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SPECIAL LAGRANGIAN LINEAR SUBSPACES IN PRODUCT SYMPLECTIC SPACE

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Abstract. The notes consist of a study of special Lagrangian linear subspaces. We will give a condition for the graph of a linear symplectomorphism $f : (\mathbb{R}^{2n}, \sigma = \sum_{i=1}^{n} dx_i \wedge dy_i) \to (\mathbb{R}^{2n}, \sigma)$ to be a special Lagrangian linear subspace in $(\mathbb{R}^{2n} \times \mathbb{R}^{2n}, \omega = \pi_2^*\sigma - \pi_1^*\sigma)$. This way a special symplectic subset in the symplectic group is introduced. A stratification of special Lagrangian Grassmannian $S\Lambda_{2n} \simeq SU(2n)/SO(2n)$ is defined.

1. Introduction. *Symplectic manifold* (X, α) is a $2n$-dimensional manifold equipped with a closed differential form α such that $(\alpha)^n$ never vanish. A k-dimensional submanifold $Y \subset X$ is said to be *isotropic* if α restricted to every tangent plane T_xY, $x \in Y$, vanish. In the case $k = n = \dim X/2$ an isotropic submanifold is called *Lagrangian*. A diffeomorphism $f : (X, \alpha) \to (X, \alpha)$ is a *symplectomorphism* if $f^*\alpha = \alpha$. Recall that the graph of a symplectomorphism is a Lagrangian submanifold in the product $X \times X$ with the standard symplectic structure $\pi_2^*\alpha - \pi_1^*\alpha$, where π_1, π_2 are projections on arbitrary factors of $X \times X$. Kähler manifolds are distinguished class of symplectic manifolds. A manifold $(X, \alpha, \mathcal{J}, g)$ is said to be *Kähler* if (X, α) is a symplectic manifold, $\mathcal{J}$ a complex structure, g a Hermitian metric on X and $\alpha(u, \mathcal{J}v)$ is equal to the imaginary part of g. Let us assume that there exists a holomorphic $(n, 0)$-form Ω on X, in local coordinates $z_1, \dots, z_n \in X$ the *complex volume form* and the symplectic form are equal to $\Omega = dz_1 \wedge \ldots \wedge dz_n$ and $\alpha = \frac{i}{2} \sum_{k=1}^{n} dz_k \wedge d\bar{z}_k$ ([MS], [Wei]).

Definition 1.1. An oriented Lagrangian submanifold $L \subset (X, \alpha, \mathcal{J}, \Omega)$ is called *special* if $\operatorname{Im} \Omega|_L = 0$.

In fact there is a more general definition involving a *phase* $\theta \in [0, 2\pi]$. Let Λ_n be the *Lagrangian Grassmannian*, i.e. a manifold consisting of all linear Lagrangian subspaces in $2n$-dimensional linear symplectic space. Recall that $\Lambda_n \simeq U(n)/O(n)$, where $U(n)$ is the

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unitary group and $O(n)$ the orthogonal group ([MS]). Let $\det : \Lambda_n \to S^1$ be a mapping which sends a matrix A representing a Lagrangian linear subspace Y to its determinant, i.e. $\det(A) = \exp(i\theta)$. The number θ is called the *phase* of Y ([HL], [Joy]). The mapping to S^1 can be defined globally if a global complex volume form Ω exists.

DEFINITION 1.2. An oriented Lagrangian submanifold is said to be *special* if every its tangent space has the phase zero.

In a more general case special Lagrangian submanifolds with the fixed phase $\theta \in [0, 2\pi]$ are considered.

EXAMPLES.

1) In $(\mathbb{R}^2, \alpha = dx \wedge dy)$ the subspace $L = \{y = 0\}$ is a linear special Lagrangian subspace with the phase 0.

2) Recall that every Lagrangian submanifold can be locally described as the graph of a function differential. In $\mathbb{C}^m$ the condition for graph df $(f : \mathbb{R}^m \to \mathbb{R})$ to be a special Lagrangian submanifold is $\operatorname{Im}\det(I + i \operatorname{Hess} f) = 0$, where I is the identity matrix and $\operatorname{Hess} f$ the Hessian of f. In general, the above condition is very difficult, this is a nonlinear second-order elliptic partial differential equation. For $m = 2$ it gives the harmonic formula, i.e. $\Delta f = 0$. For $m = 3$ it has the form $\Delta f = \det(\operatorname{Hess} f)$ and this is the equation of Monge-Ampère type and its linearization at any solution is always elliptic ([HL]).

