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ON ASYMPTOTIC CRITICAL VALUES AND THE RABIER THEOREM

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Abstract. Let $X \subset k^n$ be a smooth affine variety of dimension $n-r$ and let $f = (f_1, \ldots, f_m) : X \to k^m$ be a polynomial dominant mapping. It is well-known that the mapping f is a locally trivial fibration outside a small closed set $B(f)$. It can be proved (using a general Fibration Theorem of Rabier) that the set $B(f)$ is contained in the set $K(f)$ of generalized critical values of f. In this note we study the Rabier function. We give a few equivalent expressions for this function, in particular we compare this function with the Kuo function and with the (generalized) Gaffney function. As a consequence we give a direct short proof of the fact that f is a locally trivial fibration outside the set $K(f)$ (i.e., that $B(f) \subset K(f)$). This generalizes the previous results of the author for $X = k^r$ (see [2]).

1. Introduction. Let X be a smooth affine variety over $k = \mathbb{R}$ or $k = \mathbb{C}$ of dimension $n - r$ and let $f : X \to k^m$ be a polynomial dominant mapping. It is well-known that the mapping f is a locally trivial fibration outside a bifurcation set $B(f)$, which has a measure 0.

Let us recall that in general the set $B(f)$ is bigger than $K_0(f)$—the set of critical values of f. It contains also the set $B_\infty(f)$ of bifurcations points at infinity. Briefly speaking, the set $B_\infty(f)$ consists of points at which f is not a locally trivial fibration at infinity (i.e., outside a compact set). To control the set $B_\infty(f)$ one can use the set of *asymptotic critical values at infinity of f* (see [6]):

$$K_\infty(f) = \{y \in k^m : \text{there is a sequence } x_l \to \infty \text{ such that } f(x_l) \to y \\ \text{and } \|x_l\|\nu(\operatorname{res}_{T_{x_l}X}\, df(x_l)) \to 0\},$$

where we consider the induced Euclidean metric on X and ν is the function defined by Rabier (see Definition 2.1 below). If $y \notin K_\infty(f)$ we say also that y is Malgrange regular.

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If $m = 1$ and $X = k^n$, then there is a wide literature devoted to different regularity conditions and their comparison (e.g., [8], [9], [10]). It has been proved for instance that the Malgrange regularity is equivalent to another regularity called t-regularity, by Siersma and Tibăr (see [7]). The case $m > 1$ and $X = k^n$ was studied in [1], [2] and [4]. In this paper (and in [3]) we study the case when X is a smooth affine variety (or even a Stein submanifold of $\mathbb{C}^m$) and $m \leq \dim X$.

Let $K(f) = K_0(f) \cup K_\infty(f)$ be the set of generalized critical values of f. It can be proved that the set $K(f)$ is a proper algebraic subset of $\mathbb{C}^m$—or proper semi-algebraic in the real case (see [3]). Moreover, we have (e.g., by a general Fibration Theorem of Rabier [6], see also [1]) $B(f) \subset K(f)$. These two facts together allow us to construct effectively a Zariski open dense subset $U \subset k^m$ over which the mapping f is a locally trivial fibration.

In this note we study the Rabier function. As a consequence we give a direct proof of the fact that $B(f) \subset K(f)$ in the case when $X \subset k^n$ is a smooth submanifold and $f : X \to k^m$ is a smooth mapping (moreover, some of these results are used in [3] to study the properties of the set $K(f)$).

The fact that $B(f) \subset K(f)$ follows from a very general Theorem of Rabier (see [6]), but it is so important (e.g., in the study of polynomial mappings) that (as I believe) it is worth to have a simple direct proof of it in a special case of submanifolds of a Euclidean space.

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2. On the Rabier function ν. Here we give several equivalent expressions for ν.

