ALGEBRA, GEOMETRY AND MATHEMATICAL PHYSICS
BANACH CENTER PUBLICATIONS, VOLUME 93
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2011

A REVIEW ON δ-STRUCTURABLE ALGEBRAS

NORIAKI KAMIYA

Center for Mathematical Sciences, University of Aizu
965-8580 Aizuwakamatsu, Japan
E-mail: kamiya@u-aizu.ac.jp

DANIEL MONDOC

Centre for Mathematical Sciences, Lund University
22 100 Lund, Sweden
E-mail: Daniel.Mondoc@math.lu.se

SUSUMU OKUBO

Department of Physics, University of Rochester
Rochester, NY 14627 U.S.A
E-mail: okubo@pas.rochester.edu

Abstract. In this paper we give a review on δ-structurable algebras. A connection between Malcev algebras and a generalization of δ-structurable algebras is also given.

1. Introduction. The history of nonassociative algebras, the subject of this paper, started with Hamilton, Cayley and Hurwitz and further with Artin and Zorn, who studied alternative and nearly associative algebras. Thereafter Freudenthal, Tits ([51]), I.L. Kantor ([32]-[34]) and Koecher ([36]) studied constructions of Lie algebras from nonassociative algebras and triple systems, in particular Jordan algebras, while B.N. Allison ([1], [2]) defined the concept of structurable algebras, containing Jordan algebras. Recently, we have studied constructions of Lie superalgebras as well as Lie algebras from triple systems ([23], [25], [27]-[28]). Hence within the general framework of (ε, δ)-Freudenthal Kantor triple systems, $\varepsilon, \delta = \pm 1$, (for short (ε, δ)-FKTSs) and the standard embedding Lie (super)algebra construction studied in [6], [7], [12]-[14], [25] (see also references therein) we defined δ-structurable algebras ([27]) as a class of nonassociative algebras with involution which coincides with the class of structurable algebras for $\delta = 1$ as intro-

2010 *Mathematics Subject Classification*: Primary 17A40; Secondary 17A30, 17B60.
Key words and phrases: Malcev algebra, Lie algebra, Lie superalgebra, triple system.
The paper is in final form and no version of it will be published elsewhere.

duced and studied in [1], [2]. Structurable algebras are a class of nonassociative algebras with involution that include Jordan algebras (with trivial involution), associative algebras with involution, and alternative algebras with involution. They are related to generalized Jordan triple systems (for short GJTSs) of 2nd order, or $(-1,1)$-FKTSs, as introduced and studied in [32], [33] and further studied in [3], [4], [31], [40]-[43], [49] (see also references therein). Their importance lies with constructions of five graded Lie algebras. For $\delta=-1$ the anti-structurable algebras ([27]) are a new class of nonassociative algebras that may similarly shed light on the notion of $(-1,-1)$-FKTSs hence (by [6], [7], [29], [30]) on the construction of Lie superalgebras and Jordan algebras. Specially, nonassociative algebras such as Jordan and Lie (super)algebras ([11]) play an important role in many mathematical and physical subjects ([5], [8]-[12], [14], [24], [26], [35], [45], [46], [53]). We also note that the construction and characterization of these algebras can be expressed in terms of the notion of triple systems ([21], [22], [47]) by using the standard embedding method ([20], [37], [39], [48]). As a final comment of this section, we notice that our subject will be applied to normed algebras, in particular to the case of normed triple systems containing Jordan and Banach algebras.

2. Preliminaries

2.1. (ϵ,δ)-Freudenthal Kantor triple systems, δ-Lie triple systems, Lie (super)algebras. We are concerned in this paper with triple systems which have finite dimension over a field Φ of characteristic $\neq 2$ or 3, unless otherwise specified.

Definition 2.1. A vector space V over a field Φ endowed with a trilinear operation $V\times V\times V\to V$, $(x,y,z)\mapsto(xyz)$ is said to be a *GJTS of 2nd order* if

$$(ab(xyz))=((abx)yz)-(x(bay)z)+(xy(abz)) \tag{1}$$

$$K(K(a,b)x,y)-L(y,x)K(a,b)-K(a,b)L(x,y)=0 \tag{2}$$

where $L(a,b)c:=(abc)$ and $K(a,b)c:=(acb)-(bca)$.

Definition 2.2. A *Jordan triple system* (for short JTS) satisfies (1) and $(abc)=(cba)$.

