

## ALMOST-GRADED CENTRAL EXTENSIONS OF LAX OPERATOR ALGEBRAS

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**Abstract.** Lax operator algebras constitute a new class of infinite dimensional Lie algebras of geometric origin. More precisely, they are algebras of matrices whose entries are meromorphic functions on a compact Riemann surface. They generalize classical current algebras and current algebras of Krichever-Novikov type. Lax operators for  $\mathfrak{gl}(n)$ , with the spectral parameter on a Riemann surface, were introduced by Krichever. In joint works of Krichever and Sheinman their algebraic structure was revealed and extended to more general groups. These algebras are almost-graded. In this article their definition is recalled and classification and uniqueness results for almost-graded central extensions for this new class of algebras are presented. The explicit forms of the defining cocycles are given. If the finite-dimensional Lie algebra on which the Lax operator algebra is based is simple then, up to equivalence and rescaling of the central element, there is a unique non-trivial almost-graded central extension. Some results are joint work with Oleg Sheinman.

**1. Introduction.** Classical current algebras (also called loop algebras) and their central extensions, the affine Lie algebras, are of fundamental importance in quite a number of fields in mathematics and its applications. These algebras are examples of infinite dimensional Lie algebras which are still tractable. They constitute the subclass of Kac-Moody algebras of untwisted affine type [1].

In the usual approach they are presented in a purely algebraic manner. But there is a very useful geometric description behind. In fact, the classical current algebras correspond to Lie algebra valued meromorphic functions on the Riemann sphere (i.e. on the unique compact Riemann surface of genus zero) which are allowed to have poles only at two

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fixed points. If this rewriting is done, a very useful generalization (e.g. needed in string theory, but not only there) is to consider the objects over a compact Riemann surface of arbitrary genus with possibly more than two points where poles are allowed. Such objects (vector fields, functions, etc.) and central extensions for higher genus with two possible poles were introduced by Krichever and Novikov [4] and generalized by myself to the multi-point situation [8]. These objects are of importance in a global operator approach to Conformal Field Theory [11], [12], [13]

More generally, the current algebra resp. their central extensions, the affine algebras, correspond to infinite dimensional symmetries of systems. Moreover, they define objects over the moduli space of Riemann surfaces with marked points (Wess-Zumino-Novikov-Witten (WZNW) models, Knizhnik-Zamolodchikov (KZ) connections, etc., see e.g. [12]).

Their use, e.g. in quantization, regularizations, Fock space representation, force us to consider central extensions of these algebras. Note that the well-known Heisenberg algebra is a central extension of a commutative Lie algebra. For the classical current algebras the Kac-Moody algebras of affine type are obtained via central extensions.

More recently, a related class of current algebra appeared, the Lax operator algebras. Again these are algebras of current type associated to finite-dimensional Lie algebras and to compact Riemann surfaces of arbitrary genus, resp. to smooth projective curves over the complex numbers. They find applications in the theory of integrable systems and are connected to the moduli space of bundles over compact Riemann surfaces.

In 2002 Krichever [2], [3] studied  $\mathfrak{gl}(n)$  Lax operators on higher genus Riemann surfaces. In 2007 Krichever and Sheinman [5] uncovered their algebraic structure not only for  $\mathfrak{gl}(n)$  but also for more general classes of finite-dimensional algebras, e.g. for  $\mathfrak{sl}(n)$ ,  $\mathfrak{so}(n)$  and  $\mathfrak{sp}(2n)$ .

Krichever-Novikov current type algebras consist of Lie algebra valued meromorphic functions on a fixed compact Riemann surface (of genus  $g$ ) with possible poles at a finite set of points. The orders of the poles are not restricted. For the Lax operator algebra associated to  $\mathfrak{gl}(n)$  the elements are allowed to have additional poles of maximal order one at a finite set of  $n \cdot g$  additional points  $\gamma_s$ , the weak singularities. To each point  $\gamma_s$  a vector  $\alpha_s \in \mathbb{C}^n$  is assigned which enters the formulation of the required form of the expansion of the element at the point  $\gamma_s$ , see Section 3. The appearance of this additional poles is due to the fact, that the Lax operators operate on functions representing sections of a non-trivial rank  $n$  vector bundle. The additional data needed to describe the algebra are the Tyurin parameters [18] appearing in the context of the moduli space of vector bundles over Riemann surfaces. In this context the Krichever-Novikov current algebra for  $\mathfrak{gl}(n)$  corresponds to the trivial rank  $n$  vector bundle.

The classical counterparts of these algebras are graded algebras. Such a grading is important e.g. in the context of representations, for example to define highest weight representations. In higher genus the Krichever-Novikov type algebras are not graded, but only almost-graded (see Definition 2.1). Fortunately the almost-gradedness is enough for the constructions in representation theory. It turns out that Lax operator algebras are also almost-graded.

Starting from an almost-graded Lie algebra those central extensions are of particular interest for which the almost-grading can be extended to the central extension. Classification results for almost-graded extensions for the Krichever-Novikov current algebras are given in [10]. For the Krichever-Novikov current algebras associated to finite-dimensional simple Lie algebras there is up to equivalence of the extension and rescaling of the central element only one nontrivial almost-graded central extension.

For the Lax operator algebras in joint work with Oleg Sheinman we classified almost-graded central extensions. Again it turns out that in the case that the associated simple finite-dimensional Lie algebra is  $\mathfrak{sl}(n)$ ,  $\mathfrak{so}(n)$  or  $\mathfrak{sp}(2n)$  the almost-graded central extension is essentially unique. We give its explicit form. By requiring a certain invariance under the action of vector fields ( $\mathcal{L}$ -invariance, see Definition 6.3) we even fix its representing two-cocycle in its cohomology class. The results with complete quite involved technical proofs appeared in [14]. It is the goal of this contribution to report on these results.

**2. Krichever-Novikov type current algebras.** Let us first consider the classical situation. We fix  $\mathfrak{g}$  a finite-dimensional complex Lie algebra. The classical current algebra  $\bar{\mathfrak{g}}$  (sometimes also called loop algebra) is obtained by tensoring  $\mathfrak{g}$  by the (associative and commutative) algebra  $\mathbb{C}[z, z^{-1}]$  of Laurent polynomials, i.e.  $\bar{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}]$  with the Lie product

$$[x \otimes z^n, y \otimes z^m] := [x, y] \otimes z^{n+m}, \quad x, y \in \mathfrak{g}, n, m \in \mathbb{Z}.$$

By setting  $\deg(x \otimes z^n) := n$  the Lie algebra  $\bar{\mathfrak{g}}$  becomes a graded algebra. Clearly,  $\bar{\mathfrak{g}}$  is an infinite dimensional Lie algebra. These algebras appear e.g. as symmetry algebras of systems with infinitely many independent symmetries [1]. In applications quite often one is forced to consider central extensions of them. And we will do so also further down.

