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REMARKS ON F-PLANAR CURVES AND THEIR GENERALIZATIONS

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Abstract. Generalized planar curves (A-curves) are more general analogues of F-planar curves and geodesics. In particular, several well known geometries are described by more than one affinor. The best known example is the almost quaternionic geometry. A new approach to this topic (A-structures) was started in our earlier papers. In this paper we expand the concept of A-structures to projective A-structures.

1. A-structures. The concept of planar curves is a generalization of a geodesics on a smooth manifold equipped with certain structure. In [MS] authors proved a set of facts about structures equipped with two different affinors. A manifold equipped with an affine connection and a set of affinors $A = \{F_1, \dots, F_l\}$ is called an A-structure and a curve satisfying $\nabla_{\dot{c}}\dot{c} \in \langle F_1, \dots, F_l\rangle$ is called A-planar. There are some very well known structures equipped with more than one affinor based on quaternions.

Definition 1.1. Let M be a smooth manifold such that $\dim(M) = m$. Let A be a smooth ℓ-dimensional ($\ell < m$) vector subbundle in $T^*M \otimes TM$ such that the identity affinor $E = id_{TM}$ restricted to T_xM belongs to $A_xM \subset T^*_xM \otimes T_xM$ at each point $x \in M$. We say that M is equipped with an ℓ-dimensional A-*structure.*

An almost quaternionic structure ($A = \langle E, I, J, K\rangle$, $I^2 = J^2 = -id_{TM}$, $K = IJ$, $IJ = -JI$) and almost complex structure ($A = \langle E, J\rangle$, $J^2 = -id_{TM}$) are the best known examples of A-structures. Another one is e.g. an almost product structure ($A = \langle E, J\rangle$, $J^2 = id_{TM}$) or an almost para-quaternionic structure ($A = \langle E, I, J, K\rangle$, $I^2 = J^2 = id_{TM}$, $K = IJ$, $IJ = JI$) etc.

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Classically, an *F-planar* curve is a curve $c : \mathbb{R} \to M$ satisfying the condition

$$\nabla_{\dot{c}}\dot{c} \in \langle \dot{c}, F(\dot{c})\rangle.$$

Clearly geodesics are F-planar curves for all affinors F, because $\nabla_{\dot{c}}\dot{c} \in \langle \dot{c}\rangle \subset \langle \dot{c}, F(\dot{c})\rangle$.

DEFINITION 1.2. Let M be a smooth manifold equipped with an A-structure and a linear connection ∇. A smooth curve $c : \mathbb{R} \to M$ is said to be *A-planar* if

$$\nabla_{\dot{c}}\dot{c} \in A(\dot{c}).$$

DEFINITION 1.3. Let M be a smooth manifold equipped with an A-structure and a linear connection ∇. Let $\bar{M}$ be another manifold with a linear connection $\bar{\nabla}$ and a B-structure. A diffeomorphism $f : M \to \bar{M}$ is called (A, B)*-planar* if each A-planar curve c on M is mapped by f onto the B-planar curve $f_\star c$ on M.

Now, we shall prove some basic facts about A-planar curves and their morphisms.

DEFINITION 1.4. For any tangent vector $X \in T_xM$ we shall write $A_x(X)$ for the vector subspace

$$A_x(X) = \{F(X)|F \in A_xM\} \subset T_xM$$

and call it the *A-hull of the vector X*. Similarly, the *A-hull of a vector field* is a subbundle in TM obtained pointwise.

For example, A-hull for an almost quaternionic structure is

$$A_x(X) = \{aX + bI(X) + cJ(X) + dK(X)|a, b, c, d \in \mathbb{R}\}.$$

DEFINITION 1.5. Let (M, A) be a smooth manifold M equipped with an ℓ-dimensional A-structure. We say that the A-structure has *weak generic rank* ℓ if for each $x \in M$ the subset of vectors $X \in T_xM$ such that the A-hull $A_x(X)$ generates a vector subspace of dimension ℓ is open and dense.