We can generalize the concept of GJTS of 2nd order as follows ([12], [13], [16], [52]).

Definition 2.3. For $\varepsilon=\pm1$ and $\delta=\pm1$ a triple product that satisfies the identities

$$(ab(xyz))=((abx)yz)+\varepsilon(x(bay)z)+(xy(abz)) \tag{3}$$

$$K(K(a,b)x,y)-L(y,x)K(a,b)+\varepsilon K(a,b)L(x,y)=0 \tag{4}$$

where

$$L(a,b)c:=(abc),\quad K(a,b)c:=(acb)-\delta(bca) \tag{5}$$

is called an (ε,δ)-*FKTS*.

Remark. We note that $K(b,a)=-\delta K(a,b)$.

Let U be an (ε,δ)-FKTS and $V_k, k=1,2,3$, be subspaces of U. We denote by (V_1,V_2,V_3) the subspace of U spanned by elements (x_1,x_2,x_3), $x_k\in V_k, k=1,2,3$.

Definition 2.4. A subspace V of U is called an *ideal* of an (ε,δ)-FKTS U if the following relations hold $(V,U,U)\subseteq V,\ (U,V,U)\subseteq V,\ (U,U,V)\subseteq V$.

U is called *simple* if $(,,)$ is not a zero map and U has no non-trivial ideal.

We denote triple products by (xyz), $\{xyz\}$, $[xyz]$ and $\langle xyz\rangle$ depending on the case.

Remark. We note that the concept of GJTS of 2nd order coincides with that of $(-1,1)$-FKTS. Thus we can construct the simple Lie algebras by means of the standard embedding method ([6], [12]-[25], [34], [52]).

For an (ε,δ)-FKTS U and $L(a,b)$ defined by (5) we denote

$$S(a,b) := L(a,b)+\varepsilon L(b,a), \quad A(a,b) := L(a,b)-\varepsilon L(b,a).$$

Remark. We note that $S(a,b)=\varepsilon S(b,a)$.

$S(a,b)$ (respectively $A(a,b)$) is a derivation (respectively anti-derivation) of U ([28]).

Definition 2.5. For $\delta=\pm1$, a triple system $(a,b,c)\mapsto[abc], a,b,c\in V$ is called a *δ-Lie triple system* (for short δ-LTS) if the following identities are fulfilled

$$\begin{aligned}&[abc]=-\delta[bac],\\ &[abc]+[bca]+[cab]=0,\\ &[ab[xyz]]=[[abx]yz]+[x[aby]z]+[xy[abz]],\end{aligned}$$

where $a,b,x,y,z\in V$. An 1-LTS is a LTS, while a -1-LTS is called an anti-LTS, by [13].

Proposition 2.6 ([13], [20]). *Let $U(\varepsilon,\delta)$ be a (ε,δ)-FKTS. If J is an endomorphism of $U(\varepsilon,\delta)$ such that $J\langle xyz\rangle=\langle JxJyJz\rangle$ and $J^2=-\varepsilon\delta Id$, then $(U(\varepsilon,\delta),[xyz])$ is a LTS (if $\delta=1$) or an anti-LTS (if $\delta=-1$) with respect to the product (6).*

$$[xyz]:=\langle xJyz\rangle-\delta\langle yJxz\rangle+\delta\langle xJzy\rangle-\langle yJzx\rangle \tag{6}$$

Corollary 2.7. *Let $U(\varepsilon,\delta)$ be a (ε,δ)-FKTS. Then the vector space $T(\varepsilon,\delta)=U(\varepsilon,\delta)\oplus U(\varepsilon,\delta)$ becomes a LTS (if $\delta=1$) or an anti-LTS (if $\delta=-1$) with respect to the product*

$$\left[\begin{pmatrix}a\\ b\end{pmatrix}\begin{pmatrix}c\\ d\end{pmatrix}\begin{pmatrix}e\\ f\end{pmatrix}\right]=\begin{pmatrix}L(a,d)-\delta L(c,b) & \delta K(a,c)\\ -\varepsilon K(b,d) & \varepsilon(L(d,a)-\delta L(b,c))\end{pmatrix}\begin{pmatrix}e\\ f\end{pmatrix}. \tag{7}$$

