ALGEBRA, GEOMETRY AND MATHEMATICAL PHYSICS
BANACH CENTER PUBLICATIONS, VOLUME 93
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2011

THE DEFORMATION RELATION ON THE SET OF COHEN-MACAULAY MODULES ON A QUOTIENT SURFACE SINGULARITY

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Abstract. Let X be a quotient surface singularity, and define $\mathbf{G}^{\mathrm{def}}(X,r)$ as the directed graph of maximal Cohen-Macaulay (MCM) modules with edges corresponding to deformation incidences. We conjecture that the number of connected components of $\mathbf{G}^{\mathrm{def}}(X,r)$ is equal to the order of the divisor class group of X, and when X is a rational double point (RDP), we observe that this follows from a result of A. Ishii. We view this as an enrichment of the McKay correspondence. For a general quotient singularity X, we prove the conjecture under an additional cancellation assumption. We discuss the deformation relation in some examples, and in particular we give all deformations of an indecomposable MCM module on a rational double point.

1. Introduction. Suppose $X = \mathbb{C}^2/G$ is a quotient surface singularity and r a rational number. Let $\mathbf{G}^{\mathrm{def}}(X,r)$ be the directed graph with isomorphism classes $[M]$ of rank r MCM $\mathcal{O}_X$-modules M and directed edges $[M_1] \to [M_2]$ if there is a deformation of M_1 to M_2. The purpose of this note is to discuss the following conjecture:

Conjecture (A). Suppose $X = \mathbb{C}^2/G$ is a quotient surface singularity. Then the number of connected components in the graph $\mathbf{G}^{\mathrm{def}}(X,r)$ is equal to the order of the abelianization $G/[G,G]$ of G.

2010 *Mathematics Subject Classification*: Primary 14B12; Secondary 14D20, 13C14, 14J17.
Key words and phrases: quotient singularities, modules, deformation theory.
The paper is in final form and no version of it will be published elsewhere.

DOI: 10.4064/bc93-0-3

It is known that $G/[G,G]$ is isomorphic to the divisor class group $C(X)$. For any rational surface singularities $C(X)$ has order given as $|\det(E_i \cdot E_j)|$, where the E_i's are the exceptional curves of the minimal resolution.

There is a graph $\mathbf{G}^{\text{nef}}(X,r)$ defined in terms of the combinatorial properties of the minimal resolution of X. When X is a rational double point, we interpret a result of A. Ishii as an isomorphism of the two graphs;

$$\mathbf{G}^{\text{def}}(X,r) \cong \mathbf{G}^{\text{nef}}(X,r),$$

and view this as an enrichment of the McKay correspondence.

For any rational surface singularity we prove that the number of irreducible components of $\mathbf{G}^{\text{nef}}(X,r)$ is given as $|\det(E_i \cdot E_j)|$. Thus the conjecture follows in the case of rational double points. More generally, there is a well-defined map $\mathbf{G}^{\text{def}}(X,r) \to \mathbf{G}^{\text{nef}}(X,r)$ which is not an isomorphism however. We prove that $|\det(E_i \cdot E_j)|$ is a lower bound for the number of components of $\mathbf{G}^{\text{def}}(X,r)$ for all rational surface singularities, and by relating to the Grothendieck group, we prove equality for any quotient surface singularities under an additional cancellation assumption.

2. Notation. We will denote by X an isolated singularity, and we will assume that $X = \operatorname{Spec}(R)$ where R is the henselization of a local $\mathbb{C}$-algebra essentially of finite type. Let $C(X)$ denote the divisor class group of reflexive rank one R-modules. We say that X is rational if it is normal and if for any resolution $\pi : \widetilde{X} \to X$, $R^i\pi_*\mathcal{O}_{\widetilde{X}} = 0$ for $i > 0$. If X is a rational surface singularity, the exceptional set E is a union of irreducible components $E_i \cong \mathbb{P}^1$. There is a *fundamental cycle* Z supported on E, defined by $\mathfrak{m}\mathcal{O}_{\widetilde{X}}$ where $\mathfrak{m}$ is the maximal ideal of R. The divisor may be constructed as the unique smallest effective divisor $Z = \sum r_i E_i$ satisfying $Z \cdot E_i \leq 0$, see [2]. Quotient surface singularities are rational. To a rational surface singularity one attaches a graph $\Gamma(X)$ with nodes corresponding to each E_i and where there is an (undirected) edge between E_i and E_j if $E_i \cap E_j \neq \emptyset$. The only rational surface singularities that are Gorenstein, are the rational double points (RDP). These are exactly the quotient singularities $X = \mathbb{C}^2/G$ where G is a finite subgroup of $\mathrm{SL}(2,\mathbb{C})$.