We denote

$$\mathcal{V} := \{X \in T_xM|\dim A(X) = \ell\}.$$

The affinor J on an almost product structure has eigenvalues ± 1. Clearly, if $JX = \lambda X$, then $X = J^2X = \lambda JX = \lambda^2 X$, and thus $\lambda = \pm 1$ and $T_xM = V^+ \oplus V^-$. Hence $X + FX \in V^+$ and $X - FX \in V^-$ and one can easily see that

$$\dim A(X + FX) = 1,\ \dim A(2X + FX) = 2.$$

LEMMA 1.6. *Every A-structure* (M, A) *on a manifold M,* $\dim M \geq \dim A$*, where A is an algebra with inversion, has weak generic rank* $\dim A$*.*

Proof. Consider X such that $X \notin \mathcal{V}$, therefore $\exists F \in A = \langle E, G\rangle, FX = 0$, and $F^{-1}FX = 0$ implies $X = 0$. ∎

LEMMA 1.7. *Every two dimensional A-structure* (M, A) *on a manifold M,* $\dim M \geq 2$*, has weak generic rank 2.*

Proof. Consider X such that $X \notin \mathcal{V}$, therefore $\exists F \in A = \langle E, G\rangle, FX = aX + bG(X) = 0$, i.e. the vector X has to be an eigenvector of G and the vector X has to belong to one of finitely many k-dimensional ($k < \dim M$) subspaces. Finally, the complement $\mathcal{V}$ is open and dense. ∎

LEMMA 1.8. *Let (M, A) be a para-quaternionic structure on a manifold of dimension $\dim M > 4$. Then the A-structure $A = \langle E, F, G, FG\rangle$ has weak generic rank 4.*

Proof. Recall from linear algebra that two commuting diagonalizable linear maps are simultaneously diagonalizable, i.e. for two commuting product structures we have

$$F = \begin{pmatrix} E_{m_F} & 0 \\ 0 & -E_{n_F} \end{pmatrix}, G = \begin{pmatrix} E_{m_G} & 0 \\ 0 & -E_{n_G} \end{pmatrix}, FG = \begin{pmatrix} E_{k_1} & 0 & 0 \\ 0 & -E_{k_2} & 0 \\ 0 & 0 & E_{k_3} \end{pmatrix}.$$

Let $X \notin \mathcal{V}$, i.e. $\exists H$ such that $HX = 0$, where

$$H := \begin{pmatrix} c_1 E_{k_1} & 0 & 0 \\ 0 & c_2 E_{k_2} & 0 \\ 0 & 0 & c_3 E_{k_3} \end{pmatrix}$$

for all $c_1, c_2, c_3 \in \mathbb{R}$. Vector X has to be a vector inside $\ker H$, i.e. has to belong to one of finitely many k-dimensional ($k < \dim M$) subspaces. Finally, the complement $\mathcal{V}$ is open and dense. ∎

We have proved that an almost product structure and an almost complex structure have weak generic rank 2 together with the fact that an almost quaternionic structure and an almost para-quaternionic structure have weak generic rank 4.

DEFINITION 1.9. Let (M, A) be a smooth manifold M equipped with an ℓ-dimensional A-structure. We say that the A-structure has *generic rank* ℓ if for each $x \in M$ the subset of vectors $(X, Y) \in T_xM \oplus T_xM$ such that the A-hulls $A_x(X)$ and $A_x(Y)$ generate a vector subspace $A_x(X) \oplus A_x(Y)$ of dimension 2ℓ is open and dense.

THEOREM 1.10 ([HS08]). *Let (M, A) be a smooth manifold of dimension n with ℓ-dimensional A-structure such that $2\ell \le \dim M$. If A_x is an algebra (i.e. for all $f, g \in A_x$, $fg = f \circ g \in A_x$) for all $x \in M$ and A has weak generic rank ℓ, then the structure has generic rank ℓ.*

Now, we know that an almost product structure and an almost complex structure have a generic rank 2 (on a manifold M, $\dim M \ge 4$) together with fact that an almost quaternionic structure and an almost para-quaternionic structure have a generic rank 4 (on a manifold M, $\dim M \ge 8$).