3) In $\mathbb{C}^2$ with the standard complex structure $\mathcal{I}$: $z_0 = x_0 + ix_1$, $z_1 = x_2 + ix_3$, every special Lagrangian submanifold is a $\mathcal{J}$-holomorphic curve with respect to the following structure $\mathcal{J}$: $w_0 = x_0 + ix_2$, $w_1 = x_1 - ix_3$. $\mathcal{J}$ is $\mathbb{R}$-linear and antiholomorphic, i.e. $\mathcal{J}(\mathcal{I}z) = -\mathcal{I}(\mathcal{J}z)$, $z \in \mathbb{C}^2$ ([Joy]).

The definition of a submanifold for which all tangent spaces have the common phase seems to be very restrictive. Special Lagrangian submanifolds can be defined only in symplectic manifolds $(X, \alpha, \mathcal{J})$ for which the holomorphic volume form is globally defined. Calabi-Yau manifolds, i.e. Kähler manifolds with the trivial canonical bundle (with global holomorphic volume form), have natural Lagrangian submanifolds which are special. Note that every special Lagrangian submanifold is a minimal submanifold, it minimizes the volume in its homology class. The special Lagrangian Grassmannian $S\Lambda_n$ (i.e. the family of all oriented n-dimensional vector subspaces in $2n$-dimensional symplectic vector space V) can be identified with the quotient $SU(n)/SO(n)$ ([HL]), where $SU(n)$ is the special unitary group and $SO(n)$ the special orthogonal group. If we consider the special linear Lagrangian subspace L_0 in V spanned by the canonical basis $\{e_1, \dots , e_n\}$ over real numbers, then every Lagrangian vector space in V can be obtained by a unitary transformation of vectors $e_1, \dots, e_n$ and every special Lagrangian vector space in V can be produced by a special unitary transformation of $e_1, \dots , e_n$. Special Lagrangian submanifolds are expected to play a role in the eventual explanation of Mirror Symmetry between Calabi-Yau manifolds (3-dimensional). Thus they are important in String Theory.

The paper is organized as follows. In the first part we give a condition for the graph of a linear symplectomorphism to be a special Lagrangian linear subspace. A *special symplectic subset* in the symplectic group is introduced. This subset consists of matrices representing linear symplectomorphisms whose graphs are special Lagrangian subspaces. We show some features of the special symplectic subset. In Section 3 we recall the stratification of the Lagrangian Grassmannian $\Lambda_{2n} \simeq U(2n)/O(2n)$ constructed in the product of two symplectic spaces. The stratification is associated with a question when a Lagrangian subspace is or is not the graph of a linear symplectomorphism. It was defined by Janeczko ([Jan]). We introduce an analog of this partition in the special Lagrangian case. In the last part of the paper we show a partition of the special Lagrangian Grassmannian $S\Lambda_2$.

2. Special Lagrangian subspaces as graphs of linear symplectomorphisms. We will consider the special Lagrangian geometry only in the linear case. Let $(\mathbb{R}^{2n} \simeq \mathbb{C}^n, \sigma, \mathcal{J}, g)$ be a linear symplectic space with a symplectic form σ, a Hermitian metric g, a complex structure $\mathcal{J}$, and $\sigma(u, \mathcal{J}v)$ be equal to the imaginary part of g. Let us endow the product $(\mathbb{C}^n \times \mathbb{C}^n, \omega, -\mathcal{J} \times \mathcal{J})$ with the standard symplectic product structure $\omega = \pi_2^*\sigma - \pi_1^*\sigma$, the complex structure $-\mathcal{J} \times \mathcal{J}$ compatible with ω. We have the holomorphic volume form $\Omega = d\bar{z}_1 \wedge \ldots \wedge d\bar{z}_n \wedge dz_{n+1} \wedge \ldots \wedge dz_{2n}$ in local coordinates $(z_1, \ldots, z_n, z_{n+1}, \ldots, z_{2n})$.