3. The deformation relation. In this section we introduce the deformation relation on the set of isomorphism classes of MCM modules on an isolated singularity X.

DEFINITION 3.1. Let M be a coherent $\mathcal{O}_X$-module. A *deformation* (M_S, S, s) of M is a coherent $\mathcal{O}_{X\times S}$-module M_S which is S-flat. Here S is a scheme of finite type over $\mathbb{C}$ and $s \in S$ is a closed point (called the central point) such that the fiber over s is M.

Let M and N be two MCM modules on X. We say that M *locally deforms to* N, denoted $M \dashrightarrow_{\text{def}} N$, if there is a deformation (M_T, T, t) of M such that $\overline{\operatorname{Loc}}(N) \subseteq T$ strictly contains the central $\mathbb{C}$-point t corresponding to M. Here $\overline{\operatorname{Loc}}(N)$ denotes the Zariski-closure of the set of $\mathbb{C}$-points $r \in T(\mathbb{C})$ such that the pullback M_r of M_T to r is isomorphic to N.

The deformation relation is reflexive and transitive, but not symmetric. Transitivity follows from the existence of a versal family and openness of versality, see [4] and [6].

Given a module M, it is in principle possible to obtain the deformation relation by computing a versal deformation. However, this is infeasible in practice, and in general it is not easy to determine the deformation relation.

Definition 3.2. Let $\mathbf{MCM}(X)$ be the set of isomorphism classes of MCM modules on X and let $\mathbf{MCM}(X, r)$ be the set of isomorphism classes of rank r MCM modules. Denote by $\mathbf{G}^{\mathrm{def}}(X)$ and $\mathbf{G}^{\mathrm{def}}(X, r)$ the directed graphs with vertices corresponding to these sets and directed edges corresponding to $\dashrightarrow_{\mathrm{def}}$.

The (maximal) depth property is equivalent to the vanishing of Ext groups and is preserved in flat families, so that MCM modules only deforms to MCM modules.

4. Relating the deformation graph to the resolution graph. In this section we relate the deformation graph $\mathbf{G}^{\mathrm{def}}(X, r)$ to a graph defined in terms of divisors on the minimal resolution $\pi : \widetilde{X} \to X$ of a rational surface singularity X.

If M is an MCM module on X, we define the corresponding full sheaf as $\widetilde{M} := \pi^* M/\operatorname{tors}$. This is a locally free sheaf on $\widetilde{X}$, see [1]. We have:

Proposition 4.1 ([4, Corollary 4.10]). *If $M \dashrightarrow_{\mathrm{def}} N$ then $c_1(\widetilde{N}) - c_1(\widetilde{M})$ is effective with support on the exceptional set of $\pi : \widetilde{X} \to X$.*

The map $\mathrm{Pic}(\widetilde{X}) \to \mathbb{Z}^n$ given by sending $d \in \mathrm{Pic}(\widetilde{X})$ to $(d \cdot E_1, \ldots, d \cdot E_n)$ is an isomorphism, see [5, Prop. 14.4, Th. 12.1], and hence $\mathrm{Pic}(\widetilde{X})$ is the free abelian group generated by the divisors D_i with $E_j \cdot D_i = \delta_{ij}$.

We have the following exact sequence of abelian groups

$$0 \to \oplus_{i=1}^{n} \mathbb{Z}E_i \xrightarrow{\alpha} \mathrm{Pic}(\widetilde{X}) \to H \to 0 \tag{1}$$

where H is defined as the cokernel. The map α is given by the intersection matrix, when considered as a map $\mathbb{Z}^n \xrightarrow{\alpha} \mathbb{Z}^n$, and the group H has order $|\det(E_i \cdot E_j)|$. By Proposition 17.1 of [5], we have that H is isomorphic to the divisor class group $C(X)$ of X.

Definition 4.2. A divisor $d \in \mathrm{Pic}(\widetilde{X})$ is *nef* if $d = \sum n_i D_i$ with $n_i \in \{0, 1, 2, \ldots\}$ for all i. We let $\mathbf{N}(X) \subset \mathrm{Pic}(\widetilde{X})$ denote the set of nef divisors. If d and d' are *nef* divisors, let $d \dashrightarrow_{\mathrm{nef}} d'$ if $d' - d$ is effective and supported on the exceptional set E.

The relation $\dashrightarrow_{\mathrm{nef}}$ is a transitive and reflexive relation on the set $\mathbf{N}(X)$ of nef divisors. We denote also by $\dashrightarrow_{\mathrm{nef}}$ the relation on $\mathbf{N}(X) \times \mathbb{N}$ given by $(d, r) \dashrightarrow_{\mathrm{nef}} (d', r')$ if and only if $d \dashrightarrow_{\mathrm{nef}} d'$ and $r = r'$.