Remark. Thus we can obtain the standard embedding Lie algebra (if $\delta=1$) or Lie superalgebra (if $\delta=-1$), $L(\varepsilon,\delta)=D(T(\varepsilon,\delta),T(\varepsilon,\delta))\oplus T(\varepsilon,\delta)$, associated to $T(\varepsilon,\delta)$ where $D(T(\varepsilon,\delta),T(\varepsilon,\delta))$ is the set of inner derivations of $T(\varepsilon,\delta)$, i.e.

$$D(T(\varepsilon,\delta),T(\varepsilon,\delta)):=\left\{\begin{pmatrix}L(a,b) & \delta K(c,d)\\ -\varepsilon K(e,f) & \varepsilon L(b,a)\end{pmatrix}\right\}_{span},$$

$$T(\varepsilon,\delta):=\left\{\begin{pmatrix}x\\ y\end{pmatrix}\middle| x,y\in U(\varepsilon,\delta)\right\}_{span}.$$

Remark. $L(\varepsilon,\delta)=L_{-2}\oplus L_{-1}\oplus L_0\oplus L_1\oplus L_2$ is the 5-graded Lie (super)algebra such that $L_{-1}\oplus L_1=T(\varepsilon,\delta), D(T(\varepsilon,\delta),T(\varepsilon,\delta))=L_{-2}\oplus L_0\oplus L_2$ and $[L_i,L_j]\subseteq L_{i+j}$. This Lie (super)algebra construction is one of the reasons to study nonassociative algebras and triple systems.

2.2. δ-structurable algebras. Throughout the section it is assumed that $(\mathcal{A},^-)$ is a finite dimensional nonassociative unital algebra with involution (involutive anti-automorphism, i.e. $\overline{\overline{x}} = x$ and $\overline{xy} = \overline{y}\,\overline{x}$ for $x, y \in \mathcal{A}$) over Φ. The identity element of $\mathcal{A}$ is denoted by 1.

Remark. $\mathcal{A} = \mathcal{H} \oplus \mathcal{S}$, where $\mathcal{H} = \{a \in \mathcal{A} | \overline{a} = a\}$ and $\mathcal{S} = \{a \in \mathcal{A} | \overline{a} = -a\}$, by [1].

Put $[x, y] := xy - yx$, $[x, y, z] := (xy)z - x(yz), x, y, z \in \mathcal{A}$. Note that (8) is valid.

$$\overline{[x, y, z]} = -[\overline{z}, \overline{y}, \overline{x}]. \tag{8}$$

Let L_x, R_x be defined by $L_x(y) := xy, R_x(y) := yx, x, y \in \mathcal{A}$. For $\delta = \pm 1$ define:

$$^{\delta}V_{x,y} := L_{L_x(\overline{y})} + \delta(R_x R_{\overline{y}} - R_y R_{\overline{x}}), \tag{9}$$

$$^{\delta}B_{\mathcal{A}}(x, y, z) :=^{\delta} V_{x,y}(z) = (x\overline{y})z + \delta[(z\overline{y})x - (z\overline{x})y], x, y, z \in \mathcal{A}. \tag{10}$$

Definition 2.8. $^{+}B_{\mathcal{A}}(x, y, z)$ is called the *triple system obtained from the algebra* $(\mathcal{A},^-)$. We call $^{-}B_{\mathcal{A}}(x, y, z)$ the *anti-triple system obtained from the algebra* $(\mathcal{A},^-)$.

We shall write for short $V_{x,y} :=^{\delta} V_{x,y}$, $B_{\mathcal{A}} := (^{\delta}B_{\mathcal{A}}, \mathcal{A})$.

Remark. The upper left index notation is chosen in order not to be mixed with the upper right index notation of [1] which has a different meaning.

Definition 2.9. A unital non-associative algebra with involution $(\mathcal{A},^-)$ is called a δ-*structurable algebra* if the following identity is fulfilled

$$[V_{u,v}, V_{x,y}] = V_{V_{u,v}(x),y} - V_{x,V_{v,u}(y)}. \tag{11}$$

$(\mathcal{A},^-)$ is called a *structurable algebra* ([1]) if the identity (11) is fulfilled for $V_{u,v} = {}^{+}V_{u,v}, V_{x,y} = {}^{+}V_{x,y}, u, v, x, y \in \mathcal{A}$, and we will call $(\mathcal{A},^-)$ an *anti-structurable algebra* if the identity (11) is fulfilled for $V_{u,v} = {}^{-}V_{u,v}, V_{x,y} = {}^{-}V_{x,y}$.

