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REMARKS ON NON-LOCAL INVARIANTS OF MARTINET'S SINGULAR SYMPLECTIC STRUCTURES

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1. Introduction. The fundamental result for symplectic topology is Gromov's non-squeezing theorem.

THEOREM 1 (Gromov's Nonsqueezing Theorem). *Let*

$$\omega_0 = \sum_{i=1}^{n} dp_i \wedge dq_i$$

be the standard symplectic structure on $\mathbb{R}^{2n}$. *If there is a symplectic embedding*

$$B^{2n}(r) \hookrightarrow Z^{2n}(R),$$

where $B^{2n}(r) = \left\{(p,q) \in \mathbb{R}^{2n} : |p|^2 + |q|^2 \leq r^2\right\}$ *is a standard ball and*

$$Z^{2n}(R) = B^2(R) \times \mathbb{R}^{2n-2} = \left\{(p,q) \in \mathbb{R}^{2n} : p_1^2 + q_1^2 \leq R^2\right\}$$

is a symplectic cylinder, then

$$r \leq R.$$

Gromov proves this theorem using J-holomorphic curves ([9]). There are other proofs of this theorem: a proof due to Viterbo which uses generating functions ([20]) and a proof due to Hofer and Zehnder which is based on the calculus of variations ([10]).

This theorem was extended to arbitrary symplectic manifold (M, ω) by Lalonde and McDuff ([12]).

THEOREM 2. *If* (M, ω) *is any symplectic manifold of dimension* $2n$, *there is a symplectic embedding of the standard ball* $B^{2n+2}(r)$ *into the cylinder* $(B^2(R) \times M, dp \wedge dq \oplus \omega)$ *only if* $r \leq R$.

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Gromov's nonsqueezing theorem is crucial for the proof of rigidity of symplectomorphisms. It is also the most basic geometric expression of this rigidity (see [14], [10]). This theorem makes possible to define a new symplectic invariant (a symplectic capacity)—Gromov width.

Another problem which visualize symplectic invariants is the symplectic camel problem. Let

$$W = \{(p,q) \in \mathbb{R}^{2n} : p_1 = 0\}$$

and

$$H_r = \{(p,q) \in \mathbb{R}^{2n} : |p|^2 + |q|^2 < r^2\}.$$

We ask if there exists a continuous family (an isotopy) of symplectic embeddings $[0,1] \ni t \mapsto \Phi_t : B^{2n}(R) \to \mathbb{R}^{2n}$, such that $\Phi_t(B^{2n}(R)) \subset \mathbb{R}^{2n} \setminus (W \setminus H_r)$ for every $t \in [0,1]$ and $\Phi_0(B^{2n}(R))$, $\Phi_1(B^{2n}(R))$ are in different components of $\mathbb{R}^{2n} \setminus W$. The question was asked by Arnold. McDuff and Traynor in [15] and Viterbo in [20] prove that such symplectic isotopy exists if and only if $R < r$. McDuff and Traynor use Gromov's methods developed to prove the nonsqueezing theorem and Viterbo's proof uses generating functions.

In this paper we consider similar problems for Martinet's singular symplectic form $\omega = x\,dx \wedge dy + \sum_{i=1}^{n-1} dp_i \wedge dq_i$ on $\mathbb{R}^{2n}$. This closed 2-form is also called a folded symplectic form (see [2]). It is considered in [13], [17], [11], [4], [5], [3] and [2].

Now we recall some basic facts on the local classification of singularities of differential closed 2-forms on $\mathbb{R}^{2n}$ for $n \geq 2$ ([13]).

Let α be a germ of a closed 2-form on $\mathbb{R}^{2n}$ at 0. We define

$$\Sigma_k(\alpha) = \{z \in \mathbb{R}^{2n} : \operatorname{rank} \alpha|_z = 2n - k\}, \ k \text{ is even}.$$

Let $\alpha^n = f\Omega$, where Ω is the volume form on $\mathbb{R}^{2n}$.

(i) If $f(0) \neq 0$ then α is a germ of a symplectic form (denoted by Σ_0) and by Darboux theorem we obtain

$$\alpha = \sum_{i=1}^{n} dx_i \wedge dy_i \tag{1}$$

in local coordinates around $0 \in \mathbb{R}^{2n}$.

(ii) Next we assume $f(0) = 0$ while $(df)(0) \neq 0$. We have $\Sigma_2(\alpha) = \{f = 0\}$. If $\left(\alpha|_{\Sigma_2(\alpha)}\right)^{n-1}(0) \neq 0$ then in local coordinates around $0 \in \mathbb{R}^{2n}$

$$\alpha = x_1\,dx_1 \wedge dy_1 + \sum_{i=2}^{n} dx_i \wedge dy_i \tag{2}$$

and this type of singularity is denoted by $\Sigma_{2,0}$ (and called Martinet's singular symplectic form).