2. Projective A-structures. Let M be a smooth manifold equipped with an A-structure and a linear connection ∇. The connection is said to be an *A-connection* if it belongs to the class of connections

$$[\nabla]_A = \nabla + \sum_{i=1}^{k=\dim A} \Upsilon_i \odot F_i, \tag{1}$$

where $\langle F_1, \dots, F_k\rangle = A$ as a vector space, $\odot$ is symmetric tensor product and Υ_i are one forms on M.

THEOREM 2.1. *Let $(M, A = \langle E, J\rangle)$ be an almost complex structure and ∇ be a linear connection preserving J, i.e. $\nabla J = 0$. Then the class of A-connections $[\nabla]_A$ equals the* *class*

$$[\nabla] = \nabla + \Upsilon \odot E - (\Upsilon \circ J) \odot J, \tag{2}$$

where Υ *is any one form on* M.

Proof. First, let us consider the difference tensor $P(X,Y) = \bar{\nabla}_X(Y) - \nabla_X(Y)$ and one can see that its value is symmetric in each tangent space because both connections share the same torsion. Since both ∇ and $\bar{\nabla}$ preserve J, the difference tensor P is complex linear in the second variable. By symmetry it is thus complex bilinear and we can compute:

$$\begin{aligned}
\tilde{\nabla}_X Y - \nabla_X Y &= \Upsilon_1(X)JY + \Upsilon_1(Y)X - \Upsilon_2(X)JY - \Upsilon_2(Y)X,\\
\tilde{\nabla}_{JX} JY - \nabla_{JX} JY &= J^2(\tilde{\nabla}_X Y - \nabla_X Y) = -(\tilde{\nabla}_X Y - \nabla_X Y),\\
-(\tilde{\nabla}_X Y - \nabla_X Y) &= \Upsilon_1(JX)JY + \Upsilon_1(JY)JX - \Upsilon_2(JX)Y - \Upsilon_2(JY)X.
\end{aligned}$$

The sum of the first and third row implies

$$\begin{aligned}
&-\Upsilon_1(X)Y - \Upsilon_1(Y)X - \Upsilon_2(X)JY - \Upsilon_2(Y)JX\\
&= \Upsilon_1(JX)JY + \Upsilon_1(JY)JX - \Upsilon_2(JX)Y - \Upsilon_2(JY)X.
\end{aligned}$$

Thus $(\Upsilon_2(X) + \Upsilon_1(JX)) = 0$ because we can suppose that X, Y, JX, JY are linearly independent without loss of generality. ∎

THEOREM 2.2. *Let* $(M, Q = \langle I, J, K\rangle)$ *be an almost quaternionic structure and* ∇ *be a linear connection preserving* I, J, K, *i.e.* $\nabla I = \nabla J = \nabla K = 0$. *Then the class of connections* $[\nabla]_A$ *equals the class*

$$[\nabla] = \nabla + \Upsilon \odot E - (\Upsilon \circ I) \odot I - (\Upsilon \circ J) \odot J - (\Upsilon \circ K) \odot K, \tag{3}$$

where Υ *is any one form on* M.

Proof. First, let us consider the difference tensor $P(X,Y) = \bar{\nabla}_X(Y) - \nabla_X(Y)$ and one can see that its value is symmetric in each tangent space because both connections share the same torsion. Since both ∇ and $\bar{\nabla}$ preserve I, J, K the difference tensor P is quaternionic linear in the second variable. By symmetry it is thus quaternionic bilinear and we can compute:

$$\begin{aligned}
P(X,Y) &= \Upsilon_1(X)Y + \Upsilon_1(Y)X + \Upsilon_2(X)IY + \Upsilon_2(Y)IX + \Upsilon_3(X)JY\\
&\quad + \Upsilon_3(Y)JX + \Upsilon_4(X)KY + \Upsilon_4(Y)KX,\\
P(IX,IY) &= \Upsilon_1(IX)IY + \Upsilon_1(IY)IX - \Upsilon_2(IX)Y - \Upsilon_2(IY)X + \Upsilon_3(IX)KY\\
&\quad + \Upsilon_3(IY)KX - \Upsilon_4(IX)JY - \Upsilon_4(IY)JX,\\
P(JX,JY) &= \Upsilon_1(JX)JY + \Upsilon_1(JY)JX - \Upsilon_2(JX)KY - \Upsilon_2(JY)KX\\
&\quad - \Upsilon_3(JX)Y - \Upsilon_3(JY)X + \Upsilon_4(JX)IY - \Upsilon_4(JY)IX,\\
P(KX,KY) &= \Upsilon_1(KX)KY + \Upsilon_1(KY)KX + \Upsilon_2(KX)JY - \Upsilon_2(KY)JX\\
&\quad - \Upsilon_3(KX)IY - \Upsilon_3(KY)IX - \Upsilon_4(KX)Y - \Upsilon_4(KY)X.
\end{aligned}$$

The sum of three times the first row and the last three rows implies a system of linear equations because we can suppose that $X, Y, IX, IY, JX, JY, KX, KY$ are linearly independent without loss of generality:

$$
\begin{aligned}
-3\Upsilon_1(X) - \Upsilon_2(IX) - \Upsilon_3(JX) - \Upsilon_4(KX) &= 0,\\
-3\Upsilon_2(X) + \Upsilon_1(IX) + \Upsilon_4(JX) - \Upsilon_3(KX) &= 0,\\
-3\Upsilon_3(X) - \Upsilon_4(IX) + \Upsilon_1(JX) + \Upsilon_2(KX) &= 0,\\
-3\Upsilon_4(X) + \Upsilon_3(IX) - \Upsilon_2(JX) + \Upsilon_1(KX) &= 0.
\end{aligned}
$$

Hence

$$
\begin{aligned}
-3\Upsilon_1(IX) + \Upsilon_2(X) - \Upsilon_3(KX) + \Upsilon_4(JX) &= 0,\\
3\Upsilon_2(X) - \Upsilon_1(IX) - \Upsilon_4(JX) + \Upsilon_3(KX) &= 0,
\end{aligned}
$$

and finally

$$\Upsilon_2(X) = -\Upsilon_1(IX).$$

One can compute that $\Upsilon_3(X) = -\Upsilon_1(JX)$, $\Upsilon_4(X) = -\Upsilon_1(KX)$ in the same way. ∎

THEOREM 2.3. *Let $(M, A = \langle E, P\rangle)$ be an almost product structure and ∇ be a linear connection preserving P, i.e. $\nabla P = 0$. Then the class of connections $[\nabla]_A$ equals the class $[\nabla]_A$.*

$$[\nabla] = \nabla + \Upsilon \odot E + (\Upsilon \circ P) \odot P, \tag{4}$$

where Υ is any one form on M.

Proof. First, let us consider the difference tensor $P(X,Y) = \bar{\nabla}_X(Y) - \nabla_X(Y)$ and one can see that its value is symmetric in each tangent space because both connections share the same torsion. Since both ∇ and $\bar{\nabla}$ preserve P, the difference tensor P is complex linear in the second variable. By symmetry it is thus complex bilinear and we can compute:

$$
\begin{aligned}
\tilde{\nabla}_X Y - \nabla_X Y &= \Upsilon_1(X)Y + \Upsilon_1(Y)X + \Upsilon_2(X)PY + \Upsilon_2(Y)PX,\\
\tilde{\nabla}_X Y - \nabla_X Y &= \tilde{\nabla}_{PX}PY - \nabla_{PX}PY = \Upsilon_1(PX)PY + \Upsilon_1(PY)PX\\
&\quad + \Upsilon_2(PX)Y + \Upsilon_2(PY)X
\end{aligned}
$$

and therefore

$$
\begin{aligned}
&\Upsilon_1(X)Y + \Upsilon_1(Y)X + \Upsilon_2(X)PY + \Upsilon_2(Y)PX\\
&= \Upsilon_1(PX)PY + \Upsilon_1(PY)PX + \Upsilon_2(PX)Y + \Upsilon_2(PY)X
\end{aligned}
$$

Thus $(\Upsilon_2(X) - \Upsilon_1(PX)) = 0$ because we can suppose that X, Y, PX, PY are linearly independent without loss of generality. ∎

THEOREM 2.4. *Let $(M, A = \langle E, I, J, K\rangle)$ be an almost para-quaternionic structure and ∇ be a linear connection preserving $I, J,$ and K then the class of connections $[\nabla]_A$ equals the class*

$$[\nabla] = \nabla + \Upsilon \odot E + (\Upsilon \circ I) \odot I + (\Upsilon \circ J) \odot J + (\Upsilon \circ K) \odot K, \tag{5}$$

where Υ is any one form on M.