Remark. If $(\mathcal{A},^-)$ is structurable then, in the terminology of [33], the triple system $B_{\mathcal{A}}$ is called a GJTS and by [8], $B_{\mathcal{A}}$ is a GJTS of 2nd order, i.e. satisfies (3) and (4).

Definition 2.10. If $(\mathcal{A},^-)$ is anti-structurable then we call $B_{\mathcal{A}}$ an *anti-GJTS*.

Put $T_x := V_{x,1}, x \in \mathcal{A}$. Then, by (9), $T_x = L_x + \delta R_{x-\overline{x}}$ for $x \in \mathcal{A}$ thus $T_h = L_h, h \in \mathcal{H}$.

Remark. (i) If $u = h \in \mathcal{H}$ and $x, y \in \mathcal{A}$, (11) becomes

$$[L_h, V_{x,y}] = V_{hx,y} - V_{x,hy}. \tag{12}$$

(ii) Suppose $^-$ is the identity map and hence $\mathcal{A}$ is commutative. If $(\mathcal{A},^-)$ is δ-structurable then $\mathcal{A}$ is a Jordan algebra, by [27]. Conversely, by [36]§3, any Jordan algebra satisfies (12) if $V_{x,y} = {}^{+}V_{x,y}$ for $x, y \in \mathcal{A}$, hence it is structurable. By [27], any Jordan algebra is anti-structurable if $((hx)y)z - h((xy)z) = (x(yh))z - (xy)(hz)$, for $h, x, y, z \in \mathcal{A}$.

Clearly, the last identity is fulfilled by an associative algebra.

Definition 2.11. For $s \in \mathcal{S}$ and $h \in \mathcal{H}$ we say that $(\mathcal{A},^-)$ is $\mathcal{S}$ *skew-alternative* if $[s, x, y] = -[x, s, y]$ while $(\mathcal{A},^-)$ is $\mathcal{H}$ *skew-alternative* if $[h, x, y] = -[x, h, y]$ for $x, y \in \mathcal{A}$.

Remark. If $(\mathcal{A},^-)$ is $\mathcal{S}$ skew-alternative then by [1], $[s, x, y] = -[x, s, y] = [x, y, s]$, $s \in \mathcal{S}, x, y \in \mathcal{A}$. If $(\mathcal{A},^-)$ is $\mathcal{H}$ skew-alternative then by (8), $[h, x, y] = -[x, h, y] = [x, y, h], h \in \mathcal{H}, x, y \in \mathcal{A}$.

PROPOSITION 2.12 ([27]). *If* $(\mathcal{A}, ^-)$ *is structurable, then* $(\mathcal{A}, ^-)$ *is* $\mathcal{S}$ *skew-alternative. If* $(\mathcal{A}, ^-)$ *is anti-structurable, then* $(\mathcal{A}, ^-)$ *is* $\mathcal{H}$ *skew-alternative.*

REMARK. Let $(\mathcal{A}, ^-)$ be a δ-structurable algebra and let $\mathrm{Der}(\mathcal{A}, ^-)$ be the set of derivations of $\mathcal{A}$ that commute with $^-$. By [27], $T_{\mathcal{A}} \cap \mathrm{Der}(\mathcal{A}, ^-) = 0$ and so we may define the *structure algebra* $\mathrm{Str}(\mathcal{A}, ^-) := T_{\mathcal{A}} \oplus \mathrm{Der}(\mathcal{A}, ^-)$. This algebra plays an important role in the structure study of structurable algebras ([1]) and may play a role in the structure study of anti-structurable algebras, a theory to be presented elsewhere.

2.3. Examples. For examples of structurable algebras we refer to [1] and [2].

Let $\mathcal{M}_{m,n}(\Phi)$ denote the vector space of $m \times n$ matrices over Φ and for $x \in \mathcal{M}_{m,n}(\Phi)$ denote by $x^\top$ the transposed matrix.