Definition 4.3. Let the *graph of nef divisors* $\mathbf{G}^{\mathrm{nef}}(X)$ be the directed graph with pairs $(d, r) \in \mathbf{N}(X) \times \mathbb{N}$ as vertices and edges given by $(d, r) \dashrightarrow_{\mathrm{nef}} (d', r')$.

If $M \dashrightarrow_{\mathrm{def}} N$ then $c_1(\widetilde{M}) \dashrightarrow_{\mathrm{nef}} c_1(\widetilde{N})$, by Proposition 4.1, i.e. there is a map of directed graphs

$$(c_1, r) : \mathbf{G}^{\mathrm{def}}(X, r) \to \mathbf{G}^{\mathrm{nef}}(X).$$

This will be an important tool in studying $\mathbf{G}^{\mathrm{def}}(X, r)$.

Consider the composition $\mathbf{N}(X) \hookrightarrow \mathrm{Pic}(\widetilde{X}) \to H$, and denote by $\sim^{\mathrm{nef}}$ the equivalence relation generated by $\dashrightarrow_{\mathrm{nef}}$. Forming the quotient $\mathbf{N}(X)/\sim^{\mathrm{nef}}$, we have the following.

Proposition 4.4. *The map*

$$\mathbf{N}(X) \hookrightarrow \operatorname{Pic}(\widetilde{X}) \to H$$

induces an isomorphism $\mathbf{N}(X)/{\sim}^{\mathrm{nef}} \cong H \cong C(X)$.

Proof. Set $H' := \mathbf{N}(X)/{\sim}^{\mathrm{nef}}$. Then H' is a monoid with a well defined sum $[u]+[v] = [u+v]$, and there is a natural monoid homomorphism $H' \to H$ induced by the composition $\mathbf{N}(X) \hookrightarrow \operatorname{Pic}(\widetilde{X}) \to H$ since $u {\sim}^{\mathrm{nef}} v$ implies $u - v$ is supported on the exceptional set, i.e. $u-v$ is in the image of α in (1). We claim that $H' \to H$ is an isomorphism.

For surjectivity, suppose $[x] \in H$ with $x \in \operatorname{Pic}(\widetilde{X})$, then $x = y - z$ with $y, z \in \mathbf{N}(X)$. Let $x' = y + (h-1)z$ where $h = |\det(E_i \cdot E_j)|$ is equal to the order of the group H. Then $x' \in \mathbf{N}(X)$ and $[x'] \in H'$ maps to $[x] \in H$.

For injectivity, suppose $[x] \in \mathbf{N}(X)$ such that $[x]$ maps to zero in H. Then $x = \sum r_i E_i$. We claim that $r_i \leq 0$ for all i. By collecting the positive r_i we may write x as $x = d_1 - d_2$ where $\operatorname{Supp} d_1 \cap \operatorname{Supp} d_2 = \emptyset$ and both d_1 and d_2 are effective. Then $x \cdot d_1 = d_1^2 - d_1 d_2 \leq 0$ but since $[x] \in \mathbf{N}(\mathbf{X})$ we must have $x \cdot d_1 \geq 0$. From $d_1^2 - d_1 d_2 = 0$, we conclude $d_1^2 = 0$, and since the intersection form is definite, we conclude that $d_1 = 0$, so all $r_i \leq 0$. Thus if $[x] \in H'$, let $[z] = (h-1)[x]$. Then $[x] + [z] \mapsto 0$ in H. By the argument above, we have that $[x] + [z] = 0$ in H'. It follows that $[x]$ has an inverse, and hence that H' is a group. Now, it follows that the map $H' \to H$ is injective. ∎

5. An enrichment of the McKay correspondence. In this section X is a rational double point, i.e. X is a quotient of $\mathbb{C}^2$ by a finite subgroup $G \subset \mathrm{SL}(2, \mathbb{C})$. For any rational surface singularity, we have that $M \dashrightarrow_{\mathrm{def}} N$ implies $M \dashrightarrow_{\mathrm{nef}} N$. For rational double points the converse is shown by A. Ishii, see [4], where an important step is the classification of the minimal differences $c_1(\widetilde{N}) - c_1(\widetilde{M})$. The following proposition is stated without proof in [4].