Both types of forms Σ_0, $\Sigma_{2,0}$ are locally stable (see [13]).

Let $\omega = x\,dx \wedge dy + \sum_{i=1}^{n-1} dp_i \wedge dq_i$ denote Martinet's singular symplectic structure on $\mathbb{R}^{2n}$. Then

$$\Sigma = \Sigma_2(\omega) = \{z \in \mathbb{R}^{2n} : \omega^n|_z = 0\} = \{z \in \mathbb{R}^{2n} : x = 0\}$$

is a hypersurface of degeneration of ω.

2. Nonsqueezing for Martinet's singular symplectic structure on $\mathbb{R}^{2n}$. Let

$$B^{2n}(r) = \left\{ z = (x, y, p, q) \in \mathbb{R}^{2n} : (x, y) \in \mathbb{R}^2,\ |z| \le r \right\}$$

be the ball of radius r in $\mathbb{R}^{2n}$ and

$$Z^{2n}(R) = \left\{ z = (x, y, p, q) \in \mathbb{R}^{2n} : p_1^2 + q_1^2 \le R^2 \right\}$$

be the cylinder in $\mathbb{R}^{2n}$. Then it is easy to prove that

PROPOSITION 1. *If there is an embedding $\Phi : B^{2n}(r) \hookrightarrow Z^{2n}(R)$ preserving ω then $r \le R$.*

Proof. It is obvious that Φ must preserve the hypersurface

$$\Sigma = \left\{ z \in \mathbb{R}^{2n} : x = 0 \right\},$$

because Φ preserves ω. Let us consider $\phi = \Phi|_\Sigma$. Let $B^{2n-1}(r) = B^{2n}(r) \cap \Sigma$, $Z^{2n-1}(R) = Z^{2n}(R) \cap \Sigma$ and $\omega_1 = \omega|_\Sigma = \sum_{i=1}^{n-1} dp_i \wedge dq_i$. The kernel of ω_1 is spanned by $\partial/\partial y$. It is tangent to the boundary of $Z^{2n-1}(R)$ and it is tangent to the boundary of $B^{2n-1}(r)$ on the set

$$S^{2n-3}(r) = \left\{ (y, p, q) \in \Sigma : y = 0,\ |p|^2 + |q|^2 = r^2 \right\}.$$

Let us consider $B^{2n-2}(r) = B^{2n-1}(r) \cap \{(y, p, q) \in \Sigma : y = 0\}$. Its boundary is $S^{2n-3}(r)$ and the kernel of ω_1 is transversal to it. Let us consider $\psi = \pi_y \circ \phi|_{B^{2n-2}(r)}$ where π_y is the projection of $Z^{2n-1}(R)$ onto $Z^{2n-2}(R) = Z^{2n-1}(R) \cap \{(y, p, q) \in \Sigma : y = 0\}$ along y-axis. It is an embedding, because $\partial/\partial y$ is transversal to $\phi(B^{2n-2}(r))$. ψ preserves the symplectic form $\sum_{i=1}^{n-1} dp_i \wedge dq_i$ on $\mathbb{R}^{2n-2}$ and maps $B^{2n-2}(r)$—the standard ball of radius r into $Z^{2n-2}(R)$—the standard symplectic cylinder of radius R. Therefore $r \le R$ by Gromov's nonsqueezing theorem. ∎

Proposition 1 is true for every cylinder Z, such that the kernel of $\omega|_\Sigma$ is tangent to $\partial Z \cap \Sigma$. But this is not a typical position. The kernel of $\omega|_\Sigma$ is transversal to $\partial Z \cap \Sigma$ for a typical position of a cylinder Z. It is an open problem if the nonsqueezing theorem is true for a typical position of a cylinder Z. The method of restriction to Σ does not work in this case. This is a consequence of the following

PROPOSITION 2. *If $\omega_1 = \sum_{i=1}^{n-1} dp_i \wedge dq_i$ is a closed 2-form on $\mathbb{R}^{2n-1}$ then for any $R, r > 0$ there exists an embedding preserving ω_1 of*

$$B^{2n-1}(r) = \left\{ z = (y, p, q) \in \mathbb{R}^{2n-1} : |z| \le r \right\}$$

into

$$Z^{2n-1}(R) = \left\{ z = (y, p, q) \in \mathbb{R}^{2n-1} : y^2 + q_1^2 \le R^2 \right\}.$$