Proof. First, let us consider the difference tensor $P(X,Y) = \bar{\nabla}_X(Y) - \nabla_X(Y)$ and one can see that its value is symmetric in each tangent space because both connections share the same torsion. Since both ∇ and $\bar{\nabla}$ preserve $I.J.K$, the difference tensor P is complex linear in the second variable. By symmetry it is thus quaternionic bilinear and we can compute:

$$\begin{aligned}
P(X,Y) &= \Upsilon_1(X)Y + \Upsilon_1(Y)X + \Upsilon_2(X)IY + \Upsilon_2(Y)IX \\
&\quad + \Upsilon_3(X)JY + \Upsilon_3(Y)JX + \Upsilon_4(X)KY + \Upsilon_4(Y)KX, \\
P(IX,IY) &= \Upsilon_1(IX)IY + \Upsilon_1(IY)IX + \Upsilon_2(IX)Y + \Upsilon_2(IY)X \\
&\quad + \Upsilon_3(IX)KY + \Upsilon_3(IY)KX + \Upsilon_4(IX)JY + \Upsilon_4(IY)JX, \\
P(JX,JY) &= \Upsilon_1(JX)JY + \Upsilon_1(JY)JX + \Upsilon_2(JX)KY - \Upsilon_2(JY)KX \\
&\quad + \Upsilon_3(JX)Y + \Upsilon_3(JY)X + \Upsilon_4(JX)IY + \Upsilon_4(JY)IX, \\
P(KX,KY) &= \Upsilon_1(KX)KY + \Upsilon_1(KY)KX + \Upsilon_2(KX)JY + \Upsilon_2(KY)JX \\
&\quad + \Upsilon_3(KX)IY + \Upsilon_3(KY)IX + \Upsilon_4(KX)Y + \Upsilon_4(KY)X.
\end{aligned}$$

The sum of three times the first row and the next three rows together implies a system of linear equations because $X, Y, IX, IY, JX, JY, KX, KY$ are linearly independent without loss of generality:

$$\begin{aligned}
3\Upsilon_1(X) + \Upsilon_2(IX) + \Upsilon_3(JX) + \Upsilon_4(KX) &= 0, \\
3\Upsilon_2(X) + \Upsilon_1(IX) + \Upsilon_4(JX) + \Upsilon_3(KX) &= 0, \\
3\Upsilon_3(X) + \Upsilon_4(IX) + \Upsilon_1(JX) + \Upsilon_2(KX) &= 0, \\
3\Upsilon_4(X) + \Upsilon_3(IX) + \Upsilon_2(JX) + \Upsilon_1(KX) &= 0.
\end{aligned}$$

Hence

$$3\Upsilon_1(IX) + \Upsilon_2(X) + \Upsilon_3(KX) + \Upsilon_4(JX) = 0$$

and

$$\Upsilon_2(X) = \Upsilon_1(IX) + \Upsilon_3(KX) + \Upsilon_4(JX).$$

Finally one can compute that $\Upsilon_3(X) = \Upsilon_1(JX)$, $\Upsilon_4(X) = \Upsilon_1(KX)$ in the same way. ∎

Definition 2.5. Let M be a smooth manifold of dimension m. A *projective A-structure on M* is a triple $(M, A, [\nabla]_A)$, where the couple (M, A) is an A-structure and $[\nabla]_A$ is a class of A-connections

$$[\nabla]_A = \nabla + \sum_{i=1}^{k=\dim A} \Upsilon \odot F_i,$$

for any one form Υ.