Proposition 5.1. *Assume X is a rational double point. If $d \dashrightarrow_{\mathrm{nef}} d'$ is a minimal relation then either $d' - d = E_i$ for some i, or $d' - d = Z_\Gamma$ where Z_Γ is the fundamental cycle of a connected sub-graph Γ of the intersection graph such that $d'E_i = 0$ for all E_i corresponding to a vertex in Γ.*

Proof. Suppose $d \dashrightarrow_{\mathrm{nef}} d'$ is minimal and $d' = d + F$ with F effective with support on E. By the negative definiteness there is an i with $FE_i < 0$ and if $(d+F)E_i > KE_i$ then $dE_i \geq -E_i^2$, thus $d + E_i$ is nef, $E_i \subseteq \operatorname{Sup} F$ and $F = E_i$. Assume therefore $(d+F)E_i \leq KE_i = -E_i^2 - 2 = 0$. Let I be the largest connected sub-graph of nodes containing i such that $FE_j \leq 0$ and $(d+F)E_j \leq -E_j^2 - 2$ for all $j \in I$. Let Z_I be the smallest element among the effective divisors with support on E with $\operatorname{Sup} Z_I \subseteq I$ such that $Z_I E_j \leq 0$ for all $j \in I$, remark that F satisfies this, so in particular $Z_I \leq F$. Also remark that $\operatorname{Sup} Z_I = I$. Therefore $G = F - Z_I$ is effective with support on E. We have $(d+G)E_j = (d+F)E_j - Z_I E_j \geq 0$ for all $j \notin \operatorname{adj} I$ where $\operatorname{adj} I$ is the set of nodes adjacent to (but not in) I. If $j \in \operatorname{adj} I$, then

$$(d+G)E_j = (d+F)E_j - \Big(\sum_{j' \in \operatorname{adj}\{j\} \cap I} r_{j'} E_{j'} \Big) E_j \tag{2}$$

where $Z_I = \sum_{j\in I} r_j E_j$. It follows by inspection of the fundamental cycles for the double points, that $\sum_{j'\in \mathrm{adj}\{j\}\cap I} r_{j'} \leq 1$. Since either $FE_j \geq 1$ or $(d+F)E_j \geq -E_j^2 - 1 \geq 1$ it follows from the minimality of F that $G = 0$ and $F = Z_I$. Since there are no cycles in the graph, $|\,\mathrm{adj}\{j\} \cap I| = 1$. ∎

In both of the two cases of the proposition, the existence of a deformation is proven by identifying the minimal stratum in the versal deformation space, see Theorem 5.5 in [4].

We define $\mathbf{G}^{\mathrm{nef}}(X, r)$ to be the full subgraph of $\mathbf{G}^{\mathrm{nef}}(X)$ corresponding to pairs (d, r) where r is fixed and where $d \cdot Z \leq r$ where Z is the fundamental cycle of X.

Theorem 5.2. *Let X be a rational double point. The map $(c_1, \mathrm{rank}) : \mathbf{G}^{\mathrm{def}}(X, r) \to \mathbf{G}^{\mathrm{nef}}(X)$ induces an isomorphism*

$$\mathbf{G}^{\mathrm{def}}(X, r) \cong \mathbf{G}^{\mathrm{nef}}(X, r)\,.$$

Proof. It follows from Theorem 1.11 in [1], that two MCM modules M and N of the same rank r are isomorphic if and only if $c_1(\widetilde{N}) = c_1(\widetilde{M})$, and from Proposition 4.1, it follows that the map (c_1, rank) realizes $\mathbf{G}^{\mathrm{def}}(X, r)$ as a subgraph of $\mathbf{G}^{\mathrm{nef}}(X)$. To identify this subgraph, consider a MCM module $M \cong R^{\alpha_0} \oplus M_1^{\alpha_1} \oplus \cdots \oplus M_n^{\alpha_n}$ of rank $r = \alpha_0 + \sum_{i=1}^n \alpha_i \,\mathrm{rank}\, M_i$, where M_i is the unique indecomposable with $c_1(\widetilde{M}) = D_i$. Then $d := c_1(\widetilde{M}) = \sum_{i=1}^n \alpha_i D_i$ and we get that $d \cdot Z = \sum_{i=1}^n \alpha_i D_i \cdot Z = \sum_{i=1}^n \alpha_i \cdot \mathrm{rank}\, M_i = r - \alpha_0$. Thus $d \cdot Z \leq r$. On the other hand if $d = \sum_{i=1}^n \alpha_i D_i$ and $d \cdot Z \leq r$ then $M \cong R^{\alpha_0} \oplus M_1^{\alpha_1} \oplus \cdots \oplus M_n^{\alpha_n}$ with $\alpha_0 := r - d\cdot Z$ maps to (d, r) under the map (c_1, rank). By Theorem 5.5 in [4] it follows hat $M \dashrightarrow_{\mathrm{nef}} N$ implies $M \dashrightarrow_{\mathrm{def}} N$, hence the image of (c_1, rank) is the full graph $\mathbf{G}^{\mathrm{nef}}(X, r)$. ∎

The classical McKay correspondence is the bijection of the vertices of the two graphs. The theorem enriches this correspondence by introducing a relation (arrows) on the vertices and by proving that they correspond to the deformation relation on the MCM modules.