We shall find a condition for a linear symplectomorphism to have the graph being a special Lagrangian linear subspace in $(\mathbb{C}^{2n}, \omega)$.

LEMMA 2.1. *If $\mathcal{I}d : (\mathbb{C}^n, \sigma) \to (\mathbb{C}^n, \sigma)$ is the identity symplectomorphism, then*

$$\mathrm{phase}(\mathrm{graph}\, \mathcal{I}d) = n\,\frac{\pi}{2} \pmod{2\pi}.$$

Proof. Let $(e_1, e_2, \ldots, e_n, -ie_1, -ie_2, \ldots, -ie_n)$ be the standard orthogonal basis of the domain $(\mathbb{C}^n \simeq \mathbb{R}^{2n}, \sigma, \mathcal{J})$ over $\mathbb{R}$, then $(e_1, \ldots, e_n, ie_1, \ldots, ie_n)$ is the image of the basis. Thus $L = \mathrm{graph}\, \mathcal{I}d$ is a real linear subspace spanned (over $\mathbb{R}$) by the columns of the matrix $\begin{pmatrix} I & -iI \\ I & iI \end{pmatrix}$, where the block I is the identity matrix of dimension $n \times n$. We calculate: $\mathrm{phase}(\mathrm{graph}\, \mathcal{I}d) = \arg(\det \mathrm{graph}\, \mathcal{I}d) = \arg((2i)^n) = n\frac{\pi}{2} \pmod{2\pi}$. ∎

The above result permits us to fix the phase $\theta = n\frac{\pi}{2} \pmod{2\pi}$ for special Lagrangian subspaces and submanifolds.

PROPOSITION 2.2. *Let $\Phi : (\mathbb{C}^n, \sigma) \to (\mathbb{C}^n, \sigma)$ be a linear real symplectomorphism and let the symplectic matrix $\Phi \in \mathrm{Sp}(n)$ have the block form $\Phi = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where every block is a submatrix of dimension $n \times n$. Then* graph Φ *is a special Lagrangian linear subspace in $(\mathbb{C}^n \times \mathbb{C}^n, \omega)$ if and only if*

$$\arg\det\big((A + D) + i(C - B)\big) = 0.$$

Proof. We consider the orthogonal basis $(e_1, \ldots, e_n, -ie_1, \ldots, -ie_n)$ as in the above lemma, then the matrix representing $\mathrm{graph}\, \Phi$ as a real linear subspace in $\mathbb{C}^n \times \mathbb{C}^n$ is

$$\mathrm{graph}\, \Phi = \begin{pmatrix} I & -iI \\ A + iC & B + iD \end{pmatrix}$$

We calculate $\det(\mathrm{graph}\, \Phi) = \det\big((B - C) + i(A + D)\big) = \det(iI)\det\big((A + D) + i(C - B)\big) = (i)^n \det((A + D) + i(C - B))$, thus $\arg\big(\det(\mathrm{graph}\, \Phi)\big) = n\frac{\pi}{2} \pmod{2\pi}$ if and only if $\det\big((A + D) + i(C - B)\big) \in \mathbb{R}_+$. ∎

A special Lagrangian subspace should have fixed orientation. If we consider $\mathrm{graph}\, \Phi$ with the opposite orientation we deduce that $\mathrm{graph}\, \Phi$ is a special Lagrangian subspace if and only if $\det\big((A + D) + i(C - B)\big) \in \mathbb{R}_-$.

Remark 2.3. The graph of a linear symplectomorphism is a special Lagrangian linear subspace if after choosing an arbitrary orientation the determinant of the matrix $\big((A+D)+i(C-B)\big)$ is a real number.

We can express the conditions in terms of complex structure $\mathcal{J}$. Define

$$\Phi+\mathcal{J}\Phi\mathcal{J}^{-1}=\Phi+(\Phi^T)^{-1}=\begin{pmatrix}A+D & B-C\\ C-B & A+D\end{pmatrix}\simeq\big((A+D)+i(C-B)\big),$$

where $\Phi=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in \mathrm{Sp}(n)$. We have used the identification between matrices over $\mathbb{R}$ and over $\mathbb{C}$, i.e. $(X+iY)\simeq\begin{pmatrix}X & -Y\\ Y & X\end{pmatrix}$, $\det(X+iY)=\left|\det\begin{pmatrix}X & -Y\\ Y & X\end{pmatrix}\right|^2$.