For almost complex, product, quaternionic and para-quaternionic structures the class of A-connections $[\nabla]_A$ looks as follows:

$$\begin{aligned}
[\nabla]_A &= \nabla + \Upsilon \odot E - (\Upsilon \circ J) \odot J, \\
[\nabla]_A &= \nabla + \Upsilon \odot E + (\Upsilon \circ P) \odot P, \\
[\nabla]_A &= \nabla + \Upsilon \odot E - (\Upsilon \circ I) \odot I - (\Upsilon \circ J) \odot J - (\Upsilon \circ K) \odot K, \\
[\nabla]_A &= \nabla + \Upsilon \odot E + (\Upsilon \circ I) \odot I + (\Upsilon \circ J) \odot J + (\Upsilon \circ K) \odot K.
\end{aligned}$$

THEOREM 2.6. *Let $(M, A, [\nabla]_A)$ be a smooth projective A-structure. A curve $c : \mathbb{R} \to M$ is A-planar with respect to at least one A-connection $\bar{\nabla}$ on M if and only if $c : \mathbb{R} \to M$ is a geodesic of some A-connection. Moreover this happens if and only if c is A-planar with respect to all A-connections.*

Proof. Consider a curve $c : \mathbb{R} \to M$ such that $\nabla_{\dot{c}}\dot{c} \in A(\dot{c})$, where $\nabla \in [\nabla]_A$. Then

$$\bar{\nabla}_{\dot{c}}\dot{c} = \nabla_{\dot{c}}\dot{c} + \sum_{i=1}^{\dim A} 2\Upsilon_i^1(\dot{c})F_i(\dot{c}),$$

$$\bar{\nabla}_{\dot{c}}\dot{c} = \sum_{i=1}^{\dim A} \xi_i F_i(\dot{c}) + \sum_{i=1}^{\dim A} 2\Upsilon_i^1(\dot{c})F_i(\dot{c}),$$

$$\bar{\nabla}_{\dot{c}}\dot{c} = \sum_{i=1}^{\dim A} (2\Upsilon_i^1(\dot{c}) + \xi_i)F_i(\dot{c}).$$

The set of equations $2\Upsilon_i^1(\dot{c}) + \xi_i = 0$ has solutions, i.e. there exists $\Upsilon_i^1 \in \Omega^1(M)$ such that c is a geodesic curve for the A-connection $\bar{\nabla}$. The rest of the proof is easy. ∎

THEOREM 2.7. *Let M be a smooth manifold of dimension $2n$, where $n > 1$ and let $(M, A, [\nabla])$ be a projective A-structure on M of dimension n with generic rank n, where A is an algebra. Let $\bar{\nabla}$ be a linear connection on M such that ∇ and $\bar{\nabla}$ preserve any $F \in A$ and they have the same torsion. If any geodesic of ∇ is A-planar for $\bar{\nabla}$, then $\bar{\nabla}$ lies in the projective equivalence class of ∇.*

Proof. First, let us consider the difference tensor $P(X, Y) = \bar{\nabla}_X(Y) - \nabla_X(Y)$ and one can see that its value is symmetric in each tangent space because both connections share the same torsion. Since both ∇ and $\bar{\nabla}$ preserve A, the difference tensor P is A-linear in the second variable. By symmetry it is thus A-bilinear.

Consider a curve $c : \mathbb{R} \to M$ such that $X = \dot{c} \in \mathcal{V}$ and such that c is geodesics with respect to ∇ and A-planar with respect to $\bar{\nabla}$. In this case the deformation $P(X, X) := \bar{\nabla}_X(X) - \nabla_X(X)$ equals $\sum_{i=1}^{k=\dim A} \Upsilon_i(X)F_i(X)$, and