6. Analysis of the conjecture. We have defined $\sim^{\mathrm{nef}}$ on $\mathbf{N}(X)$ and $\mathbf{N}(X) \times \mathbb{N}$ as the equivalence relation generated by $\dashrightarrow_{\mathrm{nef}}$. This may also be viewed as an equivalence relation on $\mathbf{MCM}(X)$ and $\mathbf{MCM}(X, r)$ by defining $[M] \sim^{\mathrm{nef}} [N]$ if and only if $\mathrm{rank}\, M = \mathrm{rank}\, N$ and $c_1(\widetilde{M}) \sim^{\mathrm{nef}} c_1(\widetilde{N})$.

We have a map $(c_1, \mathrm{rank}) : \mathbf{MCM}(X) \to \mathbf{N}(X) \oplus \mathbb{N}$ given by $[M] \mapsto (c_1(\widetilde{M}), \mathrm{rank}\, M)$. This is a monoid homomorphism where $\mathbf{MCM}(X)$ is viewed as a monoid under the direct sum operation and where the operation on $\mathbf{N}(X) \oplus \mathbb{N}$ is given as $(d, r) + (d', r') = (d + d, r + r')$, the sum in the first summand being addition of divisors and the sum in the second summand being addition of integers. From proposition 4.1, if $M \dashrightarrow_{\mathrm{def}} N$ then $c_1(\widetilde{M}) \dashrightarrow_{\mathrm{nef}} c_1(\widetilde{N})$. Note also that $M \dashrightarrow_{\mathrm{def}} N$ implies $M \oplus M' \dashrightarrow_{\mathrm{def}} N \oplus M'$ for any MCM module M'. Since rank is preserved in deformation, we get a homomorphism of monoids

$$\mathbf{MCM}(X)/{\sim^{\mathrm{def}}} \to \mathbf{N}(X)/{\sim^{\mathrm{nef}}} \oplus \mathbb{N} \cong C(X) \oplus \mathbb{N}.$$

Proposition 6.1. *Let X be any rational surface singularity. The homomorphism*

$$\mathbf{MCM}(X)/{\sim^{\mathrm{def}}} \to C(X) \oplus \mathbb{N}$$

is surjective, and composing with the projection on the first summand, we also have a surjective homomorphism of monoids

$$\mathbf{MCM}(X, r)/\sim^{\mathrm{def}} \to C(X).$$

Proof. For any element in $[d] \in H \cong C(X)$ there corresponds a MCM module M of rank one so that $c_1(\widetilde{M}) \in \mathrm{Pic}(\widetilde{X})$ maps to $[d] \in C(X)$ under the map $\mathrm{Pic}(\widetilde{X}) \to H$, see [5].

If $r \in \mathbb{N}$ then the module $c_1(\widetilde{\mathcal{O}}^{r-1} \oplus \widetilde{M}\ ) = c_1(\widetilde{M})$ and $R^{r-1} \oplus M$ has rank r. Thus $[M] \mapsto (c_1(\widetilde{M}), r)$. ∎

LEMMA 6.2. *If we have an exact sequence*

$$0 \to M' \to M \to M'' \to 0, \tag{3}$$

then $M' \oplus M'' \dashrightarrow_{\mathrm{def}} M$.

Proof. Let $S = \mathrm{Spec}(\mathbb{C}[t])$ and consider the inclusions $j_\alpha : X \hookrightarrow X \times S$ defined by $j_\alpha(x) = (x, \alpha)$ for $\alpha \in \mathbb{C}$. We claim that there is an S-flat family M_S on $X \times S$ such that $j_\alpha^* M_S \cong M$ for all $\alpha \neq 0$ and $j_0^* M_S \cong M' \oplus M''$. In fact, let

$$d = \begin{pmatrix} d' & t\eta \\ 0 & d'' \end{pmatrix}$$

and consider the $R[t]$-free complex $(F'[t] \oplus F''[t], d)$ where (F', d') and (F'', d'') are R-free resolutions of M' and M'' and η is a 1-cocycle in the Yoneda complex $\mathrm{Hom}(F'', F')$ representing the extension (3). Define $M_S = H^0(d)$. If $p : X \times S \to X$ is the projection, then the maps of resolutions give a short exact sequence of $\mathcal{O}_{X\times S}$-modules

$$0 \to p^*M' \to M_S \to p^*M'' \to 0$$

and hence M_S is S-flat. ∎

Let $\mathbf{K}_0(X)$ denote the Grothendieck group of (MCM) modules on X, and let $\mathbf{K}_0^+(X)$ be the subsemigroup of $\mathbf{K}_0(X)$ generated by elements of the form $\sum r_i M_i$ where $r_i \geq 0$ and M_i are MCM modules. It is known that $\mathbf{K}_0(X) \cong C(X) \oplus \mathbb{Z}$. For a rational surface singularity $C(X)$ is finite, see [5]. It follows that $\mathbf{K}_0^+(X) \to \mathbf{K}_0(X) \cong C(X) \oplus \mathbb{Z}$ has image $C(X) \oplus \mathbb{N}$.