LEMMA 2.13 ([27]). $(\mathcal{M}_{m,n}(\Phi), \{x,y,z\})$ *is a* $(-1,\delta)$*-FKTS, where* $\{x,y,z\}$ *is defined by*

$$\{x,y,z\} := xy^\top z + \delta(zy^\top x - zx^\top y), \quad x,y,z \in \mathcal{M}_{m,n}(\Phi) \tag{13}$$

THEOREM 2.14 ([27]). $\mathcal{M}_{n,n}(\Phi)$ *with the involution* $x \mapsto x^\top$ *is a* δ*-structurable algebra.*

EXAMPLE 2.15. $(\mathcal{M}_{m,n}(\mathbf{C}), \{x,y,z\})$ is a $(-1,\delta)$-FKTS, where $\{x,y,z\}$ is defined by

$$\{x,y,z\} := x\overline{y}^\top z + \delta(z\overline{y}^\top x - z\overline{x}^\top y), \quad x,y,z \in \mathcal{M}_{m,n}(\mathbf{C}) \tag{14}$$

Indeed, it is straightforward calculation to show that the identities (3) and (4) hold. Hence $\mathcal{M}_{n,n}(\mathbf{C})$ with the involution $x \mapsto \overline{x}^\top$ is a δ-structurable algebra.

REMARK. By [25], the following construction of Lie superalgebras is obtained by the standard embedding method. If $U(-1,-1) := \mathcal{M}_{2n,m}(\Phi)$ with the product (13) then the corresponding standard embedding Lie superalgebra is $osp\,(2n|2m) = D(n,m)$ (as defined by [11]), hence the standard embedding Lie superalgebra of the anti-structurable algebra $\mathcal{M}_{2n,2n}(\Phi)$ is $osp\,(2n|4n)$. Similarly, if $U(-1,-1) := \mathcal{M}_{2n+1,m}(\Phi)$ with the product (13) then the corresponding standard embedding Lie superalgebra is $osp\,(2n+1|2m) = B(n,m)$ (as defined by [11]), hence the standard embedding Lie superalgebra of the anti-structurable algebra $\mathcal{M}_{2n+1,2n+1}(\Phi)$ is $osp\,(2n+1|4n+2)$.

3. Malcev algebras and generalized quasi δ-structurable algebras. We give in this section a connection between Malcev algebras ([38], [50]) and a generalization of δ-structurable algebras ([17], [18]).

DEFINITION 3.1. An algebra $(\mathcal{A}, ^-)$ over Φ is called *quasi* δ*-structurable* if it is a δ-structurable algebra with no assumption of existence of identity element.

DEFINITION 3.2. An algebra $\mathcal{A}$ over Φ is called *generalized structurable* ([17]) if it is a nonassociative algebra equipped with a non-trivial derivation $D(x,y)$, $x,y \in \mathcal{A}$, such that the following conditions are fulfilled:

(i) $D(x,y) = D(y,x)$,
(ii) $D(xy,z) + D(yz,x) + D(zx,y) = 0, \quad x,y,z \in \mathcal{A}$.

REMARK ([17]). A structurable algebra $(\mathcal{A}, ^-)$ is a generalized structurable algebra.

Proof. Put $D(x,y)z := \frac{1}{3}[[x,y]+[\overline{x},\overline{y}],z]+[z,y,x]-[z,\overline{x},\overline{y}]$, where $[x,y]=xy-yx$, $[x,y,z]=(xy)z-x(yz), x,y,z\in\mathcal{A}$. Then, by [17], the identities (i) and (ii) of definition 3.2 are fulfilled. ■

DEFINITION 3.3. An algebra A with multiplication $[x,y], x,y\in A$ over a field of arbitrary characteristic, is called a *Malcev algebra* ([44]) if it satisfies the anticommutative law

$$[x,x]=0, \tag{15}$$

and

$$J(x,y,[x,z])=[J(x,y,z),x], \quad x,y,z\in A, \tag{16}$$

where $J(x,y,z):=[[x,y],z]+[[y,z],x]+[[z,x],y]$ is the *Jacobian.*

REMARK ([17]). A Malcev algebra A is a generalized structurable algebra.

