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ON A CLASS OF DIFFEOMORPHISMS DEFINED BY INTEGRO-DIFFERENTIAL OPERATORS

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Abstract. We study an integro-differential operator $\Phi : \bar{H}^1 \to L^2$ of Fredholm type and give sufficient conditions for Φ to be a diffeomorphism. An application to functional equations is presented.

1. Introduction. Let us consider the Fredholm nonlinear integro-differential operator

$$\Phi : \bar{H}^1 \ni u(\,\cdot\,) \mapsto u'(\,\cdot\,) - \int_a^b F(\cdot, \tau, u(\tau))\, d\tau \in L^2, \tag{1}$$

where $[a, b] \subseteq \mathbb{R}$, $F : [a, b] \times [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$, $n \in \mathbb{N}$ and $\bar{H}^1$ is the space of absolutely continuous functions $u : [a, b] \to \mathbb{R}^n$ such that $u(a) = 0$ and $u'(\,\cdot\,) \in L^2 = L^2([a, b], \mathbb{R}^n)$, i.e.

$$\bar{H}^1 = \big\{u \in AC([a, b], \mathbb{R}^n) : u(a) = 0,\ u' \in L^2\big\}.$$

The operator (1) leads to the integro-differential equation

$$u'(t) = \int_a^b F(t, \tau, u(\tau))\, d\tau + \alpha(t), \quad t \in [a, b] \text{ a.e.}, \tag{2}$$

with the initial condition

$$u(a) = 0, \tag{3}$$

where $\alpha \in L^2$, which is quite frequently used in mathematical biology, electrodynamics and economics (see [4, 7]).

The space $\bar{H}^1$ with the inner product

$$\langle u_1, u_2\rangle = \int_a^b \big\langle u_1'(t), u_2'(t)\big\rangle\, dt$$

is a Hilbert space.

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It is easy to see that the problem (2)–(3) can be written in an equivalent form as

$$u(t) = \int_a^t \int_a^b F(s,\tau,u(\tau))\,d\tau\,ds + y(t), \quad t \in [a,b],$$

where $y(t) = \int_a^t \alpha(s)\,ds$.

In the next theorem and later we shall use Palais–Smale (PS for short) condition. Let X be a Banach space and $\psi : X \to \mathbb{R}$ be a functional of class C^1 in the Fréchet sense. We say that (see [6]) the functional ψ satisfies the PS-condition if every sequence $\{u_j\}$ in X such that $\{\psi(u_j)\}$ is bounded and $\psi'(u_j) \to 0$ in X^* has a convergent subsequence ($\psi'(u_j)$ is the Fréchet differential of ψ at u_j).

Our main tool is the following

THEOREM 1.1. *If $f : \bar{H}^1 \to \bar{H}^1$ is of class C^1 and the linear equation*

$$f'(u_0)h = g \tag{4}$$

has a unique solution for every $u_0, g \in \bar{H}^1$ and the functional $\varphi_y : \bar{H}^1 \to \mathbb{R}$ given by

$$\varphi_y(u) = \tfrac{1}{2}\|f(u) - y\|^2_{\bar{H}^1} \tag{5}$$

satisfies the PS-condition for every $y \in \bar{H}^1$, then the mapping f is a diffeomorphism, i.e. the nonlinear equation

$$f(u) = y \tag{6}$$

has a unique solution $u_y = f^{-1}(y)$ for every $y \in \bar{H}^1$ and the operator $y \mapsto u_y$ is Fréchet-differentiable.

Proof. The above theorem is an obvious consequence of Theorem 3.1 from the paper [3] for $X = H = \bar{H}^1$ (cf. also [3, Remark 3.1]). ■

Theorem 1.1 asserts that if the operator f is a local diffeomorphism (guaranteed by (4)) and the corresponding functional φ_y, given by (5), satisfies the Palais–Smale condition, then f is in fact a global diffeomorphism. Using this theorem we show that, under assumptions (a1)–(a3) given in Section 2:

1. the Cauchy problem (2)–(3) has a unique solution u_α for every $\alpha \in L^2$,
2. the solution u_α depends continuously on the parameter α, i.e. the system (2)–(3) is stable,
3. the operator $L^2 \ni \alpha \mapsto u_\alpha \in \bar{H}^1$ is Fréchet-differentiable.