$$\begin{aligned}
P(X, Y) &= \frac{1}{2}\Big(\sum_{i=1}^{k=\dim A} \Upsilon_i(X+Y)F_i(X+Y) - \sum_{i=1}^{k=\dim A} \Upsilon_i(X)F_i(X) - \sum_{i=1}^{k=\dim A} \Upsilon_i(Y)F_i(Y)\Big) \\
&= \frac{1}{2}\Big(\sum_{i=1}^{k=\dim A} \Upsilon_i(X+Y)F_i(X) \\
&+ \sum_{i=1}^{k=\dim A} \Upsilon_i(X+Y)F_i(Y) - \sum_{i=1}^{k=\dim A} \Upsilon_i(X)F_i(X) - \sum_{i=1}^{k=\dim A} \Upsilon_i(Y)F_i(Y)\Big) \\
&= \frac{1}{2}\Big(\sum_{i=1}^{k=\dim A} (\Upsilon_i(X+Y) - \Upsilon_i(X))F_i(X) \\
&+ \sum_{i=1}^{k=\dim A} (\Upsilon_i(X+Y) - \Upsilon_i(Y))F_i(Y)\Big)
\end{aligned}$$

by polarization.

It is clear by construction that $\Upsilon_i(tX) = t\Upsilon_i(X)$ for $t \in \mathbb{R}$ and that $P(sX, tY) = stP(X, Y)$ for any $s, t \in \mathbb{R}$. Assuming that X and Y are A-linearly independent we compare the coefficients of X in the expansions of $P(sX, tY) = stP(X, Y)$ as above to get

$$s\Upsilon_i(sX + tY) - s\Upsilon_i(sX) = st(\Upsilon_i(X + Y) - \Upsilon_i(X)).$$

Dividing by s and putting $t = 1$ and taking the limit $s \to 0$, we conclude that $\Upsilon_1(X+Y) = \Upsilon_1(X) + \Upsilon_1(Y)$.

We have proved that the form Υ_i is linear in X and

$$(X, Y) \to \sum_{i=1}^{k=\dim A} (\Upsilon_i(Y)F_i(X) + \Upsilon_i(X)F_i(Y))$$

is a symmetric A-bilinear map which agrees with $P(X, Y)$. If both arguments coincide, it always agrees with P by polarization and $\bar{\nabla}$ lies in the projective equivalence class $[\nabla]_A$. ■

Theorem 2.8 ([HS06]). *Let (M, A), (M', A') be smooth manifolds of dimension m equipped with A-structure and A'-structure of the same generic rank $\ell \leq 2m$ and assume that the A-structure satisfies the property*

$$\forall X \in T_xM,\ \forall F \in A, \exists c_X \mid \dot{c}_X = X,\ \nabla_{\dot{c}_X}\dot{c}_X = \beta(X)F(X), \tag{6}$$

where $\beta(X) \neq 0$. If $f : M \to M'$ is an (A, A')-planar mapping. Then f is a morphism of A-structures, i.e. $f^*A' = A$.

Theorem 2.9. *Let $(M, A, [\nabla]_A)$, $(M', A', [\nabla]_{A'})$ be smooth manifolds of dimension m equipped with projective A-structure and projective A'-structure of the same generic rank $\ell \leq 2m$, where A, A' are algebras. If $f : M \to M'$ is an (A, A')-planar mapping. Then f is a morphism of A-structures, i.e. $f^*A' = A$.*

Proof. We have to prove (6). Let us consider $F \in A$, an A-connection ∇, and a curve $c : \mathbb{R} \to M$ such that $\dot{c} = X$ and $\nabla_X X = 0$ for any $X \in T_xM$ exists. We shall find a connection $\bar{\nabla} \in [\nabla]_A$ such that $\bar{\nabla}_X X = \beta(X)F(X)$, but the connection $\bar{\nabla} = \nabla + \beta \otimes F$ belongs to $[\nabla]_A$ directly. ■

Corollary 2.10. *Let $(M, A, [\nabla])$, $(M', A, [\bar{\nabla}]_A)$ be smooth manifolds of dimension $2m$ equipped with projective A-structures of the generic rank m. Let $f : M \to M'$ be a diffeomorphism between two projective A-structures. Then f is a morphism of A-structures if and only if it preserves the class of unparameterized geodesics of all A-connections on M and M'.*

The corollary above holds for an almost product structure on a manifold M, $\dim M \geq 4$, an almost complex structure on a manifold M, $\dim M \geq 4$, an almost quaternionic structure on a manifold M, $\dim M \geq 8$ and an almost para-quaternionic structure on a manifold M, $\dim M \geq 8$.

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