Definition 2.4. A *special symplectic subset* is the subset in the symplectic group $\mathrm{Sp}(n)$ consisting of the matrices whose graph is a special Lagrangian subspace, we denote it by $\mathrm{SSp}(n)$.

Obviously $\mathrm{SSp}(n)$ is not a subgroup in $\mathrm{Sp}(n)$.

Example 2.5. We consider the symplectic group in $\mathbb{C}\simeq\mathbb{R}^2$, i.e. $\mathrm{Sp}(1)\simeq \mathrm{SL}(2,\mathbb{R})$ ($\mathrm{SL}(2,\mathbb{R})$—special linear group). The special symplectic subset in $\mathrm{Sp}(1)$ consists of symmetric and positive definite matrices: $\mathrm{SSp}(1)=\Big\{\Phi=\begin{pmatrix}a & b\\ b & d\end{pmatrix} : a+d>0,\ \Phi\in\mathrm{Sp}(1)\Big\}$. We see that $\mathrm{SSp}(1)$ is not a subgroup in $\mathrm{Sp}(1)$.

Example 2.6. We will show that the matrix $\Phi=\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}\in\mathrm{SSp}(1)$ represents the linear symplectomorphism $f:(\mathbb{C},\sigma)\to(\mathbb{C},\sigma)$ whose graph is the special Lagrangian vector subspace in $\big(\mathbb{C}\times\mathbb{C},\omega=\pi_2^*\sigma-\pi_1^*\sigma,-i\times i,\Omega=d\bar z_1\wedge dz_2\big)$, i.e. the phase of graph f is $\frac{\pi}{2}$.

Let $e_1=(1,0)\simeq 1$, $e_2=(0,1)\simeq i$ be the canonical basis of $\mathbb{C}\simeq\mathbb{R}^2$. We calculate that $f(e_1)=\Phi e_1=(2,1)\simeq 2+i$ and $f(e_2)=\Phi e_2=(1,1)\simeq 1+i$. Thus $\operatorname{graph}\Phi$ is a Lagrangian vector subspace in $\mathbb{C}\times\mathbb{C}$ represented by the matrix

$$\operatorname{graph}\Phi=\begin{pmatrix}1 & 0\\ 0 & 1\\ 2 & 1\\ 1 & 1\end{pmatrix}\simeq\begin{pmatrix}1 & -i\\ 2+i & 1+i\end{pmatrix}.$$

We calculate that $\omega\big([1,0,2,1],[0,1,1,1]\big)=\sigma\big([2,1],[1,1]\big)-\sigma\big([1,0],[0,1]\big)=-1+1=0$ and $\Omega\big([1,0,2,1],[0,1,1,1]\big)=\det\begin{pmatrix}1 & -i\\ 2+i & 1+i\end{pmatrix}=3i$. Thus $\mathrm{phase}(\operatorname{graph} f)=\frac{\pi}{2}$.

We will show some properties of $\mathrm{SSp}(n)$.

Remark 2.7.

1) Every special symplectic matrix $\Phi\in\mathrm{SSp}(n)$ can be decomposed as $\Phi=PQ$, where P is symmetric, symplectic and positive definite, and $Q\in SU(n)$. The subgroup $SU(n)$ is a maximal compact subset in $\mathrm{SSp}(n)$, like $U(n)$ in $\mathrm{Sp}(n)$.

2) $\mathrm{phase}(\operatorname{graph}\Phi)-n\frac{\pi}{2}=-\big(\mathrm{phase}(\operatorname{graph}\Phi^{-1})-n\frac{\pi}{2}\big)$.

3) $\mathrm{phase}(\operatorname{graph}\Phi^T)=\mathrm{phase}(\operatorname{graph}\Phi^{-1})$.

4) $\Phi SU(n) \subset \mathrm{SSp}(n)$ and $SU(n)\Phi \subset \mathrm{SSp}(n)$ for $\Phi \in \mathrm{SSp}(n)$ ($\Phi SU(n)$ and $SU(n)\Phi$ can be treated as „cosets" of $SU(n)$ in $\mathrm{SSp}(n)$).