To understand in which sense the algebras of Krichever-Novikov type are a higher genus version of both the classical current algebras and their central extensions we first have to geometrize the classical situation. The associative algebra of Laurent polynomials  $\mathbb{C}[z, z^{-1}]$  can be described as the algebra consisting of those meromorphic functions on the Riemann sphere (resp. the complex projective line  $\mathbb{P}^1(\mathbb{C})$ ) which are holomorphic outside  $z = 0$  and  $z = \infty$  ( $z$  the quasi-global coordinate). The current algebra  $\bar{\mathfrak{g}}$  can be interpreted as the Lie algebra of  $\mathfrak{g}$ -valued meromorphic functions on the Riemann sphere with possible poles only at  $z = 0$  and  $z = \infty$ .

The Riemann sphere is the unique compact Riemann surface of genus zero. From this point of view the next step is to take  $\Sigma$  any compact Riemann surface of arbitrary genus  $g$  and an arbitrary finite set  $A$  of points where poles of the meromorphic objects will be allowed. In this way we obtain the higher genus (multi-point) current algebra as the algebra of  $\mathfrak{g}$ -valued functions on  $X$  with only possible poles at  $A$ . But we will need gradings, and later on also central extensions.

Some “grading” is essential for infinite-dimensional Lie algebras to construct highest-weight representations, vacuum representations, etc. In fact, for higher genus and even for genus zero if we allow more than two points for possible poles the algebras under consideration will not be graded in the usual sense. Fortunately, a weaker concept, an *almost-grading* will be enough.

DEFINITION 2.1. Let  $V$  be an arbitrary Lie algebra. It is called *almost-graded* if

1.  $V = \bigoplus_{m \in \mathbb{Z}} V_m$  as vector space,
2.  $\dim V_m < \infty, \forall m \in \mathbb{Z}$ ,
3. there exist  $L_1, L_2 \in \mathbb{Z}$  such that

$$[V_k, V_m] \subseteq \bigoplus_{h=k+m-L_1}^{k+m+L_2} V_h, \quad \forall k, m \in \mathbb{Z}.$$

An analogous definition works for associative algebras and for modules over almost-graded algebras.

To introduce such a grading we split the set of points  $A$  where poles are allowed into two non-empty disjoint subsets  $I$  and  $O$ ,  $A = I \cup O$ . Let  $K$  be the number of points in  $I$ . The points in  $I$  are called *in-points*, the points in  $O$  *out-points*.<sup>1</sup>

Let  $\mathcal{A}$  be the associative algebra of those functions which are meromorphic on  $\Sigma$  and holomorphic outside of  $A$ . In some earlier work [8] I constructed a certain adapted basis of  $\mathcal{A}$

$$\{A_{n,p} \mid n \in \mathbb{Z}, p = 1, \dots, K\}.$$

The functions  $A_{n,p}$  are uniquely fixed by giving certain vanishing orders at the points in  $I$  and  $O$  and some normalization conditions. We will not need the exact conditions in the following. Here I only give the vanishing order at  $I$ ,

$$\text{ord}_{P_i}(A_{n,p}) = n + 1 - \delta_i^p, \quad \forall P_i \in I.$$

Of course, a positive vanishing order means a zero, a negative a pole.

We set  $\mathcal{A}_n := \langle A_{n,p} \mid p = 1, \dots, K \rangle$  for the  $K$ -dimensional subspace of  $\mathcal{A}$ . We call the elements of  $\mathcal{A}_n$  homogeneous elements of degree  $n$ . Clearly,  $\mathcal{A} = \bigoplus_{n \in \mathbb{Z}} \mathcal{A}_n$  and in [8] it is shown that there exists a constant  $L$  such that

$$\mathcal{A}_n \cdot \mathcal{A}_m \subseteq \bigoplus_{h=n+m}^{n+m+L} \mathcal{A}_h, \quad \forall n, m \in \mathbb{Z}.$$

Obviously, Definition 2.1 is fulfilled and we obtain an almost-graded structure for the algebra  $\mathcal{A}$ .

For genus zero and  $I = \{0\}$ ,  $O = \{\infty\}$  the prescription yields  $A_n = z^n$  and  $\mathcal{A} = \mathbb{C}[z, z^{-1}]$ . In this case the algebra is graded. Note that in the two-point case we have  $K = 1$  and we will drop the second subscript  $p$  in  $A_{n,p}$ .

REMARK 2.2. The notion of almost-gradedness was introduced by Krichever and Novikov [4] (they called it quasi-gradedness) and they constructed such an almost-grading in the higher genus and two point case. To find an almost-grading in the multi-point case is more difficult and has been done in [7], [8]. The constant  $L$  depends in a known manner on the genus  $g$  and the number of points in  $I$  and  $O$ .

<sup>1</sup>In the interpretation of string theory, where the Riemann surface  $\Sigma$  corresponds to the world-sheet of the string,  $I$  corresponds to the entry points of incoming free strings and  $O$  to the emission points of outgoing free strings.

DEFINITION 2.3. Given a finite-dimensional Lie algebra  $\mathfrak{g}$ , the *higher genus multi-point current algebra*  $\bar{\mathfrak{g}}$  is the tensor product  $\bar{\mathfrak{g}} = \mathfrak{g} \otimes \mathcal{A}$  with Lie product

$$[x \otimes f, y \otimes g] := [x, y] \otimes (f \cdot g)$$

and almost-grading

$$\bar{\mathfrak{g}} = \bigoplus_{n \in \mathbb{Z}} \bar{\mathfrak{g}}_n, \quad \bar{\mathfrak{g}}_n := \mathfrak{g} \otimes \mathcal{A}_n.$$

For  $g = 0$  and  $I = \{0\}$ ,  $O = \{\infty\}$  this gives exactly the classical current algebras. See [4, 16, 10].

Additionally, we will need  $\mathcal{L}$ , the *Lie algebra of meromorphic vector fields* on  $\Sigma$  holomorphic outside of  $A$ . This algebra is again an almost-graded algebra. The grading is given with the help of a certain adapted basis [8]

$$\{e_{n,p} \mid n \in \mathbb{Z}, p = 1, \dots, K\}.$$

The conditions for  $e_{n,p}$  are similar to the conditions for the  $A_{n,p}$ . Here we only note

$$\text{ord}_{P_i}(e_{n,p}) = n + 2 - \delta_i^p, \quad \forall P_i \in I.$$

For genus zero and  $I = \{0\}$ ,  $O = \{\infty\}$  we get  $e_{n,p} = z^{n+1} \frac{d}{dz}$ , and obtain in such a way as  $\mathcal{L}$  the Witt algebra (sometimes also called the Virasoro algebra without central extension).