Let $X \cong k^n$, $Y \cong k^m$ be finite-dimensional vector spaces (over k). Let us denote by $\mathcal{L}(X, Y)$ the set of linear mappings from X to Y and by $\Sigma(X, Y) \subset \mathcal{L}(X, Y)$ the set of non-surjective mappings. Let us recall the following ([6]):

Definition 2.1. Let $A \in \mathcal{L}(X, Y)$. Set

$$\nu(A) = \inf_{||\phi||=1} ||A^*(\phi)||,$$

where $A^* : \mathcal{L}(Y^*, X^*)$ is the adjoint operator and $\phi \in Y^*$.

In [4] the following characterization of ν is given: $\nu(A) = \operatorname{dist}(A, \Sigma) = \inf_{B\in\Sigma} ||A-B||$. Moreover, we have the following useful characterization ([6] and [4]):

Proposition 2.1. *Let $A \in \mathcal{L}(X, Y)$. Then*

a) $\nu(A) = \sup\{r > 0 : B(0, r) \subset A(B(0, 1))\}$, *where* $B(0, r) = \{x \in X : ||x|| \leq r\}$.

b) *if* $A \in GL(X, Y)$ *then* $\nu(A) = ||A^{-1}||^{-1}$.

Proposition 2.2. *Let $A = (A_1, \ldots, A_m) \in \mathcal{L}(X, Y)$ and let $\overline{A_i} = \operatorname{grad} A_i$. Let*

$$\kappa(A) = \min_{1\leq i\leq m} \operatorname{dist}\big(\overline{A_i}, \langle(\overline{A_j})_{j\neq i}\rangle\big),$$

be the Kuo number of A. Then $\nu(A) \leq \kappa(A) \leq \sqrt{m}\nu(A)$.

We say that $\nu(A)$ and $\kappa(A)$ are equivalent and write $\nu(A) \sim \kappa(A)$. The symbol $X \sim Y$ means that there are positive constants C_1, C_2 such that $C_1 X \leq Y \leq C_2 X$.

DEFINITION 2.2. Let $A \in \mathcal{L}(X,Y)$ and let $H \subset X$ be a linear subspace. We set

$$\nu(A,H) = \nu(\operatorname{res}_H A), \quad \kappa(A,H) = \kappa(\operatorname{res}_H A),$$

where $\operatorname{res}_H A$ denotes the restriction of A to H.

From Proposition 2.2 we get immediately:

COROLLARY 2.1. *Let $A \in \mathcal{L}(X,Y)$ and let $H \subset X$ be a linear subspace. Then*

$$\nu(A,H) \sim \kappa(A,H).$$

PROPOSITION 2.3. *Let $A = (A_1, \ldots, A_m) \in \mathcal{L}(X,Y)$ and let $H \subset X$ be a linear subspace. Assume that H is given by a system of linear equations $B_j = 0$, $j = 1, \ldots, r$. Then*

$$\kappa(A,H) = \min_{1 \leq i \leq m} \operatorname{dist}\big(\overline{A_i}, \big\langle (\overline{A_j})_{j \neq i}; (\overline{B_j})_{j=1,\ldots,r} \big\rangle\big),$$

where $\overline{A_i} = \operatorname{grad} A_i$ and $\overline{B_j} = \operatorname{grad} B_j$.

Proof. Indeed, every vector $\overline{A_i}$ can be written as $a_i + b_i$, where a_i is orthogonal to the subspace $B = \big\langle (\overline{B_j})_{j=1,\ldots,r} \big\rangle$ (which means that $a_i \in H$) and $b_i \in B$. Hence

$$\operatorname{dist}(\overline{A_i}, \big\langle (\overline{A_j})_{j \neq i}; (\overline{B_j})_{j=1,\ldots,r} \big\rangle) = \operatorname{dist}(a_i, \langle (a_j)_{j \neq i} \rangle)$$

and since $\operatorname{grad}(\operatorname{res}_H A_i) = a_i$, the proof is finished. ∎

We need also:

DEFINITION 2.3. Let $A \in \mathcal{L}(X,Y)$ (where $n \geq m + r$) and let $H \subset X$ be a linear subspace given by a system of independent linear equations $B_i = \sum b_{ij} x_j$, $i = 1, \ldots, r$. Let $\mathbf{a} = [a_{ij}]$ be the matrix of A. Let $\mathbf{c} = [c_{kl}]$ be a $((m+r) \times n)$ matrix given by the rows $A_1, \ldots, A_m; B_1, \ldots, B_r$ (we identify $A_i = \sum a_{ij} x_j$ with the vector $(a_{i1}, \ldots, a_{in})$, similarly for B_j). Let M_I, where $I = (i_1, \ldots, i_{m+r})$, denote a $((m+r) \times (m+r))$ minor of $\mathbf{c}$ given by columns indexed by I and let $|M_I|$ denote the determinant of M_I. Further, let $M_J(j)$ denote a $((m+r-1) \times (m+r-1))$ minor given by columns indexed by J and by deleting the j-th row, where $1 \leq j \leq m$. Then by the *generalized Gaffney function of A with respect to a linear subspace H*, we mean the number

$$g(A,H) = \frac{\big(\sum_I |M_I|^2\big)^{1/2}}{\big(\sum_{J,\,1 \leq j \leq m} |M_J(j)|^2\big)^{1/2}}.$$

(If this number is not defined we put $g(A,H) = 0$.)

REMARK 2.1. It is easy to see that $g(A,H)$ depends on A and H only. A particular case of this definition (for $H = X$) has been considered by Gaffney—see [1].

PROPOSITION 2.4. *Let $A \in \mathcal{L}(X,Y)$ (where $n \geq m$) and let $H \subset X$ be a linear subspace. Then $g(A,H) \sim \kappa(A,H) \sim \nu(A,H)$.*

Proof. By basic properties of the Gram determinant (see e.g., [5]) we have

$$\operatorname{dist}\big(\overline{A_i}, \big\langle(\overline{A_j})_{j\neq i}; (\overline{B_j})_{j\in\{1,\dots,r\}}\big\rangle\big) = \frac{G\big((\overline{A_j})_{j\in\{1,\dots,m\}}, (\overline{B_j})_{j\in\{1,\dots,r\}}\big)^{1/2}}{G\big((\overline{A_j})_{j\neq i}, (\overline{B_j})_{j\in\{1,\dots,r\}}\big)^{1/2}} = \frac{\big(\sum_I |M_I|^2\big)^{1/2}}{\big(\sum_J |M_J(i)|^2\big)^{1/2}}.$$

Thus $g(A,H) \le \kappa(A,H)$. On the other hand there is a number i_0 such that the sum $\big(\sum_J |M_J(i_0)|^2\big)^{1/2}$ is maximal. Since

$$\Big(\sum_{J,\,j} |M_J(j)|^2\Big)^{1/2} = \Big(\sum_r\Big(\sum_J |M_J(r)|^2\Big)\Big)^{1/2} \le \sqrt{m}\Big(\sum_J |M_J(i_0)|^2\Big)^{1/2},$$

we have

$$g(A,H) \ge C\,\frac{\big(\sum_I |M_I|^2\big)^{1/2}}{\big(\sum_J |M_J(i_0)|^2\big)^{1/2}} = C\operatorname{dist}\big(\overline{A}_{i_0}, \big\langle(\overline{A}_j)_{j\neq i_0}; (\overline{B_j})_{j\in\{1,\dots,r\}}\big\rangle\big) \ge C\kappa(A,H),$$

where $C = 1/\sqrt{m}$. ∎

Definition 2.4. Let us apply the notation from Definition 2.3. Put

$$q(A,H) = \frac{\max_I |M_I|}{\max_{I,\,J\subset I,\,j} |M_J(j)|},$$

(where we consider only numbers with $M_J(j) \neq 0$, if all numbers $M_J(j)$ are zero, we put $q(A,H) = 0$).

