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# FRACTIONAL ROESSER PROBLEM AND ITS OPTIMIZATION

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**Abstract.** In the paper, a fractional continuous Roesser model is considered. Existence and uniqueness of a solution and continuous dependence of solutions on controls of the nonlinear model are investigated. Next, a theorem on the existence of an optimal solution for linear model with variable coefficients is proved.

**1. Introduction.** In the last few decades, fractional calculus plays an essential role in the fields of mathematics, physics, electronics, mechanics, chemistry, etc. (cf. [CM], [GO], [KST], [SKM], [WG]). Many physical phenomena are modelled accurately by using fractional partial differential equations. For instance, the fractional diffusion equations have been studied by many authors (cf. [L], [MP], [SW]). Moreover, the kinetic and advection-dispersion equations have been investigated very well (cf. [SZ], [LATZ]).

In our paper we consider the following fractional nonlinear continuous control system

$$\begin{cases} \begin{cases} \big(D^{\alpha_1}_{a_1+,t_1}x_1\big)(t) = f_1(t,x_1(t),x_2(t),u(t)), \\ \big(D^{\alpha_2}_{a_2+,t_2}x_2\big)(t) = f_2(t,x_1(t),x_2(t),u(t)), \\ t=(t_1,t_2)\in P=[a_1,b_1]\times[a_2,b_2] \ a.e., \end{cases} & (1.1) \\ \begin{cases} \big(I^{1-\alpha_1}_{a_1+,t_1}x_1\big)(a_1,t_2)=0, \quad t_2\in[a_2,b_2] \text{ a.e.}, \\ \big(I^{1-\alpha_2}_{a_2+,t_2}x_2\big)(t_1,a_2)=0, \quad t_1\in[a_1,b_1] \text{ a.e.}, \end{cases} & (1.2) \\ u(t)\in M\subset\mathbb{R}^m, \quad t\in P, \end{cases} \tag{1}$$

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with the performance index

$$J(x_1, x_2, u) = \int_P f_0(t, x_1(t), x_2(t), u(t))\, dt, \tag{2}$$

where $f_i : P \times \mathbb{R}^n \times \mathbb{R}^n \times M \to \mathbb{R}^n$, $f_0 : P \times \mathbb{R}^n \times \mathbb{R}^n \times M \to \mathbb{R}$, $\alpha_i \in (0,1)$ for $i = 1, 2$, $D^{\alpha_i}_{a_i+,t_i} x_i$ denotes the left-sided Riemann–Liouville derivative of order $\alpha_i$ of the function $x_i$ with respect to variable $t_i$, $I^{1-\alpha_i}_{a_i+,t_i} x_i$—left-sided Riemann–Liouville integral of order $1 - \alpha_i$ of the function $x_i$ with respect to variable $t_i$ $(i = 1, 2)$.

If $\alpha_1 = \alpha_2 = 1$ then system (1) is an extension of 2-D continuous Roesser model of the first order, which is a counterpart of 2-D discrete Roesser model introduced by Roesser in 1975 ([Roes]). Such models (continuous and discrete), which are applied to the research of transformation of images and chemistry processes, have been investigated by many authors (cf. [I1], [I2], [W]). In paper [W] a theorem on the existence and uniqueness of solution and the maximum principle for problem (1)–(2), with $\alpha_1 = \alpha_2 = 1$, in the case when $f_0$, $f_1$, $f_2$ are linear, have been proved. In [I2] the maximum principle for linear control system and nonlinear performance index has been derived. Moreover, in [I1] for such a problem, a theorem on the existence of an optimal solution in the case when a function $f^0$ is convex with respect to variables $(x_1, x_2, u)$ has been obtained. In [I1] an existence and uniqueness of solution and a continuous dependence of solutions on controls for the nonlinear Roesser model of the first order also have been proved.

In the paper [R] the fractional linear continuous Roesser model of type (1.1) with partial Caputo derivatives is investigated. The boundary conditions are described by partial derivatives of the integer and zero order. Particularly, a general response formula for such a problem is derived. This model is applied in fractional diffusion and transmission line equations (cf. [R1]).

The aim of this paper is obtaining analogous results for problem (1)–(2) as in [I1] for the continuous Roesser model of the first order.

In Section 2 some basic definitions and facts concerning the fractional calculus of functions of two variables are given.

Next (Section 3), we prove a theorem on the existence and uniqueness of a solution to system (1) for any control $u \in L^p(P, M)$ (Theorem 3.1) and a theorem on the continuous dependence of solutions on controls (Theorem 3.2). Finally, in the case when functions $f_1$, $f_2$ are linear, we derive a theorem on the existence of an optimal solution for problem (1)–(2) (Theorem 4.4) and demonstrate a simple illustrative example (Section 5).

To the best knowledge of the author, the problems studied in Sections 3 and 4 have not been considered yet.

**2. Preliminaries.** In this section we give basic definitions and facts connected with the fractional integrals and derivatives of functions of two variables.

Let $P = [a_1, b_1] \times [a_2, b_2] \subset \mathbb{R}^2$ be a fixed bounded rectangle.

DEFINITION 2.1 ([SKM, Formula 24.4]). Let $\varphi = \varphi(t_1, t_2) \in L^1(P, \mathbb{R}^n)$ and $\alpha > 0$. The functions $I^{\alpha}_{a_1+,t_1}\varphi$ and $I^{\alpha}_{b_1-,t_1}\varphi$ of the form

$$\big(I^{\alpha}_{a_1+,t_1}\varphi\big)(t_1,t_2) := \frac{1}{\Gamma(\alpha)}\int_{a_1}^{t_1}\frac{\varphi(\tau_1,t_2)}{(t_1-\tau_1)^{1-\alpha}}\,d\tau_1, \quad (t_1,t_2)\in(a_1,b_1)\times(a_2,b_2) \text{ a.e.}$$

$$\big(I^{\alpha}_{b_1-,t_1}\varphi\big)(t_1,t_2) := \frac{1}{\Gamma(\alpha)}\int_{t_1}^{b_1}\frac{\varphi(\tau_1,t_2)}{(\tau_1-t_1)^{1-\alpha}}\,d\tau_1, \quad (t_1,t_2)\in(a_1,b_1)\times(a_2,b_2) \text{ a.e.}$$

are called *left-sided* and *right-sided Riemann–Liouville integrals of order $\alpha$ on $P$* of the function $\varphi$ with respect to variable $t_1$, respectively.