Proof. Put

$$D(x,y):=[L_x,L_y]+L_{xy}, \tag{17}$$

where $[x,y]=xy-yx$, $L_x(y):=xy, x,y\in A$. Then, by [17], the identities (i) and (ii) of definition 3.2 are fulfilled. ■

PROPOSITION 3.4. *Let A be a Malcev algebra with involution defined by $\overline{x}:=-x, x\in A$. Then A is a quasi structurable algebra.*

Proof. By (10), for $\delta=1$, we have

$$^{+}B_A(x,y,z)={}^{+}V_{x,y}(z)=(x\overline{y})z+(z\overline{y})x-(z\overline{x})y=-(xy)z-(zy)x+(zx)y, x,y,z\in A, \tag{18}$$

since $\overline{x}:=-x, x\in A$. Thus $^{+}V_{x,y}=-([L_x,L_y]+L_{xy})(z)=-D(x,y)$, by (18) and (17), hence the identity (11) is fulfilled. ■

COROLLARY 3.5. *Let A be a Lie algebra with involution defined by $\overline{x}:=-x, x\in A$. Then A is a quasi structurable algebra.*

PROPOSITION 3.6. *Let A be a Lie algebra with involution defined by $\overline{x}:=-x, x\in A$. Then A is a (trivial) quasi anti-structurable algebra.*

Proof. By (10), for $\delta=-1$, we have

$$^{-}B_A(x,y,z)={}^{-}V_{x,y}(z)=(x\overline{y})z-(z\overline{y})x+(z\overline{x})y=-(xy)z+(zy)x-(zx)y, x,y,z\in A, \tag{19}$$

since $\overline{x}:=-x, x\in A$. Now, it follows $^{-}B_A(x,y,z)=-(xy)z-(yz)x-(zx)y$, by anti-symmetry and (19), hence $^{-}B_A(x,y,z)\equiv 0$, by Jacobi identity. ■

REMARK. From the last proposition and corollary it follows that Lie algebras are quasi δ-structurable algebras.

PROPOSITION 3.7. *A quasi anti-structurable algebra $(\mathcal{A},^{-})$ with involution defined by $\overline{x}:=-x, x\in\mathcal{A}$ is a $(-1,-1)$-JTS.*

Proof. Indeed, $K(x,y)\equiv 0, x,y\in\mathcal{A}$, where $K(x,y)$ is defined by (5), for $\delta=-1$. Then clearly the identities (3) and (4) are fulfilled, for $\varepsilon=\delta=-1$. ■

Remark. If the assumption of finite dimensionality on algebras and triple systems is not required it seems that our concept can be generalized to Banach and Jordan algebras as well as to JB^* algebras and JB^* triples. In future work, we will discuss nonassociative normed algebras and triple systems containing Banach algebras.

Acknowledgements. The first author, N. Kamiya acknowledges partial support for this research by Grant-in-Aid for Scientific Research No. 19540042 (C), (2), Japan Society for the Promotion of Science.

The second author, D. Mondoc would like to thank the organizers of 5th Baltic-Nordic AGMP Workshop 2009 for their hospitality at the Stefan Banach Internatioonal Mathematical Center, Poland.

The third author, S. Okubo acknowledges partial support for this research by U.S. Department of Energy Grant No. DE-FG02-91ER40685.

References

[1] B. N. Allison, *A class of nonassociative algebras with involution containing the class of Jordan algebras*, Math. Ann. 237 (1978), 133–156.

[2] B. N. Allison, *Models of isotropic simple Lie algebras*, Commun. Algebra 7 (1979), 1835–1875.

[3] H. Asano and K. Yamaguti, *A construction of Lie algebras by generalized Jordan triple systems of second order*, Nederl. Akad. Wetensch. Indag. Math. 42 (1980), 249–253.

[4] H. Asano, *Classification of non-compact real simple generalized Jordan triple systems of the second kind*, Hiroshima Math. J. 21 (1991), 463–489.

[5] W. Bertram, *Complex and quaternionic structures on symmetric spaces - correspondence with Freudenthal-Kantor triple systems*, in: Theory of Lie Groups and Manifolds, R. Miyaoka and H. Tamaru (eds.), Sophia Kokyuroku in Math. 45 (2002), 61–80.

[6] A. Elduque, N. Kamiya and S. Okubo, *Simple* $(-1,-1)$ *balanced Freudenthal Kantor triple systems*, Glasgow Math. J. 11 (2003), 353–372.

[7] A. Elduque, N. Kamiya and S. Okubo, $(-1,-1)$ *balanced Freudenthal Kantor triple systems and noncommutative Jordan algebras*, J. Algebra 294 (2005), 19–40.

[8] J. R. Faulkner, *Structurable triples, Lie triples, and symmetric spaces*, Forum Math. 6 (1994), 637–650.

[9] N. Jacobson, *Lie and Jordan triple systems*, Amer. J. Math. 71 (1949), 149–170.

[10] N. Jacobson, *Structure and Representations of Jordan Algebras*, Amer. Math. Soc. Colloq. Publ. 39, Amer. Math. Soc., Providence, R.I., 1968.

[11] V. G. Kac, *Lie superalgebras*, Adv. Math. 26 (1977), 8–96.

[12] N. Kamiya, *A structure theory of Freudenthal-Kantor triple systems*, J. Algebra 110 (1987), 108–123.

[13] N. Kamiya, *A construction of anti-Lie triple systems from a class of triple systems*, Mem. Fac. Sci. Shimane Univ. 22 (1988), 51–62.

[14] N. Kamiya, *A structure theory of Freudenthal-Kantor triple systems. II*, Comment. Math. Univ. St. Paul. 38 (1989), 41–60.

[15] N. Kamiya, *On (ε,δ)-Freudenthal Kantor triple systems*, in: Nonassociative Algebras and Related Topics (Hiroshima, 1990), K. Yamaguti and N. Kawamoto (eds.), World Sci. Publ., River Edge, NJ, 1991, 65–75.

[16] N. Kamiya, *The construction of all simple Lie algebras over* **C** *from balanced Freudenthal-Kantor triple systems*, in: Contributions to General Algebra, 7 (Vienna 1990), D. Dorninger et al. (eds.), Hölder-Pichler-Tempsky, Vienna, 1991, 205–213.

[17] N. Kamiya, *On a generalization of structurable algebras*, Algebras, Groups, Geom. 9 (1992), 35–47.

[18] N. Kamiya, *On Freudenthal-Kantor triple systems and generalized structurable algebras*, in: Nonassociative Algebra and Its Applications (Oviedo, 1993), Santos González (ed.), Math. Appl. 303, Kluwer, Dordrecht, 1994, 198–203.

[19] N. Kamiya, *On a realization of the exceptional simple graded Lie algebras of the second kind and Freudenthal-Kantor triple systems*, Bull. Polish Acad. Sci. Math. 46 (1998), 55–65.

[20] N. Kamiya and S. Okubo, *On δ-Lie supertriple systems associated with (ε,δ)-Freudenthal Kantor supertriple systems*, Proc. Edinburgh Math. Soc. 43 (2000), 243–260.

[21] N. Kamiya and S. Okubo, *A construction of Jordan superalgebras from Jordan-Lie triple systems*, in: Nonassociative Algebra and Its Applications, R. Costa et al. (eds.), Lect. Notes Pure and Appl. Math. 211, Marcel Dekker, 2002, 171–176.

[22] N. Kamiya and S. Okubo, *A construction of simple Jordan superalgebra of F type from a Jordan-Lie triple system*, Ann. Mat. Pura Appl. (4) 181 (2002), 339–348.

[23] N. Kamiya and S. Okubo, *Construction of Lie superalgebras $D(2,1;\alpha), G(3)$ and $F(4)$ from some triple systems*, Proc. Edinburgh Math. Soc. 46 (2003), 87–98.

[24] N. Kamiya and S. Okubo, *On generalized Freudenthal-Kantor triple systems and Yang-Baxter equations*, in: Proc. XXIV Inter. Colloq. Group Theoretical Methods in Physics, IPCS, J.-P. Gazeau et al. (eds.), Paris, 2003, 815–818.

[25] N. Kamiya and S. Okubo, *A construction of simple Lie superalgebras of certain types from triple systems*, Bull. Austral. Math. Soc. 69 (2004), 113–123.

[26] N. Kamiya and S. Okubo, *Composition, quadratic, and some triple systems*, in: Nonassociative Algebra and Its Applications, L. Sabinin et al. (eds.), Lect. Notes Pure Appl. Math. 246, Chapman & Hall/CRC, Boca Raton, FL, 2006, 205–231.

[27] N. Kamiya and D. Mondoc, *A new class of nonassociative algebras with involution*, Proc. Japan Acad. Ser. A 84 (2008), 68–72.

[28] N. Kamiya and D. Mondoc, *On anti-structurable algebras and extended Dynkin diagrams*, J. Gen. Lie Theory Appl. 3 (2009), 185–192.

[29] N. Kamiya, D. Mondoc, and S. Okubo, *On Lie and Jordan structures associated with (ϵ,δ)-Freudenthal Kantor triple systems*, Mitt. Math. Gesell. Hamburg 29 (2010), 109–123.

[30] N. Kamiya, D. Mondoc and S. Okubo, *A structure theory of $(-1,-1)$-Freudenthal Kantor triple systems*, Bull. Australian Math. Soc. 81 (2010), 132–155.

[31] S. Kaneyuki and H. Asano, *Graded Lie algebras and generalized Jordan triple systems*, Nagoya Math. J. 112 (1988), 81–115.

[32] I. L. Kantor, *Graded Lie algebras*, Trudy Sem. Vekt. Tens. Anal. 15 (1970), 227–266.

[33] I. L. Kantor, *Some generalizations of Jordan algebras*, Trudy Sem. Vekt. Tens. Anal. 16 (1972), 407–499.

[34] I. L. Kantor, *Models of exceptional Lie algebras*, Soviet Math. Dokl. 14 (1973), 254–258.

[35] I. L. Kantor, *A generalization of the Jordan approach to symmetric Riemannian spaces*, in: The Monster and Lie Algebras (Columbus, OH, 1996), J. Ferrar and K. Harada (eds.), Ohio State Univ. Math. Res. Inst. Publ. 7, de Gruyter, Berlin, 1998, 221–234.

[36] M. Koecher, *Embedding of Jordan algebras into Lie algebras I. II*, Amer. J. Math. 89 (1967), 787–816 and 90 (1968), 476–510.

[37] W. G. Lister, *A structure theory of Lie triple systems*, Trans. Amer. Math. Soc. 72 (1952), 217–242.

[38] A. Malcev, *Analytic loops*, Mat. Sb. 78 (1955), 569–578.

[39] K. Meyberg, *Lectures on algebras and triple systems*, Lecture Notes, The University of Virginia, Charlottesville, Va., 1972.

[40] D. Mondoc, *Models of compact simple Kantor triple systems defined on a class of structurable algebras of skew-dimension one*, Commun. Algebra 34 (2006), 3801–3815.

[41] D. Mondoc, *On compact realifications of exceptional simple Kantor triple systems*, J. Gen. Lie Theory Appl. 1 (2007), 29–40.

[42] D. Mondoc, *Compact realifications of exceptional simple Kantor triple systems defined on tensor products of composition algebras*, J. Algebra 307 (2007), 917–929.

[43] D. Mondoc, *Compact exceptional simple Kantor triple systems defined on tensor products of composition algebras*, Commun. Algebra 35 (2007), 3699–3712.

[44] H. C. Myung, *Malcev-admissible Algebras*, Progr. Math. 64, Birkhäuser, Boston, 1986.

[45] E. Neher, *Jordan Triple Systems by the Grid Approach*, Lect. Notes Math. 1280, Springer-Verlag, Berlin, 1987.

[46] S. Okubo, *Introduction to Octonions and Other Non-associative Algebras in Physics*, Montroll Memorial Lecture Series in Mathematical Physics 2, Cambridge University Press, Cambridge, 1995.

[47] S. Okubo and N. Kamiya, *Jordan-Lie superalgebra and Jordan-Lie triple system*, J. Algebra 198 (1997), 388–411.

[48] S. Okubo and N. Kamiya, *Quasi-classical Lie superalgebras and Lie supertriple systems*, Commun. Algebra 30 (2002), 3825–3850.

[49] S. Okubo, *Symmetric triality relations and structurable algebras*, Linear Algebra Appl. 396 (2005), 189–222.

[50] A. Sagle, *Malcev algebras*, Trans. Amer. Math. Soc. 101 (1961), 426–458.

[51] J. Tits, *Une classe d'algèbres de Lie en relation avec les algèbres de Jordan*, Nederl. Acad. Wetensch. Proc. Ser. A 65 (1962), 530–535.

[52] K. Yamaguti and A. Ono, *On representations of Freudenthal-Kantor triple systems* $U(\varepsilon, \delta)$, Bull. Fac. School Ed. Hiroshima Univ. 7 (1984), 43–51.

[53] E. Zelmanov, *Primary Jordan triple systems*, Sibirsk. Mat. Zh. (1983), 23–37.

Received November 25, 2009; Revised March 5, 2011