The systems satisfying condition (3) are called robust systems and are frequently used in the technical literature (see [8]).

2. Fundamental lemmas.

We start with the following

LEMMA 2.1. *If the function F satisfies the assumptions*:

(a1) *$F(\cdot,\cdot,u)$ is measurable on $Q := [a,b] \times [a,b]$ for any $u \in \mathbb{R}^n$ and $F(t,\tau,\cdot)$ is of class C^1 on $\mathbb{R}^n$ for a.e. $(t,\tau) \in Q$;*

(a2) *there exists a function $w \in L^2(Q,\mathbb{R}^+)$ such that $\|w\|_{L^2} < \frac{\sqrt{2}}{(b-a)}$ and*

$$|F_u(t,\tau,u)| \le w(t,\tau)$$

for a.e. $(t,\tau) \in Q$, $u \in \mathbb{R}^n$, *then, for any $u_0, g \in \bar{H}^1$, the integral equation*

$$h(t) = g(t) + \int_a^t \left[\int_a^b F_u(s, \tau, u_0(\tau)) h(\tau)\, d\tau \right] ds, \quad t \in [a, b], \tag{7}$$

has a unique solution in $\bar{H}^1$.

Proof. Let $u_0, g \in \bar{H}^1$ and $K : \bar{H}^1 \to \bar{H}^1$ be the operator given by

$$(Kh)(t) = \int_a^t \Big[\int_a^b F_u(s, \tau, u_0(\tau)) h(\tau)\, d\tau \Big]\, ds. \tag{8}$$

The equation (7) can be written as

$$h = g + Kh. \tag{9}$$

Using an iterative method we shall prove that the above equation has a unique solution in the space $\bar{H}^1$.

Indeed, let $h_0 = 0$ and

$$h_{j+1} = g + Kh_j \tag{10}$$

for $j = 0, 1, 2, \ldots$. Of course,

$$\begin{aligned} h_1 &= g, \\ h_2 &= g + Kh_1 = g + Kg, \\ h_3 &= g + Kh_2 = g + Kg + K^2 g, \\ &\vdots \\ h_{j+1} &= g + Kg + \ldots + K^j g, \\ &\vdots \end{aligned} \tag{11}$$

where $K^j g = K(K^{j-1} g)$ for $j = 1, 2, \ldots$ and $K^0 g = g$. So, h_j is the partial sum of the Neumann series (see [1])

$$\sum_{i=0}^{\infty} K^i g. \tag{12}$$

Now, we shall show that there exist constants M_i such that

$$\|K^i g\|_{\bar{H}^1} \le M_i, \quad i = 0, 1, \ldots$$

and the series $\sum_{i=0}^{\infty} M_i$ is convergent. Indeed, from (8) we have

$$\begin{aligned} \|Kg\|_{\bar{H}^1}^2 &= \int_a^b \Big| \int_a^b F_u(s, \tau, u_0(\tau)) g(\tau)\, d\tau \Big|^2 ds \\ &\le \int_a^b \Big[\int_a^b |F_u(s, \tau, u_0(\tau))|\, |g(\tau)|\, d\tau \Big]^2 ds \\ &\le \int_a^b \Big[\Big(\int_a^b \big|F_u(s, \tau, u_0(\tau))\big|^2 d\tau \Big)^{1/2} \Big(\int_a^b |g(\tau)\|^2\, d\tau \Big)^{1/2} \Big]^2 ds \\ &= \|g\|_{L^2}^2 \int_a^b \int_a^b \big|F_u(s, \tau, u_0(\tau))\big|^2 d\tau\, ds. \end{aligned} \tag{13}$$