If $\alpha = 0$ then we put

$$I^{\alpha}_{a_1+,t_1}\varphi := \varphi \quad \text{and} \quad I^{\alpha}_{b_1-,t_1}\varphi := \varphi.$$

REMARK 2.2. It is easy to show that the functions

$$(a_1,b_1)\times(a_2,b_2)\ni(t_1,t_2)\mapsto\big(I^{\alpha}_{a_1+,t_1}\varphi\big)(t_1,t_2)\in\bar{\mathbb{R}}^n$$
$$(a_1,b_1)\times(a_2,b_2)\ni(t_1,t_2)\mapsto\big(I^{\alpha}_{b_1-,t_1}\varphi\big)(t_1,t_2)\in\bar{\mathbb{R}}^n$$

are defined almost everywhere, summable and consequently almost everywhere finite on $(a_1,b_1)\times(a_2,b_2)$. Moreover (cf. [M, Lemma 1.4]), for every $\alpha_1, \alpha_2 > 0$ we have

$$\big(I^{\alpha_1}_{a_1+,t_1}I^{\alpha_2}_{a_1+,t_1}\varphi\big)(t_1,t_2) = \big(I^{\alpha_1+\alpha_2}_{a_1+,t_1}\varphi\big)(t_1,t_2), \quad (t_1,t_2)\in(a_1,b_1)\times(a_2,b_2) \text{ a.e.,} \tag{3}$$

$$\big(I^{\alpha_1}_{b_1-,t_1}I^{\alpha_2}_{b_1-,t_1}\varphi\big)(t_1,t_2) = \big(I^{\alpha_1+\alpha_2}_{b_1-,t_1}\varphi\big)(t_1,t_2), \quad (t_1,t_2)\in(a_1,b_1)\times(a_2,b_2) \text{ a.e.} \tag{4}$$

REMARK 2.3. We identify functions that are equal a.e. on $P$.

Analogously, one can define fractional Riemann–Liouville integrals of functions $\varphi \in L^1(P, \mathbb{R}^n)$ with respect to variable $t_2$.

We shall formulate next theorems and lemmas in this section for the function $I^{\alpha}_{a_1+,t_1}\varphi$. The other fractional integrals introduced above have analogous properties.

Similarly, as in the case of functions of one variable (cf. [K, Lemma 1]) we can prove the following lemma.

LEMMA 2.4. *If $\varphi \in L^p(P, \mathbb{R}^n)$, $1 \leqslant p < \infty$, $\alpha > 0$, then*

$$\big|\big(I^{\alpha}_{a_1+,t_1}\varphi\big)(t_1,t_2)\big|^p \leqslant c_1\big(I^{\alpha}_{a_1+,t_1}|\varphi|^p\big)(t_1,t_2), \quad (t_1,t_2)\in P \text{ a.e.,}$$

*where $c_1 = \big((b_1-a_1)^{\alpha}/\Gamma(\alpha+1)\big)^{p-1}$. Consequently, $I^{\alpha}_{a_1+,t_1}\varphi \in L^p(P, \mathbb{R}^n)$.*

Now, we shall prove two lemmas.

LEMMA 2.5. *If $g \in L^1(P, \mathbb{R}^n)$, $\alpha > 0$ and $\big(I^{\alpha}_{a_1+,t_1}g\big)(t_1,t_2) = 0$ for a.e. $(t_1,t_2)\in P$, then $g(t_1,t_2) = 0$ for a.e. $(t_1,t_2)\in P$.*

*Proof.* Condition (3) implies that

$$0 = \big(I^{1-\alpha}_{a_1+,t_1}0\big)(t_1,t_2) = \big(I^{1-\alpha}_{a_1+,t_1}I^{\alpha}_{a_1+,t_1}g\big)(t_1,t_2) = \big(I^{1}_{a_1+,t_1}g\big)(t_1,t_2) = \int_{a_1}^{t_1} g(s,t_2)\,ds$$

for a.e. $(t_1,t_2)\in P$. It means that $g(t_1,t_2) = 0$ for a.e. $(t_1,t_2)\in P$ and the proof is completed. ∎

LEMMA 2.6. *Let* $\alpha > 0$ *and* $1 \leqslant p < \infty$. *Then the operator* $I^{\alpha}_{a_1+,t_1} : L^p(P,\mathbb{R}^n) \to L^p(P,\mathbb{R}^n)$ *is bounded; precisely, for any function* $\varphi \in L^p(P,\mathbb{R}^n)$

$$\left\| I^{\alpha}_{a_1+,t_1}\varphi \right\|_{L^p} \leqslant K_1 \left\|\varphi\right\|_{L^p},$$

*where* $K_1 = (b_1 - a_1)^{\alpha}/\Gamma(\alpha+1)$.

*Proof.* From Lemma 2.4 and the Fubini Theorem it follows that

$$\begin{aligned}
\left\| I^{\alpha}_{a_1+,t_1}\varphi \right\|^p_{L^p} &= \int_P \left| \left(I^{\alpha}_{a_1+,t_1}\varphi\right)(t_1,t_2)\right|^p dt_1\, dt_2 \leqslant c_1 \int_P \left(I^{\alpha}_{a_1+,t_1}\left|\varphi\right|^p\right)(t_1,t_2)\, dt_1\, dt_2 \\
&= \frac{c_1}{\Gamma(\alpha)} \int_{a_2}^{b_2} \left( \int_{a_1}^{b_1} \left( \int_{a_1}^{t_1} \frac{|\varphi(\tau_1,t_2)|^p}{(t_1-\tau_1)^{1-\alpha}}\, d\tau_1 \right) dt_1 \right) dt_2 \\
&= \frac{c_1}{\Gamma(\alpha)} \int_{a_2}^{b_2} \left( \int_{a_1}^{b_1} \left( |\varphi(\tau_1,t_2)|^p \int_{\tau_1}^{b_1} \frac{1}{(t_1-\tau_1)^{1-\alpha}}\, dt_1 \right) d\tau_1 \right) dt_2 \\
&= \frac{c_1}{\alpha\Gamma(\alpha)} \int_{a_2}^{b_2} \int_{a_1}^{b_1} |\varphi(\tau_1,t_2)|^p \,(b_1-\tau_1)^{\alpha}\, d\tau_1\, dt_2 \\
&\leqslant \frac{c_1(b_1-a_1)^{\alpha}}{\Gamma(\alpha+1)} \left\|\varphi\right\|^p_{L^p} = \left( \frac{(b_1-a_1)^{\alpha}}{\Gamma(\alpha+1)} \right)^p \left\|\varphi\right\|^p_{L^p}
\end{aligned}$$

for any $\varphi \in L^p(P,\mathbb{R}^n)$, where $c_1$ is the constant from Lemma 2.4. ∎

DEFINITION 2.7. By $AC(t_1)$ ($AC(t_2)$) we denote the set of all functions $z : P \to \mathbb{R}^n$ such that

$$z(t_1,t_2) = \int_{a_1}^{t_1} l(\tau_1,t_2)\, d\tau_1 + p(t_2) \quad \text{for a.e. } (t_1,t_2) \in P$$

$$\left( z(t_1,t_2) = \int_{a_2}^{t_2} l(t_1,\tau_2)\, d\tau_2 + p(t_1) \quad \text{for a.e. } (t_1,t_2) \in P \right)$$

with $l \in L^1(P,\mathbb{R}^n)$ and $p \in L^1([a_2,b_2],\mathbb{R}^n)$ $\left(p \in L^1([a_1,b_1],\mathbb{R}^n)\right)$.