Proof. We know that $\Phi \in \mathrm{SSp}(n)$ like every symplectic matrix is a product of two matrices, where the first is symmetric, symplectic and positive definite and the second is a unitary matrix ([MS]); $\det(\Phi + \mathcal{J}\Phi\mathcal{J}^{-1}) = \det(P + \mathcal{J}P\mathcal{J}^{-1}) \det Q$. Analyzing eigenvalues of P and $\mathcal{J}P\mathcal{J}^{-1}$, we see that $\det(P + \mathcal{J}P\mathcal{J}^{-1}) \in \mathbb{R}_+$ thus $\det Q = 1$. ∎

3. Stratification of the special Lagrangian Grassmannian. We will explore the Lagrangian Grassmannian in the Cartesian product of two copies of a linear symplectic space. We recall very natural stratification of Λ_{2n} introduced in [Jan]. The partition is associated to a question when a Lagrangian subspace is or is not the graph of a linear symplectomorphism. Next we introduce an analogous stratification in the special Lagrangian Grassmannian $S\Lambda_{2n}$.

A very easy and useful observation leads us to the stratification of Λ_{2n} ([Jan]).

REMARK 3.1. If $L \subset (\mathbb{C}^n \times \mathbb{C}^n, \omega = \pi_2^*\sigma - \pi_1^*\sigma, -\mathcal{J} \times \mathcal{J})$ is a linear Lagrangian subspace, then there are two possibilities:

1) L is transversal to $\mathbb{C}^n \times \{0\}$ and to $\{0\} \times \mathbb{C}^n$ simultaneously

or

2) L is transversal neither to $\mathbb{C}^n \times \{0\}$ nor to $\{0\} \times \mathbb{C}^n$

and always $\operatorname{codim} \pi_1(L) = \operatorname{codim} \pi_2(L)$.

This condition divides Grassmannian Λ_{2n} into two parts: the regular part consisting of the graphs of linear symplectomorphisms and the critical stratum which contains the graphs of linear symplectic correspondences.

In fact the stratification is $\Lambda_{2n} = R\Lambda_{2n} + \sum_{k=1}^{n} C_k\Lambda_{2n}$, where

– $R\Lambda_{2n}$ is the regular stratum, if $L \in R\Lambda_{2n}$ then $\operatorname{codim} \pi_1(L) = \operatorname{codim} \pi_2(L) = 0$,

– $\Sigma_{k=1}^{n} C_k\Lambda_{2n}$ is the critical set; if $L \in C_k\Lambda_{2n}$ then $\operatorname{codim} \pi_1(L) = \operatorname{codim} \pi_2(L) = k$.

PROPOSITION 3.2. *In the case of special Lagrangian Grassmannian we have an analogous partition, only the deepest stratum is different*:

$$S\Lambda_{2n} = RS\Lambda_{2n} + \sum_{k=1}^{n-1} C_k S\Lambda_{2n} + C_n S\Lambda_{2n}.$$

1) *The regular stratum $RS\Lambda_{2n}$ consists of the graphs of special linear symplectomorphisms $\mathbb{C}^n \to \mathbb{C}^n$, therefore it can be identified with two copies of the special symplectic subset* (*two orientations are possible*)

$$RS\Lambda_{2n} \simeq \mathrm{SSp}^+(n) \sqcup \mathrm{SSp}^-(n).$$

2) *Every stratum $C_k S\Lambda_{2n}$, $k = 1, \dots, n-1$, is fibered in the following way*:

$$\begin{array}{ccc} \mathrm{SSp}^+(n-k) \sqcup \mathrm{SSp}^-(n-k) & \hookrightarrow & C_k S\Lambda_{2n} \\ & & \downarrow \\ & & \mathcal{I}_k^{2n} \times \mathcal{I}_k^{2n} \end{array}$$

where $\mathcal{I}_k^{2n}$ denotes the isotropic Grassmannian, i.e. the set of all k-dimensional isotropic linear subspaces in $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. Recall that $\mathcal{I}_k^{2n} \simeq U(n)/(O(k) \oplus U(n-k))$, where $U(n)$

and $U(n-k)$ are the unitary groups and $O(k)$ the orthogonal group ([MS]). *In the above bundle we have the projection on symplectic polars and fibers consist of linear symplectomorphisms between reduced symplectic spaces $\pi_1(L)/\pi_1(L)^{\perp} \to \pi_2(L)/\pi_2(L)^{\perp}$.*