Using (a2) we get
$$\|Kg\|_{\bar{H}^1}^2 \le \|w\|_{L^2}^2 \|g\|_{L^2}^2.$$
In general,
$$\|K^j g\|_{\bar{H}^1}^2 = \int_a^b \Big| \int_a^b F_u(s,\tau,u_0(\tau))(K^{j-1}g)(\tau)\,d\tau \Big|^2 ds \le \|w\|_{L^2}^2 \|K^{j-1}g\|_{L^2}^2$$
and
$$\begin{aligned}
\|K^j g\|_{L^2}^2 &= \int_a^b \big|K(K^{j-1}g)(t)\big|^2\,dt \\
&= \int_a^b \Big| \int_a^t \Big[\int_a^b F_u(s,\tau,u_0(\tau))(K^{j-1}g)(\tau)\,d\tau \Big] ds \Big|^2 dt \\
&\le \int_a^b \Big(\int_a^t \Big[\int_a^b \big|F_u(s,\tau,u_0(\tau))\big|\big|(K^{j-1}g)(\tau)\big|\,d\tau \Big] ds \Big)^2 dt \\
&\le \|K^{j-1}g\|_{L^2}^2 \int_a^b \Big(\int_a^t \Big[\int_a^b \big|F_u(s,\tau,u_0(\tau))\big|^2 d\tau \Big]^{1/2} ds \Big)^2 dt \\
&\le \|K^{j-1}g\|_{L^2}^2 \int_a^b (t-a) \Big(\int_a^t \int_a^b \big|F_u(s,\tau,u_0(\tau))\big|^2 d\tau\,ds \Big) dt \\
&\le \|K^{j-1}g\|_{L^2}^2 \|w\|_{L^2}^2 \,\frac{(b-a)^2}{2}
\end{aligned}$$
for $j = 1, 2, \ldots$. So,
$$\begin{aligned}
\|Kg\|_{L^2}^2 &\le \|g\|_{L^2}^2 \|w\|_{L^2}^2 \,\frac{(b-a)^2}{2}\,, \\
\|K^2 g\|_{L^2}^2 &\le \|g\|_{L^2}^2 \|w\|_{L^2}^4 \,\frac{(b-a)^4}{2^2}\,, \\
\|K^3 g\|_{L^2}^2 &\le \|g\|_{L^2}^2 \|w\|_{L^2}^6 \,\frac{(b-a)^6}{2^3}\,, \\
&\ \ \vdots \\
\|K^j g\|_{L^2}^2 &\le \|g\|_{L^2}^2 \|w\|_{L^2}^{2j} \,\frac{(b-a)^{2j}}{2^j}\,, \\
&\ \ \vdots
\end{aligned}$$
Consequently,
$$\|K^j g\|_{\bar{H}^1} \le \|g\|_{L^2} \|w\|_{L^2} \Big(\frac{\sqrt{2}}{2}(b-a)\|w\|_{L^2} \Big)^{j-1} =: M_j \tag{14}$$
for $j = 1, 2, \ldots$. Additionally, we put $M_0 = \|g\|_{\bar{H}^1}$. From assumption (a2) and inequality (14) it follows that the sequence $\{h_j\}$ of functions defined by (10) is convergent in $\bar{H}^1$ to some function h_0. It is easy to see (cf. (13) and $\|g\|_{L^2}^2 \le (b-a)^2 \|g\|_{\bar{H}^1}^2$ for $g \in \bar{H}^1$) that the operator K given by (8) is continuous. Hence, using (10) we conclude that h_0 satisfies the equation (9), i.e.
$$h_0 = g + Kh_0.$$

To begin the proof of the uniqueness of the solution h_0, we suppose that there exists another solution $h_1 \in \bar{H}^1$ of the equation (9). Then,

$$h_1 - h_0 = K(h_1 - h_0) = K^2(h_1 - h_0) = \ldots = K^j(h_1 - h_0)$$

for $j = 1, 2, \ldots$. So, by (14),

$$\|h_1 - h_0\|_{\bar{H}^1} \leq \|h_1 - h_0\|_{L^2} \|w\|_{L^2} \left(\frac{\sqrt{2}}{2}(b-a)\|w\|_{L^2} \right)^{j-1}$$

for $j = 1, 2, \ldots$. Since the right hand side of the inequality tends to 0 as $j \to \infty$ (cf. (a2)),

$$h_1(t) = h_0(t) \quad \text{for } t \in [a, b].$$

Our solution is thus unique and the proof is completed. ■

Let $f : \bar{H}^1 \to \bar{H}^1$ be the operator given by

$$f(u)(t) = u(t) - \int_a^t \int_a^b F(s, \tau, u(\tau))\, d\tau\, ds, \tag{15}$$