REMARK 2.8. From the above definition it follows that the function $z \in AC(t_1)$ ($z \in AC(t_2)$) is summable on $P$ and satisfies the condition $z(a_1,t_2) = p(t_2)$ for a.e. $t_2 \in [a_2,b_2]$ $\left(z(t_1,a_2) = p(t_1) \text{ for a.e. } t_1 \in [a_1,b_1]\right)$. Moreover, there exists the partial derivative $\frac{\partial z}{\partial t_1}$ ($\frac{\partial z}{\partial t_2}$) a.e. on $P$ and

$$\frac{\partial z}{\partial t_1}(t_1,t_2) = l(t_1,t_2), \quad (t_1,t_2) \in P \text{ a.e.}$$

$$\left( \frac{\partial z}{\partial t_2}(t_1,t_2) = l(t_1,t_2), \quad (t_1,t_2) \in P \text{ a.e.} \right).$$

REMARK 2.9. From the previous remark it follows that the representation of $z$ from Definition 2.7 is unique.

DEFINITION 2.10. Let $p \geq 1$. By $I^{\alpha}_{a_1+,t_1}(L^p)$ we shall denote the set

$$I^{\alpha}_{a_1+,t_1}(L^p) := \left\{ f : P \to \mathbb{R}^n : f(t) = (I^{\alpha}_{a_1+,t_1}\varphi)(t),\ t \in P \text{ a.e., where } \varphi \in L^p(P,\mathbb{R}^n) \right\}.$$

Analogously, one can define the sets $I^{\alpha}_{a_2+,t_2}(L^p)$ and $I^{\alpha}_{b_i-,t_i}(L^p)$, $i = 1, 2$.

Now, we shall prove the following

PROPOSITION 2.11. *Let $f \in L^1(P, \mathbb{R}^n)$, $\alpha > 0$ and $1 \leqslant p < +\infty$. Then*

$$f \in I^{\alpha}_{a_1+,t_1}(L^p) \iff I^{1-\alpha}_{a_1+,t_1} f \in AC^p(t_1) \quad \textit{and} \quad \big(I^{1-\alpha}_{a_1+,t_1} f\big)(a_1, \cdot) = 0,$$

*where* $AC^p(t_1) := \big\{h \in AC(t_1) : \frac{\partial h}{\partial t_1} \in L^p(P, \mathbb{R}^n)$ *and* $h(a_1, \cdot) \in L^p([a_2, b_2], \mathbb{R}^n)\big\}$.

*Proof.* Let us assume that $f \in I^{\alpha}_{a_1+,t_1}(L^p)$. Then there exists a function $\varphi \in L^p(P, \mathbb{R}^n)$ such that $f(t_1, t_2) = \big(I^{\alpha}_{a_1+,t_1} \varphi\big)(t_1, t_2)$ for a.e. $(t_1, t_2) \in P$. Thus from property (3) we obtain

$$\big(I^{1-\alpha}_{a_1+,t_1} f\big)(t_1, t_2) = \big(I^{1-\alpha}_{a_1+,t_1} I^{\alpha}_{a_1+,t_1} \varphi\big)(t_1, t_2) = \big(I^{1}_{a_1+,t_1} \varphi\big)(t_1, t_2) = \int_{a_1}^{t_1} \varphi(\tau_1, t_2)\, d\tau_1$$

for a.e. $(t_1, t_2) \in P$. Consequently, $I^{1-\alpha}_{a_1+,t_1} f \in AC^p(t_1)$ and $\big(I^{1-\alpha}_{a_1+,t_1} f\big)(a_1, \cdot) \equiv 0$ (cf. Remark 2.8).

Now, let

$$I^{1-\alpha}_{a_1+,t_1} f \in AC^p(t_1) \quad \text{and} \quad \big(I^{1-\alpha}_{a_1+,t_1} f\big)(a_1, \cdot) \equiv 0.$$

Then there exists (Remark 2.8) a function $\varphi \in L^p(P, \mathbb{R}^n)$ such that

$$\big(I^{1-\alpha}_{a_1+,t_1} f\big)(t_1, t_2) = \int_{a_1}^{t_1} \varphi(\tau_1, t_2)\, d\tau_1 = \big(I^{1}_{a_1+,t_1} \varphi\big)(t_1, t_2), \quad (t_1, t_2) \in P \text{ a.e.}$$

Using once again property (3), we get

$$\big(I^{1-\alpha}_{a_1+,t_1} f\big)(t_1, t_2) = \big(I^{1}_{a_1+,t_1} \varphi\big)(t_1, t_2) = \big(I^{1-\alpha}_{a_1+,t_1} I^{\alpha}_{a_1+,t_1} \varphi\big)(t_1, t_2), \quad (t_1, t_2) \in P \text{ a.e.},$$

and consequently

$$\big(I^{1-\alpha}_{a_1+,t_1} \big(f - \big(I^{\alpha}_{a_1+,t_1} \varphi\big)\big)\big)(t_1, t_2) = 0, \quad (t_1, t_2) \in P \text{ a.e.}$$

From Lemma 2.5 it follows that $f = I^{\alpha}_{a_1+,t_1} \varphi$ a.e. on $P$, so $f \in I^{\alpha}_{a_1+,t_1}(L^p)$. The proof is completed. ∎

Analogously we can prove the following proposition.

PROPOSITION 2.12. *Let $f \in L^1(P, \mathbb{R}^n)$ and $\alpha > 0$. Then*

$$f \in I^{\alpha}_{a_2+,t_2}(L^p) \iff I^{1-\alpha}_{a_2+,t_2} f \in AC^p(t_2) \quad \textit{and} \quad \big(I^{1-\alpha}_{a_2+,t_2} f\big)(\cdot, a_2) = 0,$$
$$f \in I^{\alpha}_{b_1-,t_1}(L^p) \iff I^{1-\alpha}_{b_1-,t_1} f \in AC^p(t_1) \quad \textit{and} \quad \big(I^{1-\alpha}_{b_1-,t_1} f\big)(b_1, \cdot) = 0,$$
$$f \in I^{\alpha}_{b_2-,t_2}(L^p) \iff I^{1-\alpha}_{b_2-,t_2} f \in AC^p(t_2) \quad \textit{and} \quad \big(I^{1-\alpha}_{b_2-,t_2} f\big)(\cdot, b_2) = 0.$$

DEFINITION 2.13. Let $\alpha \in (0, 1)$ and $f \in L^1(P, \mathbb{R}^n)$. We say that the function $f$ possesses *the left-sided Riemann–Liouville derivative $D^{\alpha}_{a_1+,t_1} f$ of order $\alpha$ with respect to variable* $t_1$, if $I^{1-\alpha}_{a_1+,t_1} f \in AC(t_1)$. By this derivative we mean the classical partial derivative $\frac{\partial}{\partial t_1}\big(I^{1-\alpha}_{a_1+,t_1} f\big)$ (existing a.e. on $P$) of the function $\big(I^{1-\alpha}_{a_1+,t_1} f\big)$, it means

$$\big(D^{\alpha}_{a_1+,t_1} f\big)(t_1, t_2) := \frac{\partial}{\partial t_1}\big(I^{1-\alpha}_{a_1+,t_1} f\big)(t_1, t_2), \quad (t_1, t_2) \in P \text{ a.e.}$$