Proposition 2.4 can also be formulated in the following way:

Corollary 2.2. *We have $q(A,H) \sim \nu(A,H)$.*

Proof. Let A denote the number of all possible matrices of type M_I (for all I) and let B denote the number of all possible matrices of type $M_J(j)$ (for all possible I, $J\subset I$ and all $1\le j\le m$). Since the norms $||x|| = \big(\sum |x_i|^2\big)^{1/2}$ and $||x||' = \sum |x_i|$ are equivalent, we have

$$g(A,H) \sim \frac{\sum_I |M_I|}{\sum_{I,\,J\subset I,\,j} |M_J(j)|}.$$

On the other hand

$$(1/B)\,\frac{\max_I |M_I|}{\max_{I,\,J\subset I,\,j} |M_J(j)|} \le \frac{\sum_I |M_I|}{\sum_{I,\,J\subset I,\,j} |M_J(j)|} \le A\,\frac{\max_I |M_I|}{\max_{I,\,J\subset I,\,j} |M_J(j)|}$$

and consequently $g(A,H) \sim q(A,H)$. Now we finish the proof by Proposition 2.4. ∎

At the end of this section we introduce another important function (the notation is as in Definition 2.3):

Definition 2.5. We define the function

$$g'(A,H) = \max_I\Big\{\min_{J\subset I,\,1\le j\le m} \frac{|M_I|}{|M_J(j)|}\Big\},$$

(where we consider only numbers with $M_J(j) \neq 0$, if all numbers $M_J(j)$ are zero, we put $g'(A,H) = 0$).

PROPOSITION 2.5. *We have $g'(A,H) \sim g(A,H)$.*

Proof. First we prove that there is a constant $C > 0$ such that $g'(A,H) \le Cg(A,H)$. Let us fix an index $I = (i_1, \dots, i_{m+r})$ such that $|M_I| \neq 0$ and consider the numbers $|M_I|/|M_J(s)|$, where $J \subset I$ and $1 \le s \le m$. For simplicity we can assume that $I = (1, \dots, m+r)$. Let the subspace H be given by a system of independent linear equations $B_i = \sum b_{ij}x_j$, $i = 1, \dots, r$, and let $\mathbf{a} = [a_{ij}]$ be the matrix of A.

Consider the system of linear equations:

$$\begin{aligned}
\sum_{j=1}^{n} a_{1j}x_j &= y_1, \\
\dots &\quad \dots \\
\sum_{j=1}^{n} a_{mj}x_j &= y_m, \\
\sum_{j=1}^{n} b_{1j}x_j &= 0, \\
\dots &\quad \dots \\
\sum_{j=1}^{n} b_{rj}x_j &= 0, \\
x_{m+r+1} &= 0, \\
\dots &\quad \dots \\
x_n &= 0.
\end{aligned}$$

We can solve this system using the Cramer rules. Let $M_{ki} := M_J(i)$ for $J = I \setminus \{k\}$. We have

$$\begin{aligned}
x_1 &= \sum_{k=1}^{m} (-1)^{1+k} y_k M_{1k}/M_I, \\
\dots &\quad \dots\dots \\
x_{m+r} &= \sum_{k=1}^{m} (-1)^{m+r+k} y_k M_{(m+r)k}/M_I, \\
x_{m+r+1} &= 0, \\
\dots &\quad \dots \\
x_n &= 0.
\end{aligned}$$

In particular we have $||x|| \le \big(\max |M_J(i)|/|M_I|\big)||y||$. Consequently we see that the image of a unit ball in the subspace $H' = \{x \in H : x_{m+r+1} = 0, \dots, x_n = 0\}$ by the mapping A contains a ball of radius $\min_{J \subset I,\, 1 \le j \le m} |M_I|/|M_J(j)|$. Now by Proposition 2.1a), we see that $\min_{J \subset I,\, 1 \le j \le m} |M_I|/|M_J(j)| \le \nu(A, H') \le \nu(A,H)$. Finally we get

$$\nu(A,H) \ge \max_I \Big\{ \min_{J \subset I,\, 1 \le j \le m} \frac{|M_I|}{|M_J(j)|} \Big\} = g'(A,H).$$

In particular there is a constant C such that $Cg(A,H) \ge g'(A,H)$.