Proof. It is easy to check that

$$\Phi(y, p, q) = \left(\frac{Ry}{r}, \frac{rp_1}{R}, p_2, \ldots, p_{n-1}, \frac{Rq_1}{r}, q_2, \ldots, q_{n-1} \right)$$

satisfies these conditions. ∎

3. The camel problem for Martinet's singular symplectic structure on $\mathbb{R}^{2n}$. Let W be a hyperplane in $\mathbb{R}^{2n}$, transversal to Σ, and $0 \in W$. Let $H_r = \{z \in \mathbb{R}^{2n} : |z| < r\}$ (W is a "wall" and H_r is a "hole" of a radius r in the wall). We ask if there exists a continuous family (an isotopy) of embeddings $[0,1] \ni t \mapsto \Phi_t : B^{2n}(R) \to \mathbb{R}^{2n}$, such that $\Phi_t(B^{2n}(R)) \subset \mathbb{R}^{2n} \setminus (W \setminus H_r)$, $\Phi_t^*\omega = \omega$ for every $t \in [0,1]$ and $\Phi_0(B^{2n}(R))$ and $\Phi_1(B^{2n}(R))$ are in different components of $\mathbb{R}^{2n} \setminus W$. This is an analog of the camel problem for the Martinet singular symplectic structure.

Firstly we find a normal form for the hyperplane W.

In a typical position W is transversal to the kernel of $\omega|_\Sigma$ on $W \cap \Sigma$. The kernel of $\omega|_\Sigma$ is spanned by $\partial/\partial y$. If

$$W = \Big\{z \in \mathbb{R}^{2n} : Ax + By + \sum_{i=1}^{n-1} C_i p_i + D_i q_i = 0\Big\}$$

then $B \neq 0$. Therefore by a diffeomorphism of the form $\Psi(z) = (x, y + \frac{A}{B}x, p, q)$, which preserves ω, we reduce W to $\Big\{z \in \mathbb{R}^{2n} : y + \sum_{i=1}^{n-1} E_i p_i + F_i q_i = 0\Big\}$. If $E_k^2 + F_k^2 \neq 0$ we may assume that $E_k \neq 0$ (otherwise we may use a diffeomorphism

$$\Phi(z) = (x, y, p_1, \ldots, p_{k-1}, q_k, p_{k+1}, \ldots, p_n, q_1, \ldots, q_{k-1}, -p_k, q_{k+1}, \ldots, q_n)\,).$$

Now we transform W to $\Big\{z \in \mathbb{R}^{2n} : y + p_k + \sum_{i=1, i\neq k}^{n-1} E_i p_i + F_i q_i = 0\Big\}$ by a diffeomorphism

$$\Theta(z) = \Big(x, y, p_1, \ldots, p_{k-1}, E_k p_k + F_k q_k, p_{k+1}, \ldots, p_n, q_1, \ldots, q_{k-1}, \frac{q_k}{E_k}, q_{k+1}, \ldots, q_n\Big),$$

which preserves ω. Finally by a diffeomorphism

$$\Gamma(z) = \Big(x, y + p_k, p_1, \ldots, p_{k-1}, p_k, p_{k+1}, \ldots, p_n, q_1, \ldots, q_{k-1}, q_k + \frac{x_1^2}{2}, q_{k+1}, \ldots, q_n\Big),$$

which preserves ω, we reduce W to $\Big\{z \in \mathbb{R}^{2n} : y + \sum_{i=1, i\neq k}^{n-1} E_i p_i + F_i q_i = 0\Big\}$. If we repeat these transformations for each k such that $E_k^2 + F_k^2 \neq 0$ then we reduce W to $\{z \in \mathbb{R}^{2n} : y = 0\}$.

If W is not transversal to the kernel of $\omega|_\Sigma$ and is transversal to Σ then it has the form $W = \Big\{z \in \mathbb{R}^{2n} : Ax + \sum_{i=1}^{n-1} C_i p_i + D_i q_i = 0\Big\}$ where $\sum_{i=1}^{n-1} C_i^2 + D_i^2 \neq 0$. We may assume that $C_k \neq 0$ for some k (otherwise $D_k \neq 0$ for some k and we may use a diffeomorphism

$$\Phi(z) = (x, y, p_1, \ldots, p_{k-1}, q_k, p_{k+1}, \ldots, p_n, q_1, \ldots, q_{k-1}, -p_k, q_{k+1}, \ldots, q_n)\,).$$