**3. Lax operator algebras.** For higher genus there is another kind of current type algebras given by the Lax operator algebras of higher genus. They are related to integrable systems and to the moduli space of semi-stable framed vector bundles over  $\Sigma$ . Again let  $\Sigma$  be a compact Riemann surface of genus  $g$  and  $A$  a finite subset of points divided into two nonempty disjoint subset  $A = I \cup O$ . For simplicity we consider here only the *two-point* situation  $I = \{P_+\}$  and  $O = \{P_-\}$ , but the results are true in the more general setting [15].

For  $n \in \mathbb{N}$  we fix  $n \cdot g$  additional points on  $\Sigma$

$$W := \{\gamma_s \in \Sigma \setminus \{P_+, P_-\} \mid s = 1, \dots, ng\}.$$

To every point  $\gamma_s$  we assign a vector  $\alpha_s \in \mathbb{C}^n$ . The system

$$T := \{(\gamma_s, \alpha_s) \in \Sigma \setminus \{P_+, P_-\} \times \mathbb{C}^n \mid s = 1, \dots, ng\}$$

is called a *Tyurin data*. This data is related to the moduli of semi-stable framed vector bundles over  $\Sigma$  [18], see Section 4.

We fix local coordinates  $z_{\pm}$  at  $P_{\pm}$  and  $z_s$  at  $\gamma_s$ ,  $s = 1, \dots, ng$ . Let  $\mathfrak{g}$  be one of the matrix algebras  $\mathfrak{gl}(n)$ ,  $\mathfrak{sl}(n)$ ,  $\mathfrak{so}(n)$ ,  $\mathfrak{sp}(2n)$ , or  $\mathfrak{s}(n)$ , where the latter denotes the algebra of scalar matrices.

We consider meromorphic functions

$$L : \Sigma \rightarrow \mathfrak{g}, \tag{1}$$

which are holomorphic outside  $W \cup \{P_+, P_-\}$ , have at most poles of order one (resp. of order two for  $\mathfrak{sp}(2n)$ ) at the points in  $W$ , and fulfil certain conditions (described below)

at  $W$  depending on the Tyurin data  $T$  and the Lie algebra  $\mathfrak{g}$ . The singularities at  $W$  (resp. in abuse of notation the points in  $W$  themselves) are called *weak singularities*.

In this section we only give the conditions for the case  $\mathfrak{g} = \mathfrak{gl}(n)$ ,  $\mathfrak{sl}(n)$ , and  $\mathfrak{s}(n)$ . The conditions for  $\mathfrak{so}(n)$  and  $\mathfrak{sp}(2n)$  are given in Section 8. Let  $T$  be fixed. For  $s = 1, \dots, ng$  we require that there exist  $\beta_s \in \mathbb{C}^n$  and  $\kappa_s \in \mathbb{C}$  such that the function  $L$  has an expansion at  $\gamma_s \in W$  of the form

$$L(z_s)|_1 = \frac{L_{s,-1}}{z_s} + L_{s,0} + \sum_{k>0} L_{s,k} z_s^k \tag{2}$$

with

$$L_{s,-1} = \alpha_s \beta_s^t, \quad \text{tr}(L_{s,-1}) = \beta_s^t \alpha_s = 0, \quad L_{s,0} \alpha_s = \kappa_s \alpha_s. \tag{3}$$

In particular, if  $\alpha_s$ , and  $\beta_s \neq 0$  the matrix  $L_{s,-1}$  is a rank 1 matrix, and  $\alpha_s$  is an eigenvector of  $L_{s,0}$ . In [5] it is shown that the requirements (3) are independent of the chosen coordinates  $z_s$  and that the set of all such functions constitutes an associative algebra under the point-wise matrix multiplication. We denote it by  $\bar{\mathfrak{g}}$ .<sup>2</sup> As Krichever and Sheinman [5] showed  $\bar{\mathfrak{g}}$  will always be a Lie algebra under the matrix commutator. The main point is to verify that the pole orders at the weak singularities do not increase and that the expansion is of the required type. (Note that the coefficients in the expansion (2) are matrices and the conditions (3) have to be used in the verification.)

REMARK 3.1. If  $\alpha_s = 0$  for all  $s$  there are no additional singularities and we recover the usual Krichever-Novikov current algebras.

REMARK 3.2. For the subalgebra  $\mathfrak{sl}(n)$  of  $\mathfrak{gl}(n)$  by (1) all matrices  $L_{s,k}$  in (2) have to be traceless. The conditions (3) stay the same.

REMARK 3.3. For the subalgebra  $\mathfrak{s}(n)$  all matrices have to be scalar matrices. As  $L_{s,-1}$  is traceless hence, either  $\alpha_s = 0$  or  $\beta_s = 0$ . In both cases there is no pole at  $\gamma_s$ . Furthermore the eigenvalue condition for  $L_{s,0}$  is also true. Hence  $\bar{\mathfrak{s}}(n)$  coincides with the Krichever-Novikov function algebra, i.e.

$$\bar{\mathfrak{s}}(n) \cong \mathfrak{s}(n) \otimes \mathcal{A} \cong \mathcal{A},$$

as associative algebras.

In fact we have a splitting  $\mathfrak{gl}(n) = \mathfrak{s}(n) \oplus \mathfrak{sl}(n)$  given by

$$X \mapsto \left( \frac{\text{tr}(X)}{n} I_n, X - \frac{\text{tr}(X)}{n} I_n \right),$$

where  $I_n$  is the  $n \times n$  unit matrix. This splitting induces a corresponding splitting for the Lax operator algebra  $\bar{\mathfrak{gl}}(n)$ :

$$\bar{\mathfrak{gl}}(n) = \bar{\mathfrak{s}}(n) \oplus \bar{\mathfrak{sl}}(n).$$

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<sup>2</sup>With the intent not to overload the notation we use  $\bar{\mathfrak{g}}$  for any of the current algebra versions associated to  $\mathfrak{g}$ . It will be clear from the context whether it is a classical, a Krichever-Novikov, or a Lax operator algebra.