$\varphi_y : \bar{H}^1 \to \mathbb{R}$ — the functional given by (5) for any fixed $y \in \bar{H}^1$. We also put

$$\varphi := \varphi_0. \tag{16}$$

It is easy to show that under the assumptions (a1)–(a2) the operator f and, consequently, the functional φ_y are of class C^1. From the theorem on the differentiability of composite mapping it follows that the differential $\varphi'_y(u)$ at a point $u \in \bar{H}^1$ is given by

$$\varphi'_y(u)h = \langle f(u) - y, f'(u)h \rangle$$

for $h \in \bar{H}^1$ and

$$f'(u)h(\cdot) = h(\cdot) - \int_a^{\cdot} \int_a^b F_u(s, \tau, u(\tau))h(\tau)\, d\tau\, ds,$$

for $h \in \bar{H}^1$.

We now prove

Lemma 2.2. *If the function F satisfies the assumptions of Lemma* 2.1 *and*

(a3) *there exist functions $A, B \in L^2(Q, \mathbb{R}^+)$ such that $\|A\|_{L^2} < \frac{\sqrt{2}}{2(b-a)}$, and*

$$|F(t, \tau, u)| \leq A(t, \tau)|u| + B(t, \tau) \quad \text{for a.e. } (t, \tau) \in Q,\ u \in \mathbb{R}^n,$$

then the functional φ_y satisfies PS-condition for every $y \in \bar{H}^1$.

Proof. Let $y \in \bar{H}^1$. First, we prove that every PS-sequence $\{u_k\} \subset \bar{H}^1$ for the functional φ_y is bounded. This will be done if we prove that φ_y is coercive, i.e. $\varphi_y(u) \to \infty$ when $\|u\| \to \infty$. Of course, φ_y is coercive whenever φ is. We have to notice that φ is Fréchet differentiable and bounded from below. So, if it satisfies the PS-condition, then it is coercive (see [5, Theorem 7]). We have

$$\begin{aligned} 2\varphi(u) &= \|u\|^2_{\bar{H}^1} - 2\int_a^b \left\langle u'(t), \int_a^b F(t, \tau, u(\tau))\, d\tau \right\rangle dt + \int_a^b \left| \int_a^b F(t, \tau, u(\tau))\, d\tau \right|^2 dt \\ &\geq \|u\|^2_{\bar{H}^1} - 2\int_a^b \left[|u'(t)| \int_a^b \big(A(t, \tau)|u(\tau)| + B(t, \tau)\big)\, d\tau \right] dt \end{aligned} \tag{17}$$

Moreover, from the Schwarz inequality

$$|u(t)| \leq \int_a^t |u'(\tau)|\, d\tau \leq \sqrt{t-a}\, \|u\|_{\bar{H}^1}$$

for $u \in \bar{H}^1$ and $t \in [a,b]$. Hence

$$\int_a^b |u(t)|^2\, dt \leq \|u\|_{\bar{H}^1}^2 \int_a^b (t-a)\, dt = \tfrac{1}{2}(b-a)^2 \|u\|_{\bar{H}^1}^2. \tag{18}$$

Consequently, from (17), (18) and the Schwarz inequality

$$\begin{aligned}
&2\varphi(u) \geq \|u\|_{\bar{H}^1}^2 \\
&\quad - 2\int_a^b |u'(t)| \bigg(\Big(\int_a^b A^2(t,\tau)\, d\tau \Big)^{1/2} \Big(\frac{b-a}{\sqrt{2}} \|u\|_{\bar{H}^1} \Big) + \Big((b-a) \int_a^b B^2(t,\tau)\, d\tau \Big)^{1/2} \bigg)\, dt \\
&\geq \|u\|_{\bar{H}^1}^2 - \sqrt{2}(b-a)\|u\|_{\bar{H}^1} \Big(\int_a^b |u'(t)|^2 dt \Big)^{1/2} \Big(\int_b^a \Big(\int_a^b A^2(t,\tau)\, d\tau \Big)\, dt \Big)^{1/2} \\
&\qquad - 2\sqrt{b-a} \Big(\int_a^b |u'(t)|^2\, dt \Big)^{1/2} \Big(\int_a^b \Big(\int_a^b B^2(t,\tau)\, d\tau \Big)\, dt \Big)^{1/2} \\
&\geq \big(1 - \sqrt{2}(b-a)\|A\|_{L^2}\big) \|u\|_{\bar{H}^1}^2 - 2\sqrt{b-a}\, \|B\|_{L^2} \|u\|_{\bar{H}^1}.
\end{aligned}$$

