1}{K+1} \sum_{k=1}^{n+1} a_{n,n-k+1} = \frac{1}{K+1} \sum_{k=0}^{n} a_{nk} = \frac{1}{K+1}$$

and by (4.3)–(4.5) we obtain

$$|R_n(x,y)| \le K_2 \omega(|x-y|) \sum_{k=1}^{n+1} \frac{A_{n,k}}{k}\,. \tag{4.7}$$

On the other hand, by (4.2), we have

$$\begin{aligned} |R_n(x,y)| &\le \frac{2}{\pi} \int_0^{\pi} \omega(f;t) |A_n(t)|\,dt \\ &= \frac{2}{\pi} \Big(\int_0^{\pi/n} + \int_{\pi/n}^{\pi} \Big) \omega(f;t) |A_n(t)|\,dt = \frac{2}{\pi}\,(I_1' + I_2'). \end{aligned} \tag{4.8}$$

Using (4.6) and (1.9), we can estimate the quantities I_1' and I_2' as follows:

$$\begin{aligned} I_1' &\le \omega\Big(f;\frac{\pi}{n}\Big) \int_0^{\pi/n} \frac{1}{\sin(t/2)} \sum_{k=0}^{n} a_{nk} \Big|\sin\Big(k+\frac{1}{2}\Big)t\Big|\,dt \\ &\le \frac{3}{2}\,\pi^2 \omega\Big(f;\frac{\pi}{n}\Big) \sum_{k=0}^{n} a_{nk} \le 3\pi^2 (K+1) a_{nn} \sum_{k=1}^{n} \omega\Big(f;\frac{\pi}{k}\Big) \\ &\le K_3 a_{nn} \int_1^{n} \omega\Big(f;\frac{\pi}{t}\Big)\,dt = K_3 a_{nn} \int_{\pi/n}^{\pi} \frac{\omega(f;u)}{u^2}\,du = O\Big(a_{nn} H\Big(\frac{\pi}{n}\Big)\Big) \end{aligned} \tag{4.9}$$

and, by (3.2),

$$I_2' \le K_4 \int_{\pi/n}^{\pi} \omega(f;t) \frac{A_{n,t^{-1}}}{t}\,dt \le K_5 a_{nn} \int_{\pi/n}^{\pi} \frac{\omega(f;t)}{t^2}\,dt = O\Big(a_{nn} H\Big(\frac{\pi}{n}\Big)\Big). \tag{4.10}$$

Combining (4.8)–(4.10) we obtain

$$|R_n(x,y)| = O\Big(a_{nn} H\Big(\frac{\pi}{n}\Big)\Big). \tag{4.11}$$

Therefore, using (4.7) and (4.11),

$$\sup_{x,y}\{\Delta^{\omega^*} R_n(x,y)\} = \sup_{x,y}\Big\{\frac{|R_n(x,y)|^{p/q}}{\omega^*(|x-y|)}|R_n(x,y)|^{1-p/q}\Big\}$$
$$\le K_4\Big\{\sum_{k=1}^{n+1}\frac{A_{nk}}{k}\Big\}^{p/q}\Big\{a_{nn}H\Big(\frac{\pi}{n}\Big)\Big\}^{1-p/q}. \quad (4.12)$$

Since

$$|R_n(x)| \le \frac{1}{2\pi}\int_0^\pi |\phi_x(t)|\,|A_n(t)|\,dt \le \frac{1}{\pi}\int_0^\pi \omega(f;t)|A_n(t)|\,dt,$$

the inequalities (4.4), (4.5), (4.8) and (4.9) lead us to

$$\|T_n(f)-f\|_C \le \frac{1}{\pi}\Big\{\int_0^\pi \omega(f;t)|A_n(t)|\,dt\Big\}^{p/q}\Big\{\int_0^\pi \omega(f;t)|A_n(t)|\,dt\Big\}^{1-p/q}$$
$$\le \frac{1}{\pi}(\omega(f;\pi))^{p/q}\Big\{\int_0^\pi |A_n(t)|\,dt\Big\}^{p/q}\Big\{\int_0^\pi \omega(f;t)|A_n(t)|\,dt\Big\}^{1-p/q}$$
$$= \frac{1}{\pi}(\omega(f;\pi))^{p/q}\Big\{\Big(\int_0^{\pi/n}+\int_{\pi/n}^{\pi}\Big)|A_n(t)|\,dt\Big\}^{p/q}\Big\{\Big(\int_0^{\pi/n}+\int_{\pi/n}^{\pi}\Big)\omega(f;t)|A_n(t)|\,dt\Big\}^{1-p/q}$$
$$\le K_5\Big\{\sum_{k=1}^{n+1}\frac{A_{nk}}{k}\Big\}^{p/q}\Big\{a_{nn}H\Big(\frac{\pi}{n}\Big)\Big\}^{1-p/q}. \quad (4.13)$$

Collecting our partial results (4.12) and (4.13), we obtain that (2.1) holds.