Similarly, we say that the function $f$ possesses *the left-sided Riemann–Liouville derivative $D^{\alpha}_{a_2+,t_2} f$ of order $\alpha$ with respect to variable* $t_2$, if $I^{1-\alpha}_{a_2+,t_2} f \in AC(t_2)$. By this derivative we mean the classical partial derivative $\frac{\partial}{\partial t_2}\big(I^{1-\alpha}_{a_2+,t_2}f\big)$, it means

$$\big(D^{\alpha}_{a_2+,t_2}f\big)(t_1,t_2) := \frac{\partial}{\partial t_2}\big(I^{1-\alpha}_{a_2+,t_2}f\big)(t_1,t_2), \quad (t_1,t_2)\in P \text{ a.e.}$$

Remark 2.14. Analogously we define the right-sided Riemann–Liouville partial derivative of $\alpha$ of the function $f\in L^1(P,\mathbb{R}^n)$. Precisely

$$\big(D^{\alpha}_{b_i-,t_i}f\big)(t_1,t_2) := -\frac{\partial}{\partial t_i}\big(I^{1-\alpha}_{b_i-,t_i}f\big)(t_1,t_2), \quad (t_1,t_2)\in P \text{ a.e.},$$

provided $I^{1-\alpha}_{b_1-,t_1}f\in AC(t_1)$, when $i=1$ and $I^{1-\alpha}_{b_2-,t_2}f\in AC(t_2)$, when $i=2$.

Remark 2.15. From the above definition, Remark 2.8, Proposition 2.11 and its proof it follows that, if $\alpha\in(0,1)$ and $f\in I^{\alpha}_{a_i+,t_i}(L^p)$ $\big(f\in I^{\alpha}_{b_i-,t_i}(L^p)\big)$, $i=1,2$, then $f$ possesses the left-sided (right-sided) Riemann–Liouville derivative of order $\alpha$ $D^{\alpha}_{a_i+,t_i}f$ $\big(D^{\alpha}_{b_i-,t_i}f\big)$ and then $D^{\alpha}_{a_i+,t_i}f=\varphi$ $\big(D^{\alpha}_{b_i-,t_i}f=\psi\big)$ a.e. on $P$, where $\varphi$ $(\psi)$ is such that $f=I^{\alpha}_{a_i+,t_i}\varphi$ $\big(f=I^{\alpha}_{b_i-,t_i}\psi\big)$.

Now, let us define a norm in $I^{\alpha}_{a_1+,t_1}(L^p)$ $\big(I^{\alpha}_{b_1-,t_1}(L^p)\big)$ in the following way:

$$\|f\|_{I^{\alpha}_{a_1+,t_1}(L^p)} := \|D^{\alpha}_{a_1+,t_1}f\|_{L^p} \qquad \big(\|f\|_{I^{\alpha}_{b_1-,t_1}(L^p)} := \|D^{\alpha}_{b_1-,t_1}f\|_{L^p}\big).$$

Similarly, we can introduce a norm in $I^{\alpha}_{a_2+,t_2}(L^p)$, $I^{\alpha}_{b_2-,t_2}(L^p)$. It is easy to check that the spaces under consideration with the norms introduced above are completed.

Later on, we shall use a theorem being the counterpart of some facts ([KST, Lemmas 2.4, 2.5a, 2.6a]) for fractional partial derivatives.

Theorem 2.16. *Let $0<\alpha<1$, $1\leqslant p<\infty$ and $i=1,2$.*

1. *If $f\in L^p(P,\mathbb{R}^n)$, then*
$$\big(D^{\alpha}_{a_i+,t_i}I^{\alpha}_{a_i+,t_i}f\big)(t_1,t_2)=f(t_1,t_2) \quad \text{and} \quad \big(D^{\alpha}_{b_i-,t_i}I^{\alpha}_{b_i-,t_i}f\big)(t_1,t_2)=f(t_1,t_2)$$
*for a.e. $(t_1,t_2)\in P$;*
2. *if $f\in I^{\alpha}_{a_i+,t_i}(L^p)$, then*
$$\big(I^{\alpha}_{a_i+,t_i}D^{\alpha}_{a_i+,t_i}f\big)(t_1,t_2)=f(t_1,t_2), \quad (t_1,t_2)\in P \text{ a.e.};$$
3. *if $f\in I^{\alpha}_{b_i-,t_i}(L^p)$, then*
$$\big(I^{\alpha}_{b_i-,t_i}D^{\alpha}_{b_i-,t_i}f\big)(t_1,t_2)=f(t_1,t_2), \quad (t_1,t_2)\in P \text{ a.e.}$$

One can easy deduce the first property by using the definition of fractional partial derivative and conditions (3), (4). The other properties of this theorem follow immediately from the first property.

By using the above theorem and Lemma 2.6 we shall prove the following

Lemma 2.17. *Let $\alpha\in(0,1)$, $1\leqslant p<\infty$, $(x^k)_{k\in\mathbb{N}}\subset I^{\alpha}_{a_1+,t_1}(L^p)$ and $x^0\in I^{\alpha}_{a_1+,t_1}(L^p)$. If the sequence $(x^k)_{k\in\mathbb{N}}$ tends to $x^0$ in $I^{\alpha}_{a_1+,t_1}(L^p)$, then it tends to $x^0$ in $L^p(P,\mathbb{R}^n)$.*

*Proof.* Let $x^k \underset{k\to\infty}{\longrightarrow} x^0$ in $I^{\alpha}_{a_1+,t_1}(L^p)$. From Theorem 2.16 and Lemma 2.6, we get

$$\begin{aligned}\big\|x^k-x^0\big\|_{L^p} &= \big\|I^{\alpha}_{a_1+,t_1}D^{\alpha}_{a_1+,t_1}(x^k-x^0)\big\|_{L^p} \leqslant K_1\big\|D^{\alpha}_{a_1+,t_1}(x^k-x^0)\big\|_{L^p}\\ &= K_1\big\|x^k-x^0\big\|_{I^{\alpha}_{a_1+,t_1}(L^p)} \underset{k\to\infty}{\longrightarrow} 0.\end{aligned}$$

It means that $x^k \underset{k\to\infty}{\longrightarrow} x^0$ in $L^p(P,\mathbb{R}^n)$. ∎

Let $\alpha_i, \beta_i > 0$, $i = 1, 2$, $1 \leqslant p < \infty$, $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2)$, $a = (a_1, a_2)$, $b = (b_1, b_2)$. We define the set $I^{\alpha}_{a+}(L^p)(t_1, t_2)$ $\big(I^{\beta}_{b-}(L^p)(t_1, t_2)\big)$ in the following way:

$$I^{\alpha}_{a+}(L^p)(t_1, t_2) := I^{\alpha_1}_{a_1+,t_1}(L^p) \times I^{\alpha_2}_{a_2+,t_2}(L^p)$$
$$\big(I^{\beta}_{b-}(L^p)(t_1, t_2) := I^{\beta_1}_{b_1-,t_1}(L^p) \times I^{\beta_2}_{b_2-,t_2}(L^p)\big).$$