**3.1. The almost-grading.** In contrast to the Krichever-Novikov situation the almost-grading of  $\mathcal{A}$  cannot be directly used to introduce an almost-grading for the Lax operator algebras (with  $\alpha_s \neq 0$ ). But similar ideas for introducing the almost-grading in  $\mathcal{A}$  and  $\mathcal{L}$  work here too. For every  $m \in \mathbb{Z}$  a subspace  $\bar{\mathfrak{g}}_m$  inside  $\bar{\mathfrak{g}}$  is defined as the subspace where non-zero elements have matrix expansions of order  $m$  at  $P_+$ , and a corresponding order at  $P_-$  forcing the elements to be essentially unique. In [5] it is shown that  $\bar{\mathfrak{g}}$  is almost-graded. More precisely, we have

$$\bar{\mathfrak{g}} = \bigoplus_{m \in \mathbb{Z}} \bar{\mathfrak{g}}_m, \quad \dim \bar{\mathfrak{g}}_m = \dim \mathfrak{g}, \quad [\bar{\mathfrak{g}}_m, \bar{\mathfrak{g}}_k] \subseteq \bigoplus_{h=m+k}^{m+k+M} \bar{\mathfrak{g}}_h,$$

with a constant  $M$  independent of  $m$  and  $k$ . In fact, for generic situations  $M = g$ , the genus of  $\Sigma$ , will work. To give an idea, for generic  $m$  the element  $L \in \bar{\mathfrak{g}}, L \neq 0$  lies in  $\bar{\mathfrak{g}}_m$  if there exists  $X_+, X_- \in \mathfrak{g} \setminus \{0\}$  with

$$L(z_+)_| = X_+ z_+^m + O(z_+^{m+1}), \quad L(z_-)_| = X_- z_-^{-m-g} + O(z_-^{-m-g+1}). \tag{4}$$

As  $\dim \bar{\mathfrak{g}}_m = \dim \mathfrak{g}$  we obtain that given  $X \in \mathfrak{g}$  there exists a unique  $X_m \in \bar{\mathfrak{g}}_m \subset \bar{\mathfrak{g}}$  such that (see also [14, Prop. 2.4])

$$X_m(z_+)_| = X \cdot z_+^m + O(z_+^{m+1}). \tag{5}$$

**4. The geometric meaning of Tyurin parameters.** Despite the fact that in this article I will not use the geometric relevance of Tyurin parameters in relation to the moduli space of bundles, it might be interesting to recall this background information. The reader not interested in this connection might directly jump to the next section.

Let  $\Sigma$  be a compact Riemann surface (or in the language of algebraic geometry a projective smooth curve over  $\mathbb{C}$ ) of genus  $g$ . Fix a number  $n \in \mathbb{N}$ . Given a rank  $n$  holomorphic (resp. algebraic) vector bundle  $E$  its determinant  $\det E$  is defined as  $\det E = \wedge^n E$ . The degree  $\deg E$  of the bundle  $E$  is defined as  $\deg(\det(E))$ . Recall that for a line bundle  $M$  over a Riemann surface the degree of  $M$  can be determined by taking a global meromorphic section of  $M$  and counting the number of zeros minus the number of poles of this section.

For vector bundles over compact Riemann surfaces we have the Hirzebruch-Riemann-Roch formula

$$\dim H^0(\Sigma, E) - \dim H^1(\Sigma, E) = \deg E - \text{rk}(E)(g - 1). \tag{6}$$

If one wants to construct a moduli space for vector bundles one has to restrict the set of vector bundles to the subset of stable, or more general semi-stable bundles.

**DEFINITION 4.1.** A bundle  $E$  over a projective smooth curve is called *stable* if for all subbundles  $F \neq E$  one has

$$\frac{\deg F}{\text{rk } F} < \frac{\deg E}{\text{rk } E}. \tag{7}$$

The bundle  $E$  is called *semi-stable* if in (7) the strict inequality  $<$  is replaced by  $\leq$ .

In the following we consider bundles  $E$  which are of rank  $n$  and degree  $n \cdot g$ . If we evaluate (6) for such bundles we obtain the value  $n$  on the left hand side. For a generic

semi-stable bundle  $E$  one has  $\dim H^1(\Sigma, E) = 0$ , hence we get

$$\dim H^0(\Sigma, E) = n.$$

If we choose a basis  $S := \{s_1, s_2, \dots, s_n\}$  of the space of global holomorphic sections of  $E$ , their exterior power is a global holomorphic section

$$s_1 \wedge s_2 \wedge \dots \wedge s_n \in H^0(\Sigma, \det E).$$

The zeros of this section are exactly the points  $P \in \Sigma$  for which the set  $S$  fails to be a basis of the fibre of  $E$  at the point  $P$ . This fibre is denoted by  $E_P$ . As  $\deg E = ng$  there exist (counted with multiplicities) exactly  $ng$  such points. For a generic choice of the set of sections, all zeros will be simple. Hence, we will obtain  $ng$  such points. They correspond exactly to the weak singularities  $W$  appearing in the definition of the Lax operator algebras. Accordingly we denote the zero points by  $\gamma_s, s = 1, \dots, ng$ .

Furthermore, as the zero is of order one at such a  $\gamma_s$  the sections evaluated at  $\gamma_s$  span an  $(n - 1)$ -dimensional subspace of  $E_{\gamma_s}$ . We have relations

$$\sum_{i=1}^n \alpha_{s,i} s_i(\gamma_s) = 0, \quad s = 1, \dots, ng.$$

In this way to every  $\gamma_s$  a vector  $\alpha_s \in \mathbb{C}^n, \alpha_s \neq 0$  can be assigned. This vector is unique up to multiplication by a non-zero scalar, hence unique as element  $[\alpha_s] \in \mathbb{P}^{n-1}(\mathbb{C})$ . Again these vectors are exactly the vectors used in the definition of the Lax operator algebras. If one checks the conditions (3) one sees immediately that they are independent of a rescaling. Hence only the projective class  $[\alpha_s]$  matters.

Obviously everything depends on the set  $S$  of basis elements. The choice of such a basis is called a *framing* of the bundle  $E$ . In the way described above the space of Tyurin parameters parameterizes an open dense subset of semi-stable framed vector bundles of rank  $n$  and degree  $ng$ . Note also that given such a bundle  $E$  with fixed set  $S$  of basis elements of  $H^0(\Sigma, E)$  it can be trivialized over  $\Sigma \setminus W$ .

Associated to these moduli spaces integrable hierarchies of Lax equations can be constructed. See [2] and [17] for results in this direction.

**5. Central extensions.** By the applications in quantum theory (but not only there) we are forced to consider central extensions of the introduced algebras. An example how they appear is given by regularization of a not well-defined action. The regularization makes the action well-defined but now it is only a projective action. To obtain a linear action we have to pass to a suitable central extension of the Lie algebra.

In the following definition  $\bar{\mathfrak{g}}$  could be any Lie algebra of current type or even more general any almost-graded Lie algebra. Central extensions are given by Lie algebra 2-cocycles with values in the trivial module  $\mathbb{C}$ . Recall that such a 2-cocycle for  $\bar{\mathfrak{g}}$  is a bilinear form  $\gamma : \bar{\mathfrak{g}} \times \bar{\mathfrak{g}} \rightarrow \mathbb{C}$  which is (1) antisymmetric and (2) fulfils the cocycle condition

$$\gamma([L, L'], L'') + \gamma([L', L''], L) + \gamma([L'', L], L') = 0, \quad \forall L, L', L'' \in \bar{\mathfrak{g}}.$$

A 2-cocycle  $\gamma$  is a coboundary if there exists a linear form  $\phi$  on  $\bar{\mathfrak{g}}$  with

$$\gamma(L, L') = \phi([L, L']), \quad \forall L, L' \in \bar{\mathfrak{g}}.$$

Two cocycles  $\gamma$  and  $\gamma'$  are cohomologous if their difference is a coboundary.