PROPOSITION 6.3. *Let X be a rational surface singularity. Assume that $M \oplus M' \sim^{\mathrm{def}} N \oplus M'$ implies $M \sim^{\mathrm{def}} N$. Then the homomorphism*

$$\mathbf{MCM}(X)/\sim^{\mathrm{def}} \to C(X) \oplus \mathbb{N}$$

is an isomorphism.

Proof. We first show that there is a well defined map

$$\mathbf{K}_0^+(X) \to \mathbf{MCM}(X)/\sim^{\mathrm{def}}.$$

If $[M] = [N]$ in $\mathbf{K}_0^+(X)$, then $[M] - [N] = \sum m_i([M_i] - [M_i''] - [M_i'])$ where $m_i \in \mathbb{Z}$ and $0 \to M_i' \to M_i \to M_i'' \to 0$ is an exact sequence. Writing $m_i = r_i - s_i$ with $r_i \geq 0$ and $s_i \geq 0$, we write this as

$$[M] + \sum s_i[M_i] + \sum r_i([M_i''] + [M_i']) = [N] + \sum s_i([M_i''] + [M_i']) + \sum r_i[M_i].$$

By Lemma 6.2, we have that

$$[M]+\sum s_i[M_i]+\sum r_i([M_i'']+[M_i'])\sim^{\mathrm{def}}[M]+\sum(r_i+s_i)[M_i]$$

and

$$[N]+\sum s_i([M_i'']+[M_i'])+\sum r_i[M_i]\sim^{\mathrm{def}}[N]+\sum(r_i+s_i)[M_i]$$

By the assumption, it follows that $[M]\sim^{\mathrm{def}}[N]$.

Since

$$\mathbf{K}_0^+(X)\to\mathbf{MCM}(X)/\sim^{\mathrm{def}}\to C(X)\oplus\mathbb{N}$$

is an isomorphism and the first map clearly is surjective, the last map is an isomorphism. ■

PROPOSITION 6.4. *Let X be a rational surface singularity. The following are equivalent:*

1. *The two equivalence relations $\sim^{\mathrm{def}}$ and $\sim^{\mathrm{nef}}$ on $\mathbf{MCM}(X,r)$ are the same for all $r>0$.*
2. *The two equivalence relations $\sim^{\mathrm{def}}$ and $\sim^{\mathrm{nef}}$ on $\mathbf{MCM}(X)$ are the same.*
3. *The map $\mathbf{MCM}(X)/\sim^{\mathrm{def}}\to C(X)\oplus\mathbb{N}$ is an isomorphism.*
4. *The map $\mathbf{MCM}(X,r)/\sim^{\mathrm{def}}\to C(X)$ is a bijection for all r.*
5. *The number of connected components in $\mathbf{G}^{\mathrm{def}}(X,r)$ is $|\det(E_i\cdot E_j)|$ for all $r>0$.*
6. *The implication $M\oplus M'\sim^{\mathrm{def}}N\oplus M'\Rightarrow M\sim^{\mathrm{def}}N$ holds for all MCM modules M,N and M'.*

Proof. It is clear that (1)⇔(2). To show that (2)⇒(3), note that from Proposition 6.1 the map

$$\mathbf{MCM}(X)/\sim^{\mathrm{def}}\to C(X)\oplus\mathbb{N}$$

is surjective. If $[M]$ and $[N]$ maps to the same element in $C(X)\oplus\mathbb{N}$, $\operatorname{rank}M=\operatorname{rank}N$ and $c_1(\widetilde{M})\sim^{\mathrm{nef}}c_1(\widetilde{N})$ by Proposition 4.4. By assumption $[M]\sim^{\mathrm{def}}[N]$. The implication (3)⇒(2) follows from Proposition 4.4. It is clear that (3)⟺(4). To prove (4)⇒(5), note that the connected components in $\mathbf{G}^{\mathrm{def}}(X,r)$ are in one-to-one correspondence with the set $\mathbf{MCM}(X,r)/\sim^{\mathrm{def}}$ by definition. Since the group $C(X)$ has order $|\det(E_i\cdot E_j)|$, (5) follows. The implication (5)⇒(4) follows since $\mathbf{MCM}(X,r)/\sim^{\mathrm{def}}\to C(X)$ is always surjective, and hence a bijection if and only if the two sets have the same number of elements. It is clear that (2)⇒(6). Moreover; (6)⇒(3) by Proposition 6.3. ■

REMARK 6.5. Even if $\sim^{\mathrm{def}}$and $\sim^{\mathrm{nef}}$ coincide on $\mathbf{MCM}(X,r)$ it does not follow that $\dashrightarrow_{\mathrm{nef}}$ and $\dashrightarrow_{\mathrm{def}}$ coincide.