In $I^{\alpha}_{a+}(L^p)(t_1, t_2)$ $\big(I^{\beta}_{b-}(L^p)(t_1, t_2)\big)$ we introduce a norm in the following way:

$$\|f\|_{I^{\alpha}_{a+}(L^p)(t_1,t_2)} := \big\|(D^{\alpha_1}_{a_1+,t_1} f_1)\big\|_{L^p} + \big\|(D^{\alpha_2}_{a_2+,t_2} f_2)\big\|_{L^p}$$
$$\Big(\|f\|_{I^{\beta}_{b-}(L^p)(t_1,t_2)} := \big\|(D^{\beta_1}_{b_1-,t_1} f_1)\big\|_{L^p} + \big\|(D^{\beta_2}_{b_2-,t_2} f_2)\big\|_{L^p}\Big),$$

where $f = (f_1, f_2)$. From the fact that $I^{\alpha_i}_{a_i+,t_i}(L^p)$, $\big(I^{\beta_i}_{b_i-,t_i}(L^p)\big)$, $i = 1, 2$, are complete it follows that $I^{\alpha}_{a+}(L^p)(t_1, t_2)$ $\big(I^{\beta}_{b-}(L^p)(t_1, t_2)\big)$ are complete.

From Lemma 2.17 we obtain immediately

Lemma 2.18. *Let $\alpha_1, \alpha_2 \in (0, 1)$, $1 \leqslant p < \infty$, $(x^k)_{k\in\mathbb{N}} = (x^k_1, x^k_2)_{k\in\mathbb{N}} \subset I^{\alpha}_{a_1+,t_1}(L^p)(t_1, t_2)$ and $x^0 = (x^0_1, x^0_2) \in I^{\alpha}_{a_1+,t_1}(L^p)(t_1, t_2)$. If the sequence $(x^k)_{k\in\mathbb{N}}$ tends to $x^0$ in the space $I^{\alpha}_{a_1+,t_1}(L^p)(t_1, t_2)$, then it tends to $x^0$ in the space $L^p(P, \mathbb{R}^{2n})$.*

**3. The fractional Roesser control system.** In this section, we shall consider fractional control system (1).

By a solution to this problem we mean a function $x = (x_1, x_2) \in I^{\alpha}_{a+}(L^p)(t_1, t_2)$.

It is easy to see that the existence of a solution to system (1) in the set $I^{\alpha}_{a+}(L^p)(t_1, t_2)$ is equivalent to the existence of a solution to the system

$$\begin{cases} \varphi_1(t) = f_1\Big(t, \dfrac{1}{\Gamma(\alpha_1)} \displaystyle\int_{a_1}^{t_1} \dfrac{\varphi_1(\tau, t_2)}{(t_1 - \tau)^{1-\alpha_1}}\, d\tau, \dfrac{1}{\Gamma(\alpha_2)} \displaystyle\int_{a_2}^{t_2} \dfrac{\varphi_2(t_1, \tau)}{(t_2 - \tau)^{1-\alpha_2}}\, d\tau, u(t)\Big) \\ \varphi_2(t) = f_2\Big(t, \dfrac{1}{\Gamma(\alpha_1)} \displaystyle\int_{a_1}^{t_1} \dfrac{\varphi_1(\tau, t_2)}{(t_1 - \tau)^{1-\alpha_1}}\, d\tau, \dfrac{1}{\Gamma(\alpha_2)} \displaystyle\int_{a_2}^{t_2} \dfrac{\varphi_2(t_1, \tau)}{(t_2 - \tau)^{1-\alpha_2}}\, d\tau, u(t)\Big) \\ t \in P \text{ a.e.} \end{cases} \tag{5}$$

in the set $L^p(P, \mathbb{R}^{2n})$.

Indeed, if $x = (x_1, x_2) \in I^{\alpha}_{a+}(L^p)(t_1, t_2)$ is a solution to problem (1), then there exists a function $\varphi = (\varphi_1, \varphi_2) \in L^p(P, \mathbb{R}^{2n})$ such that

$$x_i(t) = \big(I^{\alpha_i}_{a_i+,t_i} \varphi_i\big)(t), \quad t \in P \text{ a.e.}, \ i = 1, 2,$$

and (cf. Theorem 2.16 p. 1)

$$\begin{aligned} \varphi_i(t) = \big(D^{\alpha_i}_{a_i+,t_i} I^{\alpha_i}_{a_i+,t_i} \varphi_i\big)(t) &= \big(D^{\alpha_i}_{a_i+,t_i} x_i\big)(t) = f_i(t, x_1(t), x_2(t), u(t)) \\ &= f_i\big(t, \big(I^{\alpha_1}_{a_1+,t_1} \varphi_1\big)(t), \big(I^{\alpha_2}_{a_2+,t_2} \varphi_2\big)(t), u(t)\big), \qquad i = 1, 2, \end{aligned}$$

for a.e. $t \in P$.

Conversely, if $\varphi = (\varphi_1, \varphi_2) \in L^p(P, \mathbb{R}^{2n})$ is a solution to problem (5), then $x = (x_1, x_2) = \big(I^{\alpha_1}_{a_1+,t_1} \varphi_1, I^{\alpha_2}_{a_2+,t_2} \varphi_2\big) \in I^{\alpha}_{a+}(L^p)(t_1, t_2)$ and (cf. Remark 2.15)

$$\big(D^{\alpha_i}_{a_i+,t_i} x_i\big)(t) = \varphi_i(t) = f_i\big(t, (I^{\alpha_1}_{a_1+,t_1} \varphi_1)(t), (I^{\alpha_2}_{a_2+,t_2} \varphi_2)(t), u(t)\big) = f_i(t, x_1(t), x_2(t), u(t))$$

for a.e. $t \in P$, $i = 1, 2$. Moreover, from Propositions 2.11 and 2.12 it follows that the boundary conditions (1.2) are satisfied.

**3.1. Existence and uniqueness of a solution.** Applying the Banach contraction principle, we shall prove a theorem on the existence and uniqueness of a solution $x = (x_1, x_2)$ of system (1) for any control $u \in L^p(P, M)$ $(1 \leqslant p < \infty)$.