Given a 2-cocycle  $\gamma$  for  $\bar{\mathfrak{g}}$ , the associated central extension  $\widehat{\mathfrak{g}}_\gamma$  is given as vector space direct sum  $\widehat{\mathfrak{g}}_\gamma = \bar{\mathfrak{g}} \oplus \mathbb{C}$  with Lie product given by

$$[\widehat{L}, \widehat{L}'] = [\widehat{L}, \widehat{L}'] + \gamma(L, L') \cdot t, \quad [\widehat{L}, t] = 0, \quad L, L' \in \bar{\mathfrak{g}}.$$

Here we used  $\widehat{L} := (L, 0)$  and  $t := (0, 1)$ . Vice versa, every central extension

$$0 \rightarrow \mathbb{C} \xrightarrow{i_2} \widehat{\mathfrak{g}} \xrightarrow{p_1} \bar{\mathfrak{g}} \rightarrow 0$$

defines a 2-cocycle  $\gamma : \bar{\mathfrak{g}} \rightarrow \mathbb{C}$  after choosing a (linear) section  $s : \bar{\mathfrak{g}} \rightarrow \widehat{\mathfrak{g}}$  of  $p_1$  by the condition

$$[s(L), s(L')] - s([L, L']) = (0, \gamma(L, L')).$$

Different sections  $s_1$  and  $s_2$  give cohomologous 2-cocycles  $\gamma_1$  and  $\gamma_2$ . Two central extensions  $\widehat{\mathfrak{g}}_\gamma$  and  $\widehat{\mathfrak{g}}_{\gamma'}$  are equivalent if the defining cocycles  $\gamma$  and  $\gamma'$  are cohomologous.

With a view to applications we are only interested in those central extensions which allow the extension of the almost-grading of  $\bar{\mathfrak{g}}$  to the central extension  $\widehat{\mathfrak{g}}_\gamma$  by assigning to the central element  $t$  a certain degree and using the degree for  $\bar{\mathfrak{g}}$  for the subspace  $(\bar{\mathfrak{g}}, 0)$  in  $\widehat{\mathfrak{g}}$ . This is possible if and only if our defining cocycle  $\gamma$  is local in the following sense:

DEFINITION 5.1. A cocycle  $\gamma$  for the almost-graded Lie algebra  $\bar{\mathfrak{g}}$  is *local* if there exist constants  $M_1, M_2$  such that

$$\gamma(\bar{\mathfrak{g}}_m, \bar{\mathfrak{g}}_k) \neq 0 \Rightarrow M_1 \leq m + k \leq M_2, \quad \forall m, k \in \mathbb{Z}.$$

A central extension obtained by a local cocycle and with the extended grading is called an *almost-graded central extension* of  $\bar{\mathfrak{g}}$ .

The question is: how to construct cocycles? For the current algebras we first fix  $\langle \cdot, \cdot \rangle$  an invariant symmetric bilinear form on the finite-dimensional Lie algebra  $\mathfrak{g}$ . Recall that invariance means that

$$\langle [a, b], c \rangle = \langle a, [b, c] \rangle \quad \text{for all } a, b, c \in \mathfrak{g}.$$

For a simple Lie algebra the Cartan-Killing form is the only such form up to rescaling. Moreover it is non-degenerate. For  $\mathfrak{sl}(n)$  the Cartan-Killing form is given by  $\alpha(A, B) = \text{tr}(AB)$ .

For the classical current algebras associated to a simple finite-dimensional Lie algebra with Cartan-Killing form  $\langle \cdot, \cdot \rangle$  a non-trivial central extension  $\widehat{\mathfrak{g}} = \bar{\mathfrak{g}} \oplus \mathbb{C}t$  is defined by

$$[x \otimes z^n, y \otimes z^m] = [x, y] \otimes z^{n+m} - \langle x, y \rangle \cdot n \cdot \delta_n^{-m} \cdot t.$$

To avoid cumbersome notation I dropped the  $\hat{\phantom{x}}$  here. It is called the (classical) affine Lie algebra associated to  $\mathfrak{g}$ . Another name is untwisted affine Lie algebra of Kac-Moody type [1]. By setting  $\text{deg } t := 0$  (and using  $n = \text{deg}(x \otimes z^n)$ ) the affine algebra is a graded algebra. The cocycle

$$\gamma(x \otimes z^n, y \otimes z^m) = -\langle x, y \rangle \cdot n \cdot \delta_n^{-m}$$

is obviously local. Indeed for  $\mathfrak{g}$  simple it is the only non-trivial extension up to equivalence and rescaling.

For the usual Krichever-Novikov algebras  $\mathfrak{g} \otimes \mathcal{A}$  a 2-cocycle can be given by

$$\gamma_C(x \otimes f, y \otimes g) := \langle x, y \rangle \frac{1}{2\pi i} \int_C f dg,$$

where  $C$  is a closed curve on  $\Sigma$  [10].

The cocycles depend crucially on the choice of the integration path  $C$  and we might obtain different non-cohomologous cocycles, hence non-equivalent central extensions. But recall that we are mainly interested in local cocycles. There is a special class of integration cycles, the so called separating cycles  $C_s$ , which separate the points in  $I$  from the points in  $O$  with multiplicity one. As all separating cycles are homologous the value of the integration does not depend on the separating cycle we take. In particular, we could take circles around the points in  $I$ , hence calculate the integral using residues there. In [10, Theorem 4.6] a complete classification of local cohomology classes, i.e. classes which admit at least one representing element which is local, is given for the case of reductive Lie algebras  $\mathfrak{g}$ . In this way a classification of almost-graded central extensions of the Krichever-Novikov current algebras is obtained. In particular, for  $\mathfrak{g}$  simple there is only one non-trivial almost-graded extension up to equivalence and rescaling. Let me stress the fact that without the condition of locality of the cocycle the statement would be wrong.