On the other hand, there exists I_0 such that the minor M_{I_0} has a maximal norm.

Since

$$g(A,H)=\frac{\left(\sum_I |M_I|^2\right)^{1/2}}{\left(\sum_{J,j} |M_J(j)|^2\right)^{1/2}} \le \binom{n}{m+r}^{1/2} \frac{|M_{I_0}|}{\left(\sum_{J,\,j} |M_J(j)|^2\right)^{1/2}}$$
$$\le \binom{n}{m+r}^{1/2} \min_{J\subset I_0,\,1\le j\le m} \frac{|M_{I_0}|}{|M_J(j)|} \le \binom{n}{m+r}^{1/2} g'(A,H),$$

we deduce that there is a constant $C'>0$ such that $g(A,H)\le C'g'(A,H)$. ■

COROLLARY 2.3. *We have $g'(A,H)\sim \nu(A,H)$.*

3. Main result. In this section we give a short direct proof of the fact $B(f)\subset K(f)$ for a smooth mapping $f : X \to k^m$, where X is a smooth submanifold of k^m. Let us recall the following basic definition:

DEFINITION 3.1. Let $k=\mathbb{C}$ or $k=\mathbb{R}$ and let X be a smooth submanifold of k^n. Let $f : X\to k^m$ be a k-smooth mapping. Then we define the *set of generalized critical values* $K(f)=K_0(f)\cup K_\infty(f)$, where $K_0(f)$ is the set of critical values of f and

$$K_\infty(f)=\big\{y\in k^m : \text{there is a sequence } x_l\to\infty \text{ such that } f(x_l)\to y \\ \text{and } \|x_l\|\nu(df(x_l),T_{x_l}X)\to 0\big\}$$

is the set of critical values at infinity.

REMARK 3.1. Note that by virtue of results of Section 2, in place of the function ν above we can put also κ, g, q or g'.

We have the following simple observation (see [2], [6]):

PROPOSITION 3.1. *Let $k=\mathbb{C}$ or $k=\mathbb{R}$ and let X be a smooth affine variety over k. Let $f : X\to k^m$ be a k-smooth mapping. Then the set $K(f)=K_0(f)\cup K_\infty(f)$ is closed.*

We need also the following lemma (see [2]):

LEMMA 3.1. *Let $U\subset k^n$ be an open set and $V : U\to k^n$ be a smooth mapping. Let $y\in U$ and let*

$$x'(t)=V(x),\ \text{with } x(0)=y,$$

be a differential equation. Let $x(y,t)$, $t\in[0,t_0)$, be a solution of this equation. Assume that for $||x(y,t)||$ large enough, we have $||V(x(y,t))||<M||x(y,t)||$. Then this trajectory is bounded. In particular this trajectory either is defined for every $t>0$ or intersects the boundary ∂U of U.

Now we give a short direct proof of the fact that $B(f)\subset K(f)$, which is a particular version of a very general result of Rabier [6] (see also [1]).

THEOREM 3.1. *Let $k=\mathbb{C}$ or $k=\mathbb{R}$ and let $X\subset k^n$ be a smooth submanifold (i.e., X is smooth for $k=\mathbb{R}$ or Stein for $k=\mathbb{C}$). Let $f : X\to k^m$ be a k-smooth mapping (i.e., f is smooth for $k=\mathbb{R}$ or holomorphic for $k=\mathbb{C}$). Then*

$$B(f)\subset K(f)=K_0(f)\cup K_\infty(f),$$

i.e., the mapping f is a locally trivial fibration outside the set $K(f)$.

Proof. It is well-known that we can assume that f can be extended to a k-smooth mapping $\overline{f}$ on the whole k^n (in real case it is an easy exercise, in complex it follows from the theory of Stein manifolds).

First assume that X is a global complete intersection, i.e. $X = \{b_1 = 0, \ldots, b_r = 0\}$ and $\operatorname{rank}\{d_x b_1, \ldots, d_x b_r\} = r$ for every $x \in X$.