Now we transform W to $\Big\{z \in \mathbb{R}^{2n} : Ax + p_k + \sum_{i=1, i\neq k}^{n-1} C_i p_i + D_i q_i = 0\Big\}$ by a diffeomorphism

$$\Theta(z) = \Big(x, y, p_1, \ldots, p_{k-1}, C_k p_k + D_k q_k, p_{k+1}, \ldots, p_n, q_1, \ldots, q_{k-1}, \frac{q_k}{C_k}, q_{k+1}, \ldots, q_n\Big),$$

which preserves ω. If $\sum_{i=1, i\neq k}^{n-1} C_i^2 + D_i^2 \neq 0$ then in the same way we may reduce W to $\Big\{z \in \mathbb{R}^{2n} : Ax + p_k + p_l + \sum_{i=1, i\neq k,l}^{n-1} C_i p_i + D_i q_i = 0\Big\}$ for some $l \neq k$. By a diffeomorphism

$$\Delta(z) = (x, y, p_1, \ldots, p_{k-1}, p_k + p_l, p_{k+1}, \ldots, p_n, q_1, \ldots, q_{l-1}, q_l - q_k, q_{l+1}, \ldots, q_n)$$

we reduce W to $\left\{z \in \mathbb{R}^{2n} : Ax + p_k + \sum_{i=1, i\neq k,l}^{n-1} C_i p_i + D_i q_i = 0\right\}$. Repeating these transformations for each l such that $C_l^2 + D_l^2 \neq 0$ we reduce W to $\left\{z \in \mathbb{R}^{2n} : Ax + p_k = 0\right\}$. If $A \neq 0$ then we may reduce W to $\left\{z \in \mathbb{R}^{2n} : x + p_1 = 0\right\}$ and if $A = 0$ then we may reduce W to $\left\{z \in \mathbb{R}^{2n} : p_1 = 0\right\}$ by diffeomorphisms which preserve ω. Thus we obtain

PROPOSITION 3. *If a hyperplane W is transversal to Σ then there exists a diffeomorphism $\Phi : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$ such that $\Phi^\star \omega = \omega$ and*

$$\Phi^{-1}(W) = \left\{z \in \mathbb{R}^{2n} : y = 0\right\}$$

(*if the kernel of $\omega|_\Sigma$ is transversal to W*) *or*

$$\Phi^{-1}(W) = \left\{z \in \mathbb{R}^{2n} : x + p_1 = 0\right\}$$

(*if the kernel of $\omega|_\Sigma$ is tangent to W and the rank at $\omega|_W$ at 0 is maximal*) *or*

$$\Phi^{-1}(W) = \left\{z \in \mathbb{R}^{2n} : p_1 = 0\right\}$$

(*if the kernel of $\omega|_\Sigma$ is tangent to W and the rank at $\omega|_W$ at 0 is not maximal*).

Now it is easy to prove

PROPOSITION 4. *If a hyperplane W is transversal to Σ and the kernel of $\omega|_\Sigma$ is tangent to W then there exists an isotopy of embeddings $[0,1] \ni t \mapsto \Phi_t : B^{2n}(R) \to \mathbb{R}^{2n}$, such that $\Phi_t(B^{2n}(R)) \subset \mathbb{R}^{2n} \setminus (W \setminus H_r)$, $\Phi_t^* \omega = \omega$ for every $t \in [0,1]$, and $\Phi_0(B^{2n}(R))$ and $\Phi_1(B^{2n}(R))$ are in different components of $\mathbb{R}^{2n} \setminus W$ if and only if $R < r$, where r is a radius of the hole H_r.*

Proof. By Proposition 3 we may assume that W is

$$\left\{z \in \mathbb{R}^{2n} : x + p_1 = 0\right\}$$

or

$$\left\{z \in \mathbb{R}^{2n} : p_1 = 0\right\}.$$

Let us assume that there exists an isotopy Φ_t which satisfies these conditions and let us consider $\phi_t = \Phi_t|_{\Sigma \cap B^{2n}(R)} : B^{2n-1}(R) \to \mathbb{R}^{2n-1}$ for $t \in [0,1]$. In both cases $W \cap \Sigma$ is $\left\{z \in \mathbb{R}^{2n} : p_1 = 0\right\}$. Now we use the same argument as in the proof of Proposition 1. Let $B^{2n-1}(R) = B^{2n}(R) \cap \Sigma$ and $\omega_1 = \omega|_\Sigma = \sum_{i=1}^{n-1} dp_i \wedge dq_i$. The kernel of ω_1 is spanned by $\partial/\partial y$. It is tangent to the boundary of $B^{2n-1}(R)$ on a set

$$S^{2n-3}(R) = \left\{(y,p,q) \in \Sigma : y = 0,\ |p|^2 + |q|^2 = R^2\right\}.$$