THEOREM 3.1. *Let $\alpha_1, \alpha_2 \in (0,1)$ and $1 \leqslant p < \infty$. If (for $i = 1, 2$)*

1. *$f_i(\cdot, x_1, x_2, u)$ is measurable on $P$ for all $x_1, x_2 \in \mathbb{R}^n$, $u \in M$ and $f_i(t, x_1, x_2, \cdot)$ is continuous on $M$ for a.e. $t \in P$ and all $x_1, x_2 \in \mathbb{R}^n$;*
2. *there exists a constant $N > 0$ such that*
$$\big|f_i(t, x_1, x_2, u) - f_i(t, y_1, y_2, u)\big| \leqslant N\big(|x_1 - y_1| + |x_2 - y_2|\big)$$
*for a.e. $t \in P$ and all $x_1, x_2, y_1, y_2 \in \mathbb{R}^n$, $u \in M$;*
3. *there exist a function $r \in L^p(P, \mathbb{R}_0^+)$ and a constant $\gamma \geqslant 0$ such that*
$$|f_i(t, 0, 0, u)| \leqslant r(t) + \gamma |u|$$
*for a.e. $t \in P$ and all $u \in M$,*

*then problem* (1) *possesses a unique solution $x \in I_{a+}^{\alpha}(L^p)(t_1, t_2)$ corresponding to any control $u \in L^p(P, M)$.*

*Proof.* To prove this theorem it suffices to show that for any control $u \in L^p(P, M)$ there exists a unique fixed point of the operator $\Phi_u = \big(\Phi_u^1, \Phi_u^2\big) : L^p(P, \mathbb{R}^{2n}) \longrightarrow L^p(P, \mathbb{R}^{2n})$,
$$\Phi_u^i\big(\varphi_1(\cdot), \varphi_2(\cdot)\big) = f_i\big(\cdot, (I_{a_1+,t_1}^{\alpha_1}\varphi_1)(\cdot), (I_{a_2+,t_2}^{\alpha_2}\varphi_2)(\cdot), u(\cdot)\big), \quad i = 1, 2.$$
First, let us notice that the operator $\Phi_u$ is well defined. Indeed, from Lemma 2.4 it follows that $\big(I_{a_i+,t_i}^{\alpha_i}\varphi_i\big)(\cdot) \in L^p(P, \mathbb{R}^n)$ $(i = 1, 2)$. In particular, it means that the functions
$$P \ni t \mapsto f_i\big(t, \big(I_{a_1+,t_1}^{\alpha_1}\varphi_1\big)(t), \big(I_{a_2+,t_2}^{\alpha_2}\varphi_2\big)(t), u(t)\big) \in \mathbb{R}^n, \quad i = 1, 2,$$
are measurable. Moreover, by assumptions 2 and 3, for $i = 1, 2$, we have
$$\begin{aligned}
&\big|f_i\big(t, (I_{a_1+,t_1}^{\alpha_1}\varphi_1)(t), (I_{a_2+,t_2}^{\alpha_2}\varphi_2)(t), u(t)\big)\big|^p \\
&\leqslant 2^{p-1}\Big(\big|f_i\big(t, (I_{a_1+,t_1}^{\alpha_1}\varphi_1)(t), (I_{a_2+,t_2}^{\alpha_2}\varphi_2)(t), u(t)\big) - f_i(t, 0, 0, u(t))\big|^p + |f_i(t, 0, 0, u(t))|^p\Big) \\
&\leqslant 2^{p-1}N^p\big(\big|(I_{a_1+,t_1}^{\alpha_1}\varphi_1)(t)\big| + \big|(I_{a_2+,t_2}^{\alpha_2}\varphi_2)(t)\big|\big)^p + 2^{p-1}\big(r(t) + \gamma|u(t)|\big)^p \\
&\leqslant 2^{2p-2}N^p\big(\big|(I_{a_1+,t_1}^{\alpha_1}\varphi_1)(t)\big|^p + \big|(I_{a_2+,t_2}^{\alpha_2}\varphi_2)(t)\big|^p\big) + 2^{2p-2}\big(r(t)^p + \gamma^p|u(t)|^p\big)
\end{aligned}$$
for a.e. $t \in P$. It means that the operator $\Phi_u$ belongs to $L^p(P, \mathbb{R}^{2n})$.

Let us consider, in the space $L^p(P, \mathbb{R}^{2n})$, the Bielecki norm given by the formula
$$\|\varphi\|_k := \left(\int_P e^{-kp(t_1+t_2)}\, |\varphi(t_1, t_2)|_{\mathbb{R}^{2n}}^p \, dt_1\, dt_2\right)^{1/p}, \tag{6}$$
where $k > 0$ is any fixed constant. It is clear that
$$e^{-k(b_1+b_2)}\|\varphi\|_{L^p(P,\mathbb{R}^{2n})} \leqslant \|\varphi\|_k \leqslant e^{-k(a_1+a_2)}\|\varphi\|_{L^p(P,\mathbb{R}^{2n})}. \tag{7}$$
Consequently, the space $L^p(P, \mathbb{R}^{2n})$ with Bielecki norm is complete.

Now, we shall show that $\Phi_u$ is contracting in the space $L^p(P, \mathbb{R}^{2n})$ with norm (6). Indeed, using [KST, Lemma 2.7a], Lemma 2.4, the Fubini Theorem and assumption 2, we obtain

$$
\begin{aligned}
\big\|\Phi_u(\varphi)-\Phi_u(\psi)\big\|_k^p &= \int_P e^{-kp(t_1+t_2)}\Big|f\big(t,(I^{\alpha_1}_{a_1+,t_1}\varphi_1)(t),(I^{\alpha_2}_{a_2+,t_2}\varphi_2)(t),u(t)\big)\\
&\qquad - f\big(t,(I^{\alpha_1}_{a_1+,t_1}\psi_1)(t),(I^{\alpha_2}_{a_2+,t_2}\psi_2)(t),u(t)\big)\Big|^p_{\mathbb{R}^{2n}}\,dt_1\,dt_2\\
&\leqslant 2^{p-1}\int_P e^{-kp(t_1+t_2)}\Big(\Big|f_1\big(t,(I^{\alpha_1}_{a_1+,t_1}\varphi_1)(t),(I^{\alpha_2}_{a_2+,t_2}\varphi_2)(t),u(t)\big)\\
&\qquad - f_1\big(t,(I^{\alpha_1}_{a_1+,t_1}\psi_1)(t),(I^{\alpha_2}_{a_2+,t_2}\psi_2)(t),u(t)\big)\Big|^p\\
&\quad + \Big|f_2\big(t,(I^{\alpha_1}_{a_1+,t_1}\varphi_1)(t),(I^{\alpha_2}_{a_2+,t_2}\varphi_2)(t),u(t)\big)\\
&\qquad - f_2\big(t,(I^{\alpha_1}_{a_1+,t_1}\psi_1)(t),(I^{\alpha_2}_{a_2+,t_2}\psi_2)(t),u(t)\big)\Big|^p\Big)\,dt_1\,dt_2\\
&\leqslant 2^pN^p\int_P e^{-kp(t_1+t_2)}\big(\big|(I^{\alpha_1}_{a_1+,t_1}(\varphi_1-\psi_1))(t)\big|+\big|(I^{\alpha_2}_{a_2+,t_2}(\varphi_2-\psi_2))(t)\big|\big)^p\,dt_1\,dt_2\\
&\leqslant 2^{2p-1}N^p\int_P e^{-kp(t_1+t_2)}\big(\big|(I^{\alpha_1}_{a_1+,t_1}(\varphi_1-\psi_1))(t)\big|^p+\big|(I^{\alpha_2}_{a_2+,t_2}(\varphi_2-\psi_2))(t)\big|^p\big)\,dt_1\,dt_2\\
&\leqslant 2^{2p-1}N^pc\int_P e^{-kp(t_1+t_2)}\big(\big(I^{\alpha_1}_{a_1+,t_1}\,|\varphi_1-\psi_1|^p\big)(t)+\big(I^{\alpha_2}_{a_2+,t_2}\,|\varphi_2-\psi_2|^p\big)(t)\big)\,dt_1\,dt_2\\
&= 2^{2p-1}N^pc\bigg(\int_{a_2}^{b_2}\Big(\int_{a_1}^{b_1} e^{-kp(t_1+t_2)}\big(\overline{I}^{\alpha_1}_{a_1+,t_1}(|\varphi_1-\psi_1|^p(\cdot,t_2))\big)(t_1)\,dt_1\Big)\,dt_2\\
&\quad + \int_{a_1}^{b_1}\Big(\int_{a_2}^{b_2} e^{-kp(t_1+t_2)}\big(\overline{I}^{\alpha_2}_{a_2+,t_2}\big(|\varphi_2-\psi_2|^p(t_1,\cdot)\big)\big)(t_2)\,dt_2\Big)\,dt_1\bigg)\\
&= 2^{2p-1}N^pc\bigg(\int_{a_2}^{b_2}\Big(\int_{a_1}^{b_1} |\varphi_1-\psi_1|^p\,(t_1,t_2)\big(\overline{I}^{\alpha_1}_{b_1-,t_1}e^{-kp(\cdot+t_2)}\big)(t_1)\,dt_1\Big)\,dt_2\\
&\quad + \int_{a_1}^{b_1}\Big(\int_{a_2}^{b_2} |\varphi_2-\psi_2|^p\,(t_1,t_2)\big(\overline{I}^{\alpha_2}_{b_2-,t_2}e^{-kp(t_1+\cdot)}\big)(t_2)\,dt_2\Big)\,dt_1\bigg),
\end{aligned}
$$