**6. Central extensions for Lax operator algebras.** For the Lax operator algebras we obviously have the problem that differentiation of our objects will increase the pole order at the weak singularities. We will not stay in the algebra. The deeper reason for this is that the objects are not really functions but representing functions of sections of a bundle. To correct this we need a connection  $\nabla$  and have to take the covariant derivative. It will be defined with the help of a connection form  $\omega$ . This form is a  $\mathfrak{g}$ -valued meromorphic 1-form, holomorphic outside  $P_+$ ,  $P_-$  and  $W$ , and with prescribed behavior at the points in  $W$ . For  $\gamma_s \in W$  with  $\alpha_s = 0$  the requirement is that  $\omega$  is also regular there. For the points  $\gamma_s$  with  $\alpha_s \neq 0$  we require that it has the expansion

$$\omega(z_s)_| = \left( \frac{\omega_{s,-1}}{z_s} + \omega_{s,0} + \sum_{k>1} \omega_{s,k} z_s^k \right) dz_s.$$

For  $\mathfrak{gl}(n)$  we require that there exist  $\tilde{\beta}_s \in \mathbb{C}^n$  and  $\tilde{\kappa}_s \in \mathbb{C}$  such that

$$\omega_{s,-1} = \alpha_s \tilde{\beta}_s^t, \quad \omega_{s,0} \alpha_s = \tilde{\kappa}_s \alpha_s, \quad \text{tr}(\omega_{s,-1}) = \tilde{\beta}_s^t \alpha_s = 1.$$

Note that compared to (3) only the last condition was modified. These conditions were introduced in [2], [5], see also [14]. We will even require that  $\omega$  is holomorphic at the point  $P_+$  (resp. at the points in  $I$ ). As we allow poles at the point  $P_-$  (resp. at the points in  $O$ ) such an  $\omega$  will always exist.

The induced connection, resp. covariant derivative for the algebra will be given by

$$\nabla^{(\omega)} = d + [\omega, \cdot], \quad \nabla_e^{(\omega)} = dz(e) \frac{d}{dz} + [\omega(e), \cdot].$$

Here  $e$  is a vector field from  $\mathcal{L}$ .

REMARK 6.1. For the subalgebras  $\overline{\mathfrak{sl}}(n)$  and  $\overline{\mathfrak{h}}(n)$  we can take the same  $\omega$  as for  $\overline{\mathfrak{gl}}(n)$ . The covariant derivative will respect the subalgebras. In fact for the scalar algebra  $\nabla^{(\omega)} = d$ . For the other algebras see Section 8.

PROPOSITION 6.2 ([14]). *The covariant derivative  $\nabla_e^{(\omega)}$  acts as a derivation on  $\overline{\mathfrak{g}}$  and makes  $\overline{\mathfrak{g}}$  to an almost-graded Lie module over  $\mathcal{L}$ .*

In [14] the following 2-cocycles for  $\overline{\mathfrak{g}}$  were given

$$\gamma_{1,\omega,C}(L, L') := \frac{1}{2\pi i} \int_C \text{tr}(L \cdot \nabla^{(\omega)} L'), \quad L, L' \in \overline{\mathfrak{g}}, \tag{8}$$

and

$$\gamma_{2,\omega,C}(L, L') := \frac{1}{2\pi i} \int_C \text{tr}(L) \cdot \text{tr}(\nabla^{(\omega)} L'), \quad L, L' \in \overline{\mathfrak{g}}. \tag{9}$$

Here  $C$  is an arbitrary closed path in  $\Sigma$ . Indeed these are cocycles. The cocycle  $\gamma_{2,\omega,C}$  does not depend on  $\omega$  and will vanish for  $\overline{\mathfrak{g}} \neq \mathfrak{gl}(n), \mathfrak{h}(n)$ . Two cocycles  $\gamma_{1,\omega,C}$  and  $\gamma_{1,\omega',C}$  with different connection forms  $\omega$  and  $\omega'$  will be cohomologous.

DEFINITION 6.3. Let the action of  $\mathcal{L}$  on  $\overline{\mathfrak{g}}$  be given via  $\nabla^{(\omega)}$ . A cocycle  $\gamma$  is called  $\mathcal{L}$ -invariant if

$$\gamma(\nabla_e^{(\omega)} L, L') + \gamma(L, \nabla_e^{(\omega)} L') = 0, \quad \forall e \in \mathcal{L}. \tag{10}$$

The 2-cocycles (8), (9) are  $\mathcal{L}$ -invariant.

The cocycles depend crucially on the choice of the integration path  $C$ . As in the Krichever-Novikov case we are interested only in local cocycles. Hence, we take again the separating cycles  $C_s$  as  $C$ . The integral does not depend on the separating cycle we take (see [14, Prop. 3.6]). Note as here we have additional poles “between  $I$  and  $O$ ” there is indeed something to prove. The integral over  $C_s$  can be determined by calculating the residue at  $P_+$  (resp. at  $P_-$ ). Indeed, for  $C = C_s$  the cocycle (8) in a modified form can already be found in [5].

If we take a separating cycle as integration path we drop the reference to  $C_s$  for the cocycle, i.e. we use  $\gamma_{1,\omega}$  and  $\gamma_2$ .

PROPOSITION 6.4 ([14]). *The cocycles  $\gamma_{1,\omega}$ , and  $\gamma_2$  are  $\mathcal{L}$ -invariant local cocycles.*

But what about the opposite, i.e. is every local and  $\mathcal{L}$ -invariant cocycle a linear combination of these two cocycles? One of the main results of [14] is that this is true. To formulate the results we need to introduce the following convention. A cohomology class will be called local, resp.  $\mathcal{L}$ -invariant if it admits a representing cocycle which is local, resp.  $\mathcal{L}$ -invariant. This does not imply that all representing cocycles will be of this type.

THEOREM 6.5 ([14, Thm. 3.8]).

(a) *If  $\mathfrak{g}$  is simple (i.e.  $\mathfrak{g} = \mathfrak{sl}(n), \mathfrak{so}(n), \mathfrak{sp}(2n)$ ) then the space of local cohomology classes for  $\overline{\mathfrak{g}}$  is one-dimensional. If we fix any connection form  $\omega$  then the space will be generated by the class of  $\gamma_{1,\omega}$ . Every  $\mathcal{L}$ -invariant (with respect to the connection  $\omega$ ) local cocycle is a scalar multiple of  $\gamma_{1,\omega}$ .*

(b) *For  $\overline{\mathfrak{g}} = \overline{\mathfrak{gl}}(n)$  the space of local cohomology classes which are  $\mathcal{L}$ -invariant having been restricted to the scalar subalgebra is two-dimensional. If we fix any connection form*

$\omega$  then the space will be generated by the classes of the cocycles  $\gamma_{1,\omega}$  and  $\gamma_2$ . Every  $\mathcal{L}$ -invariant local cocycle is a linear combination of  $\gamma_{1,\omega}$  and  $\gamma_2$ .