Given these facts, we get that $\varphi(u) \geq c\|u\|_{\bar{H}^1}^2 - d\|u\|_{\bar{H}^1}$, where

$$c = \tfrac{1}{2}\big(1 - \sqrt{2}(b-a)\|A\|_{L^2}\big),$$
$$d = \sqrt{b-a}\|B\|_{L^2}$$

with c positive (by (a3)). Therefore,

$$\varphi(u) \to \infty \quad \text{as} \quad \|u\|_{\bar{H}^1} \to \infty. \tag{19}$$

Let $\{u_k\} \subset \bar{H}^1$ be a PS-sequence for the functional φ_y. According to (19) this sequence is bounded in $\bar{H}^1$ and hence weakly compact in $\bar{H}^1$. Without loss of generality, we may assume that it is weakly convergent in $\bar{H}^1$ to some u_0. We shall show that $u_k \to u_0$ with respect to the norm.

Indeed,

$$\begin{aligned}
\varphi_y'(u)h &= \int_a^b \langle u'(t), h'(t)\rangle\, dt - \int_a^b \langle y'(t), h'(t)\rangle\, dt \\
&\quad - \int_a^b \Big\langle h'(t), \int_a^b F(t,\tau,u(\tau))\, d\tau \Big\rangle\, dt \\
&\quad - \int_a^b \Big\langle u'(t), \int_a^b F_u(t,\tau,u(\tau))h(\tau)\, d\tau \Big\rangle\, dt \\
&\quad + \int_a^b \Big\langle y'(t), \int_a^b F_u(t,\tau,u(\tau))h(\tau)\, d\tau \Big\rangle\, dt \\
&\quad + \int_a^b \Big\langle \int_a^b F(t,\tau,u(\tau))\, d\tau, \int_a^b F_u(t,\tau,u(\tau))h(\tau)\, d\tau \Big\rangle\, dt.
\end{aligned}$$

Consequently,

$$(\varphi_y'(u_k)-\varphi_y'(u_0))(u_k-u_0)=\|u_k-u_0\|_{\bar H^1}^2+\sum_{i=1}^{6}\psi_i(u_k) \tag{20}$$

where

$$\psi_1(u_k)=-\int_a^b\Big\langle u_k'(t)-u_0'(t),\int_a^b\big(F(t,\tau,u_k(\tau))-F(t,\tau,u_0(\tau))\big)\,d\tau\Big\rangle\,dt,$$

$$\psi_2(u_k)=-\int_a^b\Big\langle u_k'(t),\int_a^b F_u(t,\tau,u_k(\tau))(u_k(\tau)-u_0(\tau))\,d\tau\Big\rangle\,dt,$$

$$\psi_3(u_k)=\int_a^b\Big\langle u_0'(t),\int_a^b F_u(t,\tau,u_0(\tau))(u_k(\tau)-u_0(\tau))\,d\tau\Big\rangle\,dt,$$

$$\psi_4(u_k)=\int_a^b\Big\langle y'(t),\int_a^b\big(F_u(t,\tau,u_k(\tau))-F_u(t,\tau,u_0(\tau))\big)(u_k(\tau)-u_0(\tau))\,d\tau\Big\rangle\,dt,$$

$$\psi_5(u_k)=\int_a^b\Big\langle\int_a^b F(t,\tau,u_k(\tau))\,d\tau,\int_a^b F_u(t,\tau,u_k(\tau))(u_k(\tau)-u_0(\tau))\,d\tau\Big\rangle\,dt,$$

$$\psi_6(u_k)=-\int_a^b\Big\langle\int_a^b F(t,\tau,u_0(\tau))\,d\tau,\int_a^b F_u(t,\tau,u_0(\tau))(u_k(\tau)-u_0(\tau))\,d\tau\Big\rangle\,dt.$$