where $c=\max_{i=1,2}\{c_i\}$ and $c_i$, $i=1,2$, are constants from Lemma 2.4 applied to the operators $I^{\alpha_1}_{a_1+,t_1}$, $I^{\alpha_2}_{a_2+,t_2}$. Here $\overline{I}^{\alpha_i}_{a_i+,t_i}$ is the left-sided integral operator of order $\alpha_i$ of a function of one variable $t_i$.

It is easy to calculate that

$$
\begin{aligned}
\big(\overline{I}^{\alpha_1}_{b_1-,t_1}e^{-kp(\cdot+t_2)}\big)(t_1) &= \frac{1}{\Gamma(\alpha_1)}\int_{t_1}^{b_1}\frac{e^{-kp(\tau+t_2)}}{(\tau-t_1)^{1-\alpha_1}}\,d\tau = \frac{1}{\Gamma(\alpha_1)}\int_0^{b_1-t_1}\frac{e^{-kp(w+t_1+t_2)}}{w^{1-\alpha_1}}\,dw\\
&= \frac{e^{-kp(t_1+t_2)}}{\Gamma(\alpha_1)}\int_0^{b_1-t_1} e^{-kpw}w^{\alpha_1-1}\,dw\\
&= \frac{e^{-kp(t_1+t_2)}}{kp\Gamma(\alpha_1)}\int_0^{kp(b_1-t_1)} e^{-r}r^{\alpha_1-1}\frac{1}{(kp)^{\alpha_1-1}}\,dr\\
&= \frac{e^{-kp(t_1+t_2)}}{(kp)^{\alpha_1}\Gamma(\alpha_1)}\int_0^{kp(b_1-t_1)} e^{-r}r^{\alpha_1-1}\,dr\\
&\leqslant \frac{e^{-kp(t_1+t_2)}}{(kp)^{\alpha_1}\Gamma(\alpha_1)}\int_0^{\infty} e^{-r}r^{\alpha_1-1}\,dr = e^{-kp(t_1+t_2)}(kp)^{-\alpha_1}.
\end{aligned}
$$

Similarly, we assert that

$$\big(\overline{I}^{\alpha_2}_{b_2-,t_2} e^{-kp(t_1+\cdot)}\big)(t_2) \leqslant e^{-kp(t_1+t_2)}(kp)^{-\alpha_2}.$$

Consequently,

$$\begin{aligned}
\|\Phi_u(\varphi) - \Phi_u(\psi)\|_k^p &\leqslant 2^{2p-1}N^p c\bigg(\int_P |\varphi_1 - \psi_1|^p\,(t) e^{-kp(t_1+t_2)}(kp)^{-\alpha_1}\,dt_1\,dt_2 \\
&\qquad + \int_P |\varphi_2 - \psi_2|^p\,(t) e^{-kp(t_1+t_2)}(kp)^{-\alpha_2}\,dt_1\,dt_2\bigg) \\
&\leqslant 2^{2p-1}N^p c\max\big\{(kp)^{-\alpha_1},(kp)^{-\alpha_2}\big\} \\
&\qquad \times \int_P e^{-kp(t_1+t_2)}\big(|\varphi_1(t)-\psi_1(t)|^p + |\varphi_2(t)-\psi_2(t)|^p\big)\,dt_1\,dt_2 \\
&\leqslant 2^{2p}N^p c\max\big\{(kp)^{-\alpha_1},(kp)^{-\alpha_2}\big\} \\
&\qquad \times \int_P e^{-kp(t_1+t_2)}\,|\varphi(t)-\psi(t)|^p_{\mathbb{R}^{2n}}\,dt_1\,dt_2 \\
&= 2^{2p}N^p c\max\big\{(kp)^{-\alpha_1},(kp)^{-\alpha_2}\big\}\|\varphi-\psi\|_k^p.
\end{aligned}$$

Let us notice that for sufficiently large $k$ the constant $4N\big(\max\limits_{i=1,2}\{c_i\}\max\limits_{i=1,2}\{(kp)^{-\alpha_i}\}\big)^{1/p}$ lies in $(0,1)$. It means that the operator $\Phi_u$ is contracting in the space $L^p(P,\mathbb{R}^{2n})$. Using the Banach contraction principle, we assert that this operator possesses a unique fixed point. ∎

**3.2. Continuous dependence of solutions on controls.** In this part of the paper, we shall prove a theorem on the continuous dependence of solutions of problem (1) on controls. We have

Theorem 3.2. *Let $\alpha_1,\alpha_2 \in (0,1)$, $1 \leqslant p < \infty$. If all assumptions of Theorem* 3.1 *are satisfied and the sequence of controls $(u^l)_{l\in\mathbb{N}}$ tends to $\tilde{u}$ in the space $L^p(P,M)$, then the sequence of corresponding solutions $(x^l)_{l\in\mathbb{N}} = (x_1^l,x_2^l)_{l\in\mathbb{N}}$ of system* (1) *tends to $\tilde{x} = (\tilde{x}_1,\tilde{x}_2)$ in the space $I^\alpha_{a+}(L^p)(t_1,t_2)$.*

*Proof.* Assume that the sequence $(u^l)_{l\in\mathbb{N}}$ tends to the function $\tilde{u}$ in the space $L^p(P,M)$. Using analogous arguments as in the proof of Theorem 3.1, one can show that for any fixed $k>0$ and all $l\in\mathbb{N}$

$$\begin{aligned}
\|\varphi^l - \tilde{\varphi}\|_k = \big\|\Phi^l(\varphi^l) - \tilde{\Phi}(\tilde{\varphi})\big\|_k &\leqslant \big\|\Phi^l(\varphi^l) - \Phi^l(\tilde{\varphi})\big\|_k + \big\|\Phi^l(\tilde{\varphi}) - \tilde{\Phi}(\tilde{\varphi})|\big\|_k \\
&\leqslant \mu_k\|\varphi^l - \tilde{\varphi}\|_k + \big\|\Phi^l(\tilde{\varphi}) - \tilde{\Phi}(\tilde{\varphi})\big\|_k,
\end{aligned}$$

where $\mu_k := 4N\big(\max_{i=1,2}\{c_i\}\max_{i=1,2}\big\{(kp)^{-\alpha_i}\big\}\big)^{1/p}$, $\tilde{\Phi},\Phi^l : L^p(P,\mathbb{R}^{2n}) \to L^p(P,\mathbb{R}^{2n})$,

$$\begin{aligned}
\tilde{\Phi} \ni \varphi(\cdot) = \big(\varphi_1(\cdot),\varphi_2(\cdot)\big) \mapsto \Big(&f_1\big(\cdot,\big(I^{\alpha_1}_{a_1+,t_1}\varphi_1\big)(\cdot),\big(I^{\alpha_2}_{a_2+,t_2}\varphi_2\big)(\cdot),\tilde{u}(\cdot)\big), \\
&f_2\big(\cdot,\big(I^{\alpha_1}_{a_1+,t_1}\varphi_1\big)(\cdot),\big(I^{\alpha_2}_{a_2+,t_2}\varphi_2\big)(\cdot),\tilde{u}(\cdot)\big)\Big),
\end{aligned}$$

$$\Phi^l \ni \varphi(\cdot) = \big(\varphi_1(\cdot), \varphi_2(\cdot)\big) \mapsto \Big(f_1\big(\cdot, \big(I^{\alpha_1}_{a_1+,t_1}\varphi_1\big)(\cdot), \big(I^{\alpha_2}_{a_2+,t_2}\varphi_2\big)(\cdot), u^l(\cdot)\big),$$
$$f_2\big(\cdot, \big(I^{\alpha_1}_{a_1+,t_1}\varphi_1\big)(\cdot), \big(I^{\alpha_2}_{a_2+,t_2}\varphi_2\big)(\cdot), u^l(\cdot)\big)\Big),$$

$\varphi^l$, $\tilde{\varphi}$ are fixed points of the operators $\Phi^l$, $\tilde{\Phi}$, respectively.

Consequently (using also inequality (7)), for sufficiently large $k > 0$, we obtain

$$\begin{aligned}\|\varphi^l - \tilde{\varphi}\|_k &\leqslant \frac{1}{1-\mu_k}\big\|\Phi^l(\tilde{\varphi}) - \tilde{\Phi}(\tilde{\varphi})\big\|_k \leqslant \frac{e^{-k(a_1+a_2)}}{1-\mu_k}\,\big\|\Phi^l(\tilde{\varphi}) - \tilde{\Phi}(\tilde{\varphi})\big\|_{L^p(P,\mathbb{R}^{2n})}\\
&= \frac{e^{-k(a_1+a_2)}}{1-\mu_k}\bigg(\int_P\Big|f\big(t, \big(I^{\alpha_1}_{a_1+,t_1}\tilde{\varphi_1}\big)(t), \big(I^{\alpha_2}_{a_2+,t_2}\tilde{\varphi_2}\big)(t), u^l(t)\big)\\
&\qquad - f\big(t, \big(I^{\alpha_1}_{a_1+,t_1}\tilde{\varphi_1}\big)(t), \big(I^{\alpha_2}_{a_2+,t_2}\tilde{\varphi_2}\big)(t), \tilde{u}(t)\big)\Big|^p_{\mathbb{R}^{2n}}\,dt_1\,dt_2\bigg)^{1/p}\end{aligned}$$

for all $l \in \mathbb{N}$ and additionally $\mu_k \in (0,1)$. Moreover, using the Lebesgue dominated convergence theorem, one can prove that

$$\int_P\Big|f\big(t, \big(I^{\alpha_1}_{a_1+,t_1}\tilde{\varphi_1}\big)(t), \big(I^{\alpha_2}_{a_2+,t_2}\tilde{\varphi_2}\big)(t), u^l(t)\big)$$
$$- f\big(t, \big(I^{\alpha_1}_{a_1+,t_1}\tilde{\varphi_1}\big)(t), \big(I^{\alpha_2}_{a_2+,t_2}\tilde{\varphi_2}\big)(t), \tilde{u}(t)\big)\Big|^p_{\mathbb{R}^{2n}}\,dt_1\,dt_2 \underset{l\to\infty}{\longrightarrow} 0.$$

It means that $\|\varphi^l - \varphi_0\|_k \underset{l\to\infty}{\longrightarrow} 0$. Since

$$\begin{aligned}&\|x^l - \tilde{x}\|_{I^\alpha_{a+}(L^p)(t_1,t_2)} = \big\|x^l_1 - \tilde{x_1}\big\|_{I^{\alpha_1}_{a_1+,t_1}(L^p)} + \big\|x^l_2 - \tilde{x}_2\big\|_{I^{\alpha_2}_{a_2+,t_2}(L^p)}\\
&= \big\|I^{\alpha_1}_{a_1+,t_1}\varphi^l_1 - I^{\alpha_1}_{a_1+,t_1}\tilde{\varphi}_1\big\|_{I^{\alpha_1}_{a_1+,t_1}(L^p)} + \big\|I^{\alpha_2}_{a_2+,t_2}\varphi^l_2 - I^{\alpha_2}_{a_2+,t_2}\tilde{\varphi}_2\big\|_{I^{\alpha_2}_{a_2+,t_2}(L^p)}\\
&= \big\|D^{\alpha_1}_{a_1+,t_1}I^{\alpha_1}_{a_1+,t_1}\varphi^l_1 - D^{\alpha_1}_{a_1+,t_1}I^{\alpha_1}_{a_1+,t_1}\tilde{\varphi}_1\big\|_{L^p} + \big\|D^{\alpha_2}_{a_2+,t_2}I^{\alpha_2}_{a_2+,t_2}\varphi^l_2 - D^{\alpha_2}_{a_2+,t_2}I^{\alpha_2}_{a_2+,t_2}\tilde{\varphi}_2\big\|_{L^p}\\
&= \big\|\varphi^l_1 - \tilde{\varphi}_1\big\|_{L^p} + \big\|\varphi^l_2 - \tilde{\varphi}_2\big\|_{L^p} \leqslant 2\big\|\varphi^l - \tilde{\varphi}\big\|_{L^p(P,\mathbb{R}^{2n})} \leqslant 2e^{k(b_1+b_2)}\big\|\varphi^l - \tilde{\varphi}\big\|_k,\end{aligned}$$

the proof is completed. ■

**4. Existence of an optimal solution of