**COROLLARY 6.6.** *Let  $\mathfrak{g}$  be a simple classical Lie algebra and  $\bar{\mathfrak{g}}$  the associated Lax operator algebra. Let  $\omega$  be a fixed connection form. Then in each local cohomology class  $[\gamma]$  there exists a unique representative  $\gamma'$  which is local and  $\mathcal{L}$ -invariant (with respect to  $\omega$ ). Moreover,  $\gamma' = \alpha\gamma_{1,\omega}$ , with  $\alpha \in \mathbb{C}$ .*

**REMARK 6.7.** By this corollary for  $\bar{\mathfrak{g}}$  with  $\mathfrak{g}$  simple every local cohomology class will be  $\mathcal{L}$ -invariant. This is not true for  $\bar{\mathfrak{gl}}(n)$ , due to its abelian part  $\bar{\mathfrak{s}}(n)$ . For  $\bar{\mathfrak{s}}(n) \cong \mathcal{A}$  the coboundaries are zero and the cocycle condition reduces to the antisymmetry of the bilinear form. Hence it is possible to write down a lot of different local cocycles which are not equivalent. Only by the  $\mathcal{L}$ -invariance the cocycle will be uniquely fixed up to multiplication by a scalar. As explained in [9]  $\mathcal{L}$ -invariance is a natural condition as it is automatically true for those representations of  $\mathcal{A}$  which are in fact representations of the larger algebra of differential operators of degree  $\leq 1$ .

**REMARK 6.8.** The notion of  $\mathcal{L}$ -invariance should be compared to the notion of  $S^1$ -invariant cocycles for the classical loop algebras [6].

**7. Some ideas of the proof.** The proofs are technically quite involved and can be found in [14]. Nevertheless I want to give some of the principal ideas.

(a) We start with a local,  $\mathcal{L}$  invariant cocycle and use the almost-graded  $\mathcal{L}$ -module structure to show that everything can be reduced to level zero. This should be understood in the following sense. Let  $L_m$  and  $L'_k$  be homogenous elements of degree  $m$  and  $k$  respectively, then the level of the pair  $(L_m, L'_k)$  is the sum of their degrees  $m + k$ . We show that if the level  $l = m + k \neq 0$  the cocycle values for pairs of elements of level  $l$  can be linearly expressed by values of the cocycle evaluated for pairs at higher level with universal coefficients only depending on the algebra  $\bar{\mathfrak{g}}$ . By locality the cocycle values are zero for high enough level. Hence, they will vanish for all levels  $l > 0$ . Moreover, by recursion their values at level  $l < 0$  are fixed by knowing the values at level zero. See below for some more details on this step.

(b) Next we show that for the level zero the cocycle under consideration will be a linear combination of the cocycles  $\gamma_{1,\omega}$  and  $\gamma_2$ . Hence they will coincide everywhere. This even gives an identification on the cocycle level, not only on the cohomology class level.

(c) The abelian part of the algebra is now covered, as we put  $\mathcal{L}$ -invariance into the requirement. For the simple part, we have to show that in every cohomology class there is an  $\mathcal{L}$ -invariant cocycle. To show this we consider the Chevalley generators and the Chevalley-Serre relations of the finite-dimensional simple Lie algebra  $\mathfrak{g}$  and use the almost-graded structure inside  $\bar{\mathfrak{g}}$  and the boundedness from above of the cocycle. We make appropriate cohomological changes and end up with the fact that everything depends linearly on only one single cocycle value at level zero. In fact everything is uniquely fixed with respect to the value of  $\gamma(H_1^\alpha, H_{-1}^\alpha)$  where  $\alpha$  is one (arbitrary) fixed simple root and  $H^\alpha$  the corresponding generator in the Cartan subalgebra. Hence the space is at most

one-dimensional. But  $\gamma_{1,\omega}$  is a local and  $\mathcal{L}$ -invariant cocycle which is not a boundary and we get the result. As a side effect we obtain that in every local cohomology class (for  $\mathfrak{g}$  simple) there is a unique  $\mathcal{L}$ -invariant representing cocycle.

In the following I will indicate for the interested reader some **more facts about part (a)** above. In particular I want to show the usage of the almost-grading.

Let  $\{X^r \mid r = 1, \dots, \dim \mathfrak{g}\}$  be a basis of the finite-dimensional Lie algebra. As mentioned earlier we can find elements  $\{X_n^r \mid r = 1, \dots, \dim \mathfrak{g}\}$  in  $\bar{\mathfrak{g}}$  which are of order  $n$  at the point  $P_+$  and start with leading matrix coefficient  $X^r$ , see (5). We have the decomposition  $\bar{\mathfrak{g}} = \bigoplus_{n \in \mathbb{Z}} \bar{\mathfrak{g}}_n$  into subspaces of homogenous elements of degree  $n$  where the subspace  $\bar{\mathfrak{g}}_n$  is generated by the basis  $\{X_n^r \mid r = 1, \dots, \dim \mathfrak{g}\}$ . The vector field  $e_n \in \mathcal{L}$  of degree  $n$  starts with order  $n+1$  at the point  $P_+$ . By Proposition 6.2 the algebra  $\bar{\mathfrak{g}}$  is an almost-graded module over  $\mathcal{L}$ . Calculations in local coordinates yield

$$\nabla_{e_k}^{(\omega)} X_n^r = nX_{n+k}^r + L, \quad (11)$$

where  $L$  is an element of  $\bar{\mathfrak{g}}$  of leading order  $\geq n+k+1$  at  $P_+$ . Recall that we have chosen our connection form to be holomorphic at the point  $P_+$ .

For a cocycle  $\gamma$  evaluated for pairs of elements of level  $l$  we will use the symbol  $\equiv$  to denote that the expressions are the same on both sides of an equation up to universal expressions in the values of  $\gamma$  at higher level. This has to be understood in the following strong sense:

$$\sum \alpha_{r,s}^n \gamma(X_n^r, X_{l-n}^s) \equiv 0, \quad \alpha_{r,s}^n \in \mathbb{C}$$

means a congruence modulo a linear combination of values of  $\gamma$  at pairs of basis elements of level  $l' > l$ . The coefficients of that linear combination, as well as the  $\alpha_{r,s}^n$ , depend only on the structure of the Lie algebra  $\bar{\mathfrak{g}}$  and do not depend on  $\gamma$ .

By the  $\mathcal{L}$ -invariance (10) we have

$$\gamma(\nabla_{e_p}^{(\omega)} X_m^r, X_n^s) + \gamma(X_m^r, \nabla_{e_p}^{(\omega)} X_n^s) = 0.$$

Using the almost-graded structure (11) we obtain the formula

$$m\gamma(X_{p+m}^r, X_n^s) + n\gamma(X_m^r, X_{n+p}^s) \equiv 0, \quad (12)$$

valid for all  $n, m, p \in \mathbb{Z}$ .

If we set  $p = 0$  in (12) then we obtain

$$(m+n)\gamma(X_m^r, X_n^s) \equiv 0. \quad (13)$$

Hence, for the level  $(m+n) \neq 0$  everything is determined by the values at higher level. This implies in particular that if a cocycle is bounded from above it will be automatically bounded by zero and if it vanishes at level zero it will vanish identically.