CONJECTURE (B). Let X be a quotient surface singularity. For all $r>0$, the map of graphs

$$(c_1,r):\mathbf{G}^{\mathrm{def}}(X,r)\to\mathbf{G}^{\mathrm{nef}}(X,r)$$

is surjective.

J. Wunram showed that for each exceptional divisor E_i in the minimal resolution, there is an indecomposable MCM module M_i with $c_1(\widetilde{M_i})\cdot E_j=\delta_{ij}$ and $c_1(\widetilde{M_i})\cdot Z=\operatorname{rank}M_i$, see [7]. Hence the map in Conjecture B is surjective on the vertices.

By Proposition 6.4, Conjecture B implies Conjecture A, if in addition $c_1(\widetilde{M})=c_1(\widetilde{N})$ implies $M\sim^{\mathrm{def}}N$.

Proposition 6.6. *Assume that X is a rational double point or that X is the cone over a rational normal curve. Then Conjecture A and Conjecture B holds.*

Proof. In the case X is a rational double point the conjectures follows from Theorem 5.2. The other case follows from [3]. ∎

The proof of Theorem 5.2 relies on Theorem 5.5 in [4]. The construction splits in the two cases of Proposition 5.1, and may be at least partially generalized. We are able to prove Proposition 5.1 for rational surface singularities with almost reduced fundamental cycle (which includes quotient singularities). The first of the two cases in the proof of Theorem 5.5 in [4] is due to a rather general construction. The second case is more particular to RDP's, and a generalization, it seems, would have to make use of our results in [3]. If successful this approach would enable us to prove Conjecture B.

In the case of a cone over a rational normal curve the cancelation property is proven by exhibiting exact sequences that generates the deformation relation in terms of Lemma 6.2. It seems possible, but tedious, to extend this approach at least to cyclic quotient singularities, in order to prove Conjecture A.

Remark 6.7. Let X be a rational surface singularity. We may define the *deformation group* $\mathbf{K}_0^{\mathrm{def}}(X)$ as the free abelian group generated by the elements of $\mathbf{MCM}(X)$ modulo the subgroup generated by $[M]-[N]$ where $M \dashrightarrow_{\mathrm{def}} N$. Then an argument as in the proof of Proposition 6.3 shows that

$$\mathbf{K}_0^{\mathrm{def}}(X) \cong \mathbf{K}_0(X).$$

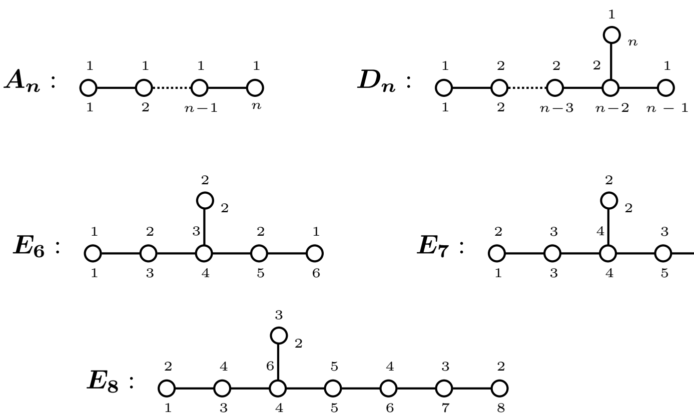

Fig. 1. The resolution graphs of the rational double points. (All weights are -2. The numbers below enumerate the vertices, and the numbers above give the multiplicity in the fundamental cycle.)

7. The deformation relation on indecomposable MCM modules on RDPs. Let X be a rational double point with minimal resolution $\widetilde{X} \to X$ and exceptional set $E = \cup E_i$. Choose divisors D_i such that $E_i \cdot D_j = \delta_{ij}$. By Theorem 1.11 in [1] there are unique indecomposable MCM modules M_i such that $c_1(M_i) = D_i$. The rank of M_i is