The left hand side of equality (20) tends to zero. Indeed,

$$\big|\varphi_y'(u_k)(u_k-u_0)\big|\le\|\varphi_y'(u_k)\|_{\mathcal{L}(\bar H^1,\mathbb{R})}\|u_k-u_0\|$$

and $\varphi_y'(u_k)(u_k-u_0)\underset{k\to\infty}{\longrightarrow}0$, because $\varphi'(u_k)\underset{k\to\infty}{\longrightarrow}0$ and the sequence $\{u_k\}$ is bounded. Furthermore, $\varphi_y'(u_0)(u_k-u_0)$ tends to zero, because the sequence $\{u_k\}$ is weakly convergent to u_0 in $\bar H^1$. To conclude the proof, we need to show that $\psi_i(u_k)\underset{k\to\infty}{\longrightarrow}0$ for $i=1,\dots,6$. As mentioned before, the sequence $\{u_k\}$ converges weakly to u_0 in $\bar H^1$, which implies the uniform convergence of $\{u_k\}$ on $[a,b]$ to u_0 and the weak convergence of $\{u_k'\}$ to u_0' in L^2.

First, consider the term $\psi_1(u_k)$. From the Lebesgue dominated convergence theorem it follows that

$$\int_a^b\big(F(t,\tau,u_k(\tau))-F(t,\tau,u_0(\tau))\big)\,d\tau\underset{k\to\infty}{\longrightarrow}0$$

for a.e. $t\in[a,b]$. Moreover, by (a3) and the Schwarz inequality

$$\begin{aligned}\Big|\int_a^b\big(F(t,\tau,u_k(\tau))-F(t,\tau,u_0(\tau))\big)\,d\tau\Big|^2&\le\Big(2\int_a^b(A(t,\tau)M+B(t,\tau))\,d\tau\Big)^2\\&\le 4(b-a)\int_a^b(A(t,\tau)M+B(t,\tau))^2\,d\tau,\end{aligned}$$

where $M>0$ is such that $\big|u_k(\tau)\big|\le M$ for $\tau\in[a,b]$, $k=0,1,\dots$. Since the function

$$[a,b]\ni t\mapsto\int_a^b\big(A(t,\tau)M+B(t,\tau)\big)^2\,d\tau$$

is integrable, therefore again by the Lebesgue dominated convergence theorem we conclude that $\int_a^b\big(F(\cdot,\tau,u_k(\tau))-F(\cdot,\tau,u_0(\tau))\big)\,d\tau\underset{k\to\infty}{\longrightarrow}0$ in L^2. Consequently, $\psi_1(u_k)$ tends to zero as a scalar product in L^2 of the functions

$$u_k'(\,\cdot\,) - u_0'(\,\cdot\,) \text{ and } \int_a^b \big(F(\cdot,\tau,u_k(\tau)) - F(\cdot,\tau,u_0(\tau))\big)\,d\tau.$$

Next, consider the term $\psi_2(u_k)$. As above,

$$\int_a^b F_u(t,\tau,u_k(\tau))(u_k(\tau)-u_0(\tau))\,d\tau \underset{k\to\infty}{\longrightarrow} 0$$

for a.e. $t\in[a,b]$ and

$$\int_a^b F_u(\cdot,\tau,u_k(\tau))(u_k(\tau)-u_0(\tau))\,d\tau \underset{k\to\infty}{\longrightarrow} 0$$

in L^2. Therefore, $\psi_2(u_k)$ tends to zero as a scalar product in L^2 of the functions $u_k'(\,\cdot\,)$ and $\int_a^b F_u(\cdot,\tau,u_k(\tau))(u_k(\tau)-u_0(\tau))\,d\tau$.