Now, we prove (2.2). By (4.2) we have

$$|R_n(x,y)| \le \frac{2}{\pi}\int_0^\pi \omega(f;t)|A_n(t)|\,dt$$
$$= \frac{2}{\pi}\Big(\int_0^{a_{nn}}+\int_{a_{nn}}^{\pi}\Big)\omega(f;t)|A_n(t)|\,dt = \frac{2}{\pi}\,(J_1+J_2). \quad (4.14)$$

Using (1.9) and (1.10), we shall estimate the quantities J_1 and J_2 similarly like the quantities I_1' and I_2', respectively. Namely, by Lemma 3.1,

$$J_1 \le \pi\int_0^{a_{nn}} \frac{\omega(f;t)}{t}\,dt = O(a_{nn}H(a_{nn}))$$

and, by (3.2),

$$J_2 \le K_6\int_{a_{nn}}^{\pi} \omega(f;t)\,\frac{A_{n,t^{-1}}}{t}\,dt \le K_7 a_{nn}\int_{a_{nn}}^{\pi}\frac{\omega(f;t)}{t^2}\,dt = O(a_{nn}H(a_{nn})).$$

From this and (4.12) we get

$$\sup_{x,y}\{\Delta^{\omega^*} R_n(x,y)\} = O\Big(\Big\{\sum_{k=1}^{n+1}\frac{A_{nk}}{k}\Big\}^{p/q}\{a_{nn}H(a_{nn})\}^{1-p/q}\Big). \quad (4.15)$$

In the same manner as in (4.13) we can show that

$$\|T_n(f)-f\|_C \le K_5\Big\{\sum_{k=1}^{n+1}\frac{A_{nk}}{k}\Big\}^{p/q}\{a_{nn}H(a_{nn})\}^{1-p/q}. \quad (4.16)$$

Combining (4.15) and (4.16) we conclude that (2.2) holds. This completes the proof. ■

Proof of Theorem 2.2. Using the same notation as in the proof of Theorem 2.1, from (4.1) and (3.4) we get

$$|R_n(x,y)| \le \frac{2C}{\pi}\,\omega(|x-y|)\int_0^\pi |A_n(t)|\,dt \le \frac{8CK(K+1)}{\pi}\,\omega(|x-y|). \tag{4.17}$$

On the other hand

$$|R_n(x,y)| \le |T_n(f;x)-f(x)| + |T_n(f;y)-f(y)|.$$

The estimate (3.3) and the inequality

$$E_n(f) \le K_1\omega\Big(f;\frac{\pi}{n+1}\Big)$$

give

$$|R_n(x,y)| \le 16(K+1)(2K+1)\sum_{k=0}^{n} a_{nk}E_k(f) \le K_2\sum_{k=0}^{n} a_{nk}\omega\Big(f,\frac{\pi}{k+1}\Big). \tag{4.18}$$

Therefore, by (4.17),

$$\begin{aligned}\sup_{x,y}\{\Delta^{\omega^*}R_n(x,y)\} &= \sup_{x,y}\Big\{\frac{|R_n(x,y)|^{p/q}}{\omega^*(|x-y|)}|R_n(x,y)|^{1-p/q}\Big\}\\ &\le K_3\Big\{\sum_{k=0}^{n} a_{nk}\omega\Big(f,\frac{\pi}{k+1}\Big)\Big\}^{1-p/q}.\end{aligned} \tag{4.19}$$

The same estimate can be shown for the deviation $T_n(f;x)-f(x)$. Namely, by (3.3) and (1.2), we get

$$\begin{aligned}\|T_n(f)-f\|_C &\le 8(K+1)(2K+1)\sum_{k=0}^{n} a_{nk}E_k(f) \le K_4\sum_{k=0}^{n} a_{nk}\omega\Big(f,\frac{\pi}{k+1}\Big)\\ &\le K_5\Big\{\sum_{k=0}^{n} a_{nk}\omega\Big(f,\frac{\pi}{k+1}\Big)\Big\}^{p/q}\Big\{\sum_{k=0}^{n} a_{nk}\omega\Big(f,\frac{\pi}{k+1}\Big)\Big\}^{1-p/q}\\ &\le K_6\Big\{\sum_{k=0}^{n} a_{nk}\omega\Big(f,\frac{\pi}{k+1}\Big)\Big\}^{1-p/q}.\end{aligned} \tag{4.20}$$

Finally, collecting our partial results (4.19) and (4.20) and using Lemma 3.5 we obtain (2.3) and (2.4). ■

References

[1] P. Chandra, *On the degree of approximation of a class of functions by means of Fourier series*, Acta Math. Hungar. 52 (1988), 199–205.

[2] P. Chandra, *A note on the degree of approximation of continuous function*, Acta Math. Hungar. 62 (1993), 21–23.

[3] L. Leindler, *On the degree of approximation of continuous functions*, Acta Math. Hungar. 104 (2004), 105–113.

[4] W. Łenski, B. Szal, *On the approximation of functions by matrix means in the generalized Hölder metric*, in: Function Spaces VIII, Banach Center Publ. 79, Polish Acad. Sci. Inst. Math., Warsaw, 2008, 119–129.

[5] R. N. Mohapatra, P. Chandra, *Degree of approximation of functions in the Hölder metric*, Acta Math. Hungar. 41 (1983), 67–76.

[6] T. Singh, *Degree of approximation to functions in a normed spaces*, Publ. Math. Debrecen 40 (1992), 261–271.

[7] X.-H. Sun, *Degree of approximation of functions in the generalized Hölder metric*, Indian J. Pure Appl. Math. 27 (1996), 407–417.