Let $a \notin K(f)$. Without loss of generality we can assume that $a = 0$. We have $a \notin K_0(f)$ and $a \notin K_\infty(f)$. This implies that there are $R > 0$, $\epsilon > 0$, $\eta > 0$, such that for every $x \in X$ with $||x|| \geq R$ and $||f(x)|| < \eta$, we have

$$\max_I \Big\{ \min_{J \subset I,\, 1 \leq j \leq m} ||x|| \, \frac{|M_I|}{|M_J(j)|} \Big\} > \epsilon. \tag{1}$$

Moreover, there is $\omega > 0$ such that for every $x \in X$ with $||x|| \leq R$ and $||f(x)|| < \eta$, we have $\max_I |M_I(x)| \geq \omega$.

Let $U = \{y \in k^m : ||y|| < \eta\}$ and let $\Gamma = f^{-1}(0)$. We show that $f^{-1}(U) \cong \Gamma \times U$ and f is a projection $\Gamma \times U \ni (\gamma, u) \mapsto u \in U$. Indeed, let us define a set

$$U_I = \Big\{ x \in \overline{f}^{-1}(U) : \text{if } ||x|| \geq R \text{ then } \min_{J \subset I,\, 1 \leq j \leq m} ||x|| \, \frac{|M_I|}{|M_J(j)|} \geq \epsilon, \\ \text{if } ||x|| \leq R \text{ then } |M_I(x)| \geq \omega \Big\}.$$

Further, let

$$V_I = \Big\{ x \in \overline{f}^{-1}(U) : \text{if } ||x|| \geq R \text{ then } \min_{J \subset I,\, 1 \leq j \leq m} ||x|| \, \frac{|M_I|}{|M_J(j)|} \leq \epsilon/2, \\ \text{if } ||x|| \leq R \text{ then } |M_I(x)| \leq \omega/2 \Big\}.$$

The sets V_I and U_I are disjoint. Consequently there is a C^∞ function $\delta_I : k^n \to [0,1]$, which is equal to 1 on U_I and to 0 on V_I. It is easy to see that the sets $H_I = \{x : \delta_I(x) > 0\}$ cover the set $f^{-1}(U)$. Now take $\delta := \sum_I \delta_I$ and let $\Delta_I = \delta_I/\delta$.

Take $y = (y_1, \ldots, y_n) \in U$. Take the index $I = (1, \ldots, m + r)$ and consider a (formal) system of differential equations:

$$\begin{aligned}
\sum_{j=1}^n \frac{\partial \overline{f}_1}{\partial x_j}(x(t)) x_j(t)' &= y_1, \\
\ldots\ldots \quad &\ldots \\
\sum_{j=1}^n \frac{\partial \overline{f}_m}{\partial x_j}(x(t)) x_j(t)' &= y_m, \\
\sum_{j=1}^n \frac{\partial b_1}{\partial x_j}(x(t)) x_j(t)' &= 0, \\
\ldots\ldots \quad &\ldots \\
\sum_{j=1}^n \frac{\partial b_r}{\partial x_j}(x(t)) x_j(t)' &= 0, \\
x_{m+r+1}(t)' &= 0, \\
\ldots \quad &\ldots \\
x_n(t)' &= 0.
\end{aligned}$$

We can solve this system using the Cramer rules (at least in U_I). Let $M_{ki} := M_J(i)$ for $J = I \setminus \{k\}$. We have

$$\begin{aligned} x_1(t)' &= \sum_{k=1}^{m} (-1)^{1+k} y_k M_{1k}/M_I, \\ \cdots &\quad \cdots\cdots \\ x_{m+r}(t)' &= \sum_{k=1}^{m} (-1)^{m+r+k} y_k M_{(m+r)k}/M_I, \\ x_{m+r+1}(t)' &= 0, \\ \cdots &\quad \cdots \\ x_n(t)' &= 0. \end{aligned}$$

We can write this system shortly as

$$x(t)' = V_I(y, x(t)).$$

By the Cramer rules, we have $df(V_I(y,x)) = y$ and $db(V_I(y,x)) = 0$. In an analogous way we can define V_I for an arbitrary index $I = (i_1, \ldots, i_m)$.