By evaluating (12) for suitable values for  $m, p, k$  and using the fact that all values in levels greater than zero vanish we obtain

$$\begin{aligned} \gamma(X_m^r, X_0^s) &= 0, \quad \forall m \geq 0. \\ \gamma(X_n^r, X_{-n}^s) &= n \cdot \gamma(X_1^r, X_{-1}^s), \quad \forall m \in \mathbb{Z}. \\ \gamma(X_1^r, X_{-1}^s) &= \gamma(X_1^s, X_{-1}^r). \end{aligned}$$

Hence everything depends only on the values of  $\gamma(X_1^r, X_{-1}^s)$ , with  $r, s = 1, \dots, \dim \mathfrak{g}$ .

Let  $X \in \mathfrak{g}$ . We denote by  $\tilde{X}_n$  any element in  $\bar{\mathfrak{g}}$  with leading term  $Xz_+^n$  at  $P_+$ . We define

$$\psi_\gamma : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C} \quad \psi_\gamma(X, Y) := \gamma(\tilde{X}_1, \tilde{Y}_{-1}). \quad (14)$$

As the cocycle vanishes for levels greater than zero,  $\psi_\gamma$  does not depend on the choice of  $\tilde{X}_1$  and  $\tilde{Y}_{-1}$ . Obviously, it is a bilinear form on  $\mathfrak{g}$ . Moreover in [14] we show that it is symmetric and invariant.

As the cocycle  $\gamma$  is fixed by the values  $\gamma(X_1^t, X_{-1}^s)$ , and they are fixed by the bilinear map  $\psi_\gamma$  we obtain:

**THEOREM 7.1** ([14]). *Let  $\gamma$  be an  $\mathcal{L}$ -invariant cocycle for  $\bar{\mathfrak{g}}$  which is bounded from above by zero. Then  $\gamma$  is completely fixed by the associated symmetric and invariant bilinear form  $\psi_\gamma$  on  $\mathfrak{g}$  defined via (14).*

In the case of a simple Lie algebra we are done, as every such form is a multiple of the Cartan-Killing form. And we obtain the uniqueness (up to multiplication with a scalar) of a local and  $\mathcal{L}$ -invariant cocycle. For  $\mathfrak{gl}(n)$  we use the splitting into  $\mathfrak{s}(n) \oplus \mathfrak{sl}(n)$  and have to refer to uniqueness results for the scalar algebra  $\mathcal{A}$  obtained in [9].

## 8. Appendix: $\mathfrak{so}(n)$ and $\mathfrak{sp}(2n)$ .

**8.1.  $\mathfrak{so}(n)$ .** In the case of  $\mathfrak{so}(n)$  we require that all  $L_{s,k}$  in (2) are skew-symmetric. In particular, they are trace-less. The set-up has to be slightly modified following [5]. First only those Tyurin parameters  $\alpha_s$  are allowed which satisfy  $\alpha_s^t \alpha_s = 0$ . Then, (3) is modified in the following way:

$$L_{s,-1} = \alpha_s \beta_s^t - \beta_s \alpha_s^t, \quad \text{tr}(L_{s,-1}) = \beta_s^t \alpha_s = 0, \quad L_{s,0} \alpha_s = \kappa_s \alpha_s. \quad (15)$$

The relations (15) do not depend on the coordinates  $z_s$ . Under the point-wise matrix commutator the set of such maps constitutes a Lie algebra, see [5].

As far as the connection form for  $\mathfrak{so}(n)$  is concerned we require that there exist  $\tilde{\beta}_s \in \mathbb{C}^n$  and  $\tilde{\kappa}_s \in \mathbb{C}$  such that

$$\omega_{s,-1} = \alpha_s \tilde{\beta}_s^t - \tilde{\beta}_s \alpha_s^t, \quad \omega_{s,0} \alpha_s = \tilde{\kappa}_s \alpha_s, \quad \tilde{\beta}_s^t \alpha_s = 1.$$

Such connection forms exist.

**8.2.  $\mathfrak{sp}(2n)$ .** For  $\mathfrak{sp}(2n)$  we consider a symplectic form  $\hat{\sigma}$  for  $\mathbb{C}^{2n}$  given by a non-degenerate skew-symmetric matrix  $\sigma$ . Without loss of generality we might even assume that this matrix is given in the standard form  $\sigma = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ . The Lie algebra  $\mathfrak{sp}(2n)$  is the Lie algebra of matrices  $X$  such that  $X^t \sigma + \sigma X = 0$ . This is equivalent to  $X^t = -\sigma X \sigma^{-1}$ , which implies that  $\text{tr}(X) = 0$ . For the standard form above,  $X \in \mathfrak{sp}(2n)$  if and only if

$$X = \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}, \quad B^t = B, \quad C^t = C.$$

At the weak singularities we have the expansion with matrices  $L_{k,s} \in \mathfrak{sp}(2n)$

$$L(z_s) = \frac{L_{s,-2}}{z_s^2} + \frac{L_{s,-1}}{z_s} + L_{s,0} + L_{s,1} z_s + \sum_{k>1} L_{s,k} z_s^k.$$

The conditions (3) are modified as follows (see [5]): there exist  $\beta_s \in \mathbb{C}^{2n}$ ,  $\nu_s, \kappa_s \in \mathbb{C}$  such that

$$L_{s,-2} = \nu_s \alpha_s \alpha_s^t \sigma, \quad L_{s,-1} = (\alpha_s \beta_s^t + \beta_s \alpha_s^t) \sigma, \quad \beta_s^t \sigma \alpha_s = 0, \quad L_{s,0} \alpha_s = \kappa_s \alpha_s, \quad (16)$$

and

$$\alpha_s^t \sigma L_{s,1} \alpha_s = 0. \quad (17)$$

Again in [5] it is shown that under the point-wise matrix commutator the set of such maps constitutes a Lie algebra.

For the connection form for  $\mathfrak{sp}(2n)$  we require that there exist  $\tilde{\beta}_s \in \mathbb{C}^{2n}$ ,  $\tilde{\kappa}_s \in \mathbb{C}$  such that

$$\omega_{s,-1} = (\alpha_s \tilde{\beta}_s^t + \tilde{\beta}_s \alpha_s^t) \sigma, \quad \omega_{s,0} \alpha_s = \tilde{\kappa}_s \alpha_s, \quad \alpha_s^t \sigma \omega_{s,1} \alpha_s = 0, \quad \tilde{\beta}_s^t \sigma \alpha_s = 1.$$

REMARK 8.1. The reader might ask why for  $\mathfrak{sp}(2n)$  there appear poles of order two at the weak singularities. By direct calculations it turns out that such a modification has to be done to retain that there will be no additional degree of freedom (compared to other points) at each weak singularity individually (due to the relations (16) and (17)), without affecting the closedness and almost-gradedness of the algebra under the commutator. There is a geometric reason behind. Note that the minimal codimension of a strict symplectic subspace of a symplectic vector space is two.

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