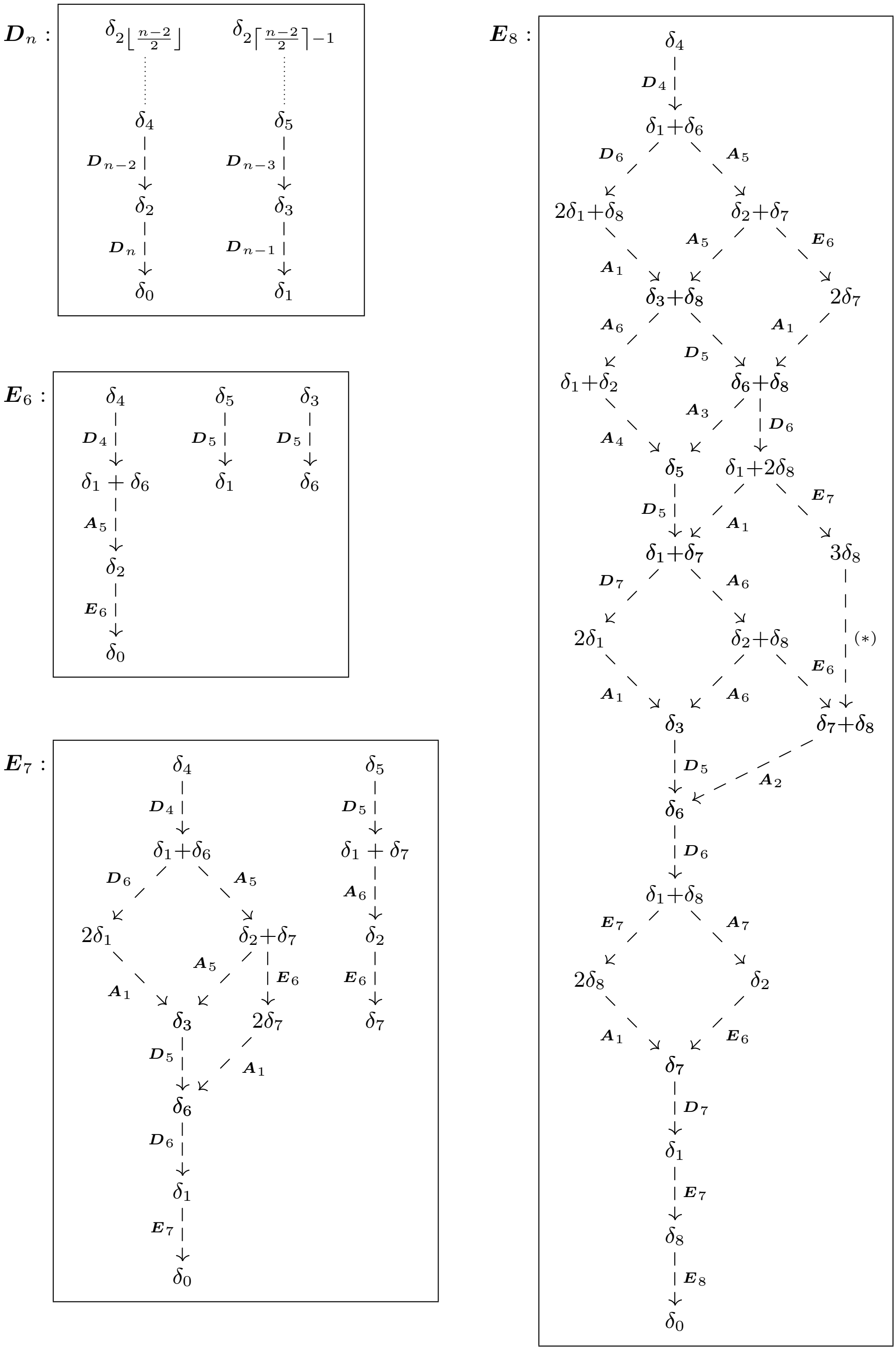

Fig. 2. Deformation graphs of the indecomposable MCM modules on the RDPs

given by the multiplicity of E_i in the fundamental cycle. By applying Proposition 5.1 we find the full sub-graph in $\mathbf{G}^{\mathrm{def}}(X, r)$ of deformations of each M_i for all the rational double points X, see Figure 2. The deformation graph corresponding to some modules is contained in the deformation graph of modules that deform to it.

In Figure 2, δ_i (for readability) denotes the divisor class of a reduced transversal divisor which intersects E_i i.e. $\delta_i = D_i$. We write δ_0 for the zero divisor. Thus δ_i corresponds to the module M_i. In Figure 1 the possible resolution graphs are shown. The number below each vertex is an enumeration and corresponds to the index i in δ_i in Figure 2. The number above each vertex gives the multiplicity in the fundamental cycle of the corresponding E_i, hence this number gives the rank of M_i.

The symbol attached to the $\rightarrow$ is the corresponding minimal stratum, see [4, 5.6]. There is an exceptional case at the $(*)$ in the $\boldsymbol{E}_8$-diagram where the corresponding minimal stratum is the 4-dimensional isolated singularity S_3.

THEOREM 7.1. *Let M be an indecomposable MCM module on a rational double point. If* $\operatorname{rk} M = 1$, *then M does not deform. If* $\operatorname{rk} M > 1$, *then the possible deformations are given in Figure 2.*

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Received January 30, 2010; Revised January 31, 2011