Now consider a vector field $V(y,x) = \sum_I \Delta_I V_I(y,x)$ in a domain $\overline{f}^{-1}(U)$. By the construction, we have $||V(x)|| \leq 2m\eta/\epsilon ||x||$ for $||x|| \geq R$ and $x \in X$. Let us consider the differential equation

$$x(t)' = V(y, x(t)), \quad x(0) = \gamma, \tag{2}$$

where $\gamma \in \Gamma$. Let us note that

$$\begin{aligned} df(V(y,x)) = df\Big(\sum_I \Delta_I V_I(y,x)\Big) &= \sum_I df(\Delta_I V_I(y,x)) \\ &= \sum_I \Delta_I df(V_I(y,x)) = \Big(\sum_I \Delta_I\Big) y = y. \end{aligned}$$

Similarly $db(V(x,y)) = 0$. Consequently, if $x(t,y,\gamma)$ is a solution of system (2), then the trajectory is contained in X and $yt = \overline{f}(x(t), y, \gamma) = f(x(t), y, \gamma)$. Since $y \in U$, we see that the trajectory $x(t,y,\gamma)$, $t \in [0, t_0)$ does not cross the border $\partial f^{-1}(U)$ for every $0 \leq t_0 \leq 1 + \delta$, for some $\delta > 0$. Consequently by Lemma 3.1 the trajectory $x(t,y,\gamma)$ is defined on the whole $[0,1]$ and is contained in X. Since $f\big(x(t,y,\gamma)\big) = yt$, the phase flow $x(t,y,\gamma)$, $t \in [0,1]$, transforms $f^{-1}(0) = \Gamma$ into $f^{-1}(y)$ (in fact, by the symmetry, it transforms Γ onto $f^{-1}(y)$). Let

$$\Phi : \Gamma \times U \ni (\gamma, y) \mapsto x(1, y, \gamma) \in f^{-1}(U).$$

It is easy to see that Φ is a diffeomorphism. Thus $0 \notin B(f)$.

In the general case we can choose a locally finite cover $\{U_i\}$ of k^n such that in each U_i the manifold $X \cap X_i$ is a complete intersection. Now we can construct vector fields V_i on U_i (construction is as above) and then glue them to one field V by a partition of unity subordinate to the cover $\{U_i\}$. The rest of the proof is the same as above. ■

At the end of this note we give two simple examples.

Example 3.1. Let us consider a Stein curve $\Gamma = \{(x,y) \in \mathbb{C}^2 : \exp(xy) = 2\}$. Let us consider the projection $f : \Gamma \ni (x,y) \mapsto y \in \mathbb{C}$. Using the generalized Gaffney function we see that

$$K_0(f) = f\big(\{(x,y) \in \Gamma : y\exp(xy) = 0\}\big) = \emptyset$$

and

$$K_\infty(f) = \big\{\lim f(x_n, y_n) = y_n;\ \text{where } (|x_n| + |y_n|) \to \infty \text{ and } |y_n| \to 0\big\} = \{0\}.$$

Hence finally $K(f) = \{0\}$ and indeed we can check directly that in this case $B(f) = K(f) = \{0\}$ (in fact f is a topological covering outside 0). Note that the mapping f has no usual critical values.

Example 3.2. Let us consider a smooth mapping

$$f : \mathbb{C}^3 \ni (x,y,z) \mapsto (x\exp(z), y\exp(z)) \in \mathbb{C}^2.$$

Using the function g' we can easily compute that $K(f) = \{0\}$. But the function f is a global fibration of $\mathbb{C}^3$—in fact it gives a fibration

$$\mathbb{C}^2 \times \mathbb{C} \ni ((x,y),z) \mapsto \big(x\exp(-z), y\exp(-z), z\big) \in \mathbb{C}^3.$$

Thus in general $B(f) \neq K(f)$.

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