In a similar way one shows that $\psi_i(u_k)\to 0$ as $k\to\infty$ for $i=3,4,5,6$. ∎

3. Main result and example. Now we are in a position to prove the main theorem of the paper

THEOREM 3.1. *If the function F satisfies the assumptions* (a1), (a2) *and* (a3), *then the operator f defined by* (15) *is a diffeomorphism between $\bar H^1$ and $\bar H^1$.*

Proof. From Lemmas 2.1 and 2.2 it follows that the operator f given by (15) satisfies the assumptions of Theorem 1.1 and consequently it is a diffeomorphism. ∎

From the above theorem we conclude that

THEOREM 3.2. *If the function F satisfies the assumptions* (a1), (a2) *and* (a3), *then the integro-differential operator Φ given by* (1) *is a diffeomorphism between the spaces $\bar H^1$ and L^2. Consequently, the Cauchy problem* (2)–(3) *has a unique solution $u_\alpha\in\bar H^1$ for every $\alpha\in L^2$ and the mapping $L^2\ni\alpha\mapsto u_\alpha\in\bar H^1$ is Fréchet-differentiable.*

To illustrate the above theorem we give some example.

EXAMPLE 3.3. Let us consider the integro-differential operator

$$\Phi:\bar H^1\ni u(\,\cdot\,)\mapsto u'(\,\cdot\,)-\int_a^b F(\cdot,\tau,u(\tau))\,d\tau\in L^2\big([0,1],\mathbb{R}\big)$$

where $F(t,\tau,u)=\frac{(4t\tau-1)u+4t\tau u^3}{3(1+u^2)}$. The function F can be written as

$$F(t,\tau,u)=\frac{4}{3}\,t\tau u-\frac{u}{3(1+u^2)}$$

and therefore $F_u(t,\tau,u)=\frac{4}{3}\,t\tau+\frac{u^2-1}{3(1+u^2)^2}$. Let us put

$$A(t,\tau)=\frac{4}{3}\,t\tau,\quad B(t,\tau)=\frac{1}{3}$$

and $w(t,\tau)=A(t,\tau)+B(t,\tau)$. Then

$$\begin{aligned}\big|F(t,\tau,u)\big|&\le A(t,\tau)|u|+B(t,\tau),\\ \big|F_u(t,\tau,u)\big|&\le w(t,\tau).\end{aligned}$$

Moreover,

$$\|A\|_{L^2}^2 = \int_0^1 \int_0^1 \frac{16}{9}\, t^2\tau^2 \, d\tau \, dt = \frac{16}{81} < \frac{1}{2} = \Big(\frac{\sqrt{2}}{2}\Big)^2$$

and

$$\|w\|_{L^2}^2 = \int_0^1 \int_0^1 \Big(\frac{4}{3}\, t\tau + \frac{1}{3}\Big)^2 d\tau \, dt = \frac{43}{81} < 2.$$

So, the function F satisfies the assumptions of Theorem 3.2. Consequently, the operator Φ is a diffeomorphism between $\bar{H}^1$ and $L^2\big([0,1],\mathbb{R}\big)$. In conclusion, the equation

$$u'(t) - \int_0^1 \frac{(4t\tau - 1)u(\tau) + 4t\tau u^3(\tau)}{3\big(1 + u^2(\tau)\big)} \, d\tau = \alpha(t), \quad t \in [0,1],$$

possesses a unique solution $u_\alpha \in \bar{H}^1$ for any $\alpha \in L^2\big([0,1],\mathbb{R}\big)$ and the mapping

$$L^2\big([0,1],\mathbb{R}\big) \ni \alpha \mapsto u_\alpha \in \bar{H}^1$$

is differentiable.

4. Concluding remarks. Example 3.3 is purely theoretical but these kinds of equations are used in electrodynamics, biomechanics and elasticity (see for instance [2, 7]). The analysis of such models could be long and quite complex, therefore we are going to devote them a separate paper.

The Cauchy problem for an integro-differential equation of Volterra type was considered in the paper [3]. Using an infinite dimensional theorem on diffeomorphisms (cf. [3, Theorem 3.1]) and Banach's contraction principle we have proved a result similar to Theorem 3.2. Unfortunately, in the case of Fredholm integro-differential operators, the contraction principle cannot be applied. In this paper we used the Neumann method instead (Lemma 2.1).

The approach proposed in our paper is quite general and works for hyperbolic operators of the form

$$\Phi(z)(x,y) = z_{xy}(x,y) + \int_a^x \int_c^y F\big(t,\tau,z_x(t,\tau),z_y(t,\tau),z(t,\tau)\big)\, dt \, d\tau,$$

where $(x,y) \in [a,b] \times [c,d] \subset \mathbb{R}^2$.

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