Let us consider the submanifold $B^{2n-2}(R) = B^{2n-1}(R) \cap \{(y,p,q) \in \Sigma : y = 0\}$. Its boundary is $S^{2n-3}(R)$ and the kernel of ω_1 is transversal to this submanifold. Let us consider $\psi_t = \pi_y \circ \phi_t|_{B^{2n-2}(R)}$ where π_y is a projection of $\mathbb{R}^{2n-1}$ onto $\mathbb{R}^{2n-2} = \{(y,p,q) \in \Sigma : y = 0\}$ along y-axis. It is an embedding, because $\partial/\partial y$ is transversal to $\phi_t(B^{2n-2}(R))$. ψ_t preserves the symplectic form $\sum_{i=1}^{n-1} dp_i \wedge dq_i$ on $\mathbb{R}^{2n-2}$. $\pi_y(W \cap \Sigma) = \left\{(p,q) \in \mathbb{R}^{2n-2} : p_1 = 0\right\}$ and $\pi_y(H_r \cap \Sigma) = \left\{(p,q) \in \mathbb{R}^{2n-2} : |p|^2 + |q|^2 < r^2\right\}$. Therefore if ψ_t exists then $R < r$ by the symplectic camel theorem. ■

If the kernel of $\omega|_\Sigma$ is transversal to W then we cannot use the same method to prove the camel theorem. But one can prove the following.

PROPOSITION 5. *If a hyperplane W is transversal to the kernel of $\omega|_\Sigma$, $R < 2$ and*

$$r < \frac{R^2}{4}$$

then there is no isotopy of embeddings $[0,1] \ni t \mapsto \Phi_t : B^{2n}(R) \to \mathbb{R}^{2n}$, such that $\Phi_t(B^{2n}(R)) \subset \mathbb{R}^{2n} \setminus (W \setminus H_r)$, $\Phi_t^*\omega = \omega$ for every $t \in [0,1]$, and $\Phi_0(B^{2n}(R))$ and $\Phi_1(B^{2n}(R))$ are in different components of $\mathbb{R}^{2n} \setminus W$, where r is a radius of the hole H_r.

Proof. By Proposition 3 we may assume that W is $\{z \in \mathbb{R}^{2n} : y = 0\}$. Let us assume that there exists an isotopy Φ_t, which satisfies these conditions. Let

$$M^+ = \{z \in \mathbb{R}^{2n} : x > 0\}, \ M^- = \{z \in \mathbb{R}^{2n} : x < 0\}.$$

It is easy to see that $\Phi_t(B^{2n}(R) \cap M^+) \subset M^+$ or $\Phi_t(B^{2n}(R) \cap M^+) \subset M^-$. We assume that $\Phi_t(B^{2n}(R) \cap M^+) \subset M^+$. Let

$$\Theta : M^+ \ni (x,y,p,q) \mapsto (\sqrt{2x}, y, p, q) \in M^+.$$

It is easy to see that $\Theta^\star\omega = \omega_0 = dx \wedge dy + \sum_{i=1}^{n-1} dp_i \wedge dq_i$,

$$P(R) = \Theta^{-1}(B^{2n}(R) \cap M^+) = \{(x,y,p,q) \in \mathbb{R}^{2n} : 2x + y^2 + |p|^2 + |q|^2 < R^2,\ x > 0\}$$

and

$$P(r) = \Theta^{-1}(H_r \cap M^+) = \{(x,y,p,q) \in \mathbb{R}^{2n} : 2x + y^2 + |p|^2 + |q|^2 < r^2,\ x > 0\}.$$

It is obvious that the ball $B^{2n}(R^2/4)$ is symplectically embedded in $P(R)$, because $R < 2$. Let Ψ denote such an embedding. On the other hand $P(r)$ is symplectically embedded in the ball $B^{2n}(r)$. Thus the mapping

$$\Theta^{-1} \circ \Phi_t \circ \Theta \circ \Psi : B^{2n}(R^2/4) \to \mathbb{R}^{2n}$$

defines an isotopy of symplectic embeddings such that $\Phi_t(B^{2n}(R^2/4)) \subset \mathbb{R}^{2n} \setminus (W \setminus H_r)$ for every $t \in [0,1]$, and $\Phi_0(B^{2n}(R^2/4))$, $\Phi_1(B^{2n}(R^2/4))$ are in different components of $\mathbb{R}^{2n} \setminus W$. By the symplectic camel theorem we get that such isotopy does not exist if $r < R^2/4$. ∎

It is an open problem if the camel theorem for Martinet's singular symplectic structures is true for $R^2/4 \le r < R$.

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