3) *Supercritical stratum $C_nS\Lambda_{2n}$ is a subset of $\Lambda_n \times \Lambda_n$. If $L \in C_nS\Lambda_{2n}$ then $L = L_1 \times L_2$, $L_i \in \Lambda_n$, $i = 1, 2$, and* $\mathrm{phase}(L_1 \times L_2) = n\frac{\pi}{2} \pmod{2\pi}$ *or* $\mathrm{phase}(L_1 \times L_2) = n\frac{\pi}{2} + \pi \pmod{2\pi}$ *if an orientation of the subspace L_1 or L_2 is changed.*

Proof. Items 1), 2) are obvious ([Jan]). In 3) $\mathrm{phase}(L_1 \times L_2) = \arg(\overline{\det L_1} \det L_2) = \arg(\exp(-i\theta_1)\exp i\theta_2) = \mathrm{phase}\, L_2 - \mathrm{phase}\, L_1 = \theta_2 - \theta_1 = n\frac{\pi}{2} \pmod{2\pi}$. ∎

4. Example. We will explore the stratification of the special Lagrangian Grassmannian $S\Lambda_{2n}$ in the smallest interesting dimension, $2n = 2$. Recall that $S\Lambda_2$ is the family of special real Lagrangian subspaces in $(\mathbb{C}^2, \mathcal{J}, \omega)$, $\dim S\Lambda_2 = 2$ and $S\Lambda_2 \simeq SU(2)/SO(2) \simeq S^3/S^1 \simeq S^2$.

The Grassmannian $S\Lambda_2$ can be divided into two strata:

1) the regular stratum $RS\Lambda_2$ which consists of the graphs of linear symplectomorphisms $\Phi : \mathbb{C} \to \mathbb{C}$,

2) the critical stratum $C_1S\Lambda_2$ which is included in the Cartesian product $\Lambda_1 \times \Lambda_1$, where $\Lambda_1 \simeq U(1)/O(1) \simeq S^1$ denotes the Lagrangian Grassmannian in $\mathbb{R}^2 \simeq \mathbb{C}$.

Using our results from Proposition 3.2 we see that the open stratum consists of two copies of special symplectic subset $\mathrm{SSp}(1)$ (see Example 2.5), i.e. $RS\Lambda_2 \simeq \mathrm{SSp}^+(1) \sqcup \mathrm{SSp}^-(1)$. If $L \in C_1S\Lambda_2$ then $L = L_1 \times L_2$, $\mathrm{phase}(L_1 \times L_2) = \pi/2$, thus the matrix $\Phi \in U(1) \times U(1)$ representing L is

$$\Phi = \begin{pmatrix} \exp(-i\theta_1) & 0 \\ 0 & \exp(i\theta_2) \end{pmatrix}, \quad \theta_2 - \theta_1 = \frac{\pi}{2}\,.$$

The stratum $C_1S\Lambda_2$ can be identified with the circle S^1 on the torus $\Lambda_1 \times \Lambda_1 \simeq S^1 \times S^1 \simeq T^2$. How are the strata located on the sphere $S^2 \simeq S\Lambda_2$? The stratum $C_1S\Lambda_2 \simeq S^1$ is the equator and $RS\Lambda_2$ two hemispheres. If we use coordinates s, δ, b which describe the set $\mathrm{SSp}(1)$ in Example 2.5 ($s = \frac{a+d}{2}$, $a = s - \delta$, $d = s + \delta$) and we parametrize $\mathrm{SSp}(1)$ by t and γ: $s = \cosh t$, $\delta = \sinh t \cos\gamma$, $b = \sinh t \sin\gamma$, we can show that if $t \to \infty$ then $\mathrm{graph}\,\Phi(t, \gamma) \to (-\gamma, \gamma + \frac{\pi}{2})$. It means that the deepest stratum is attached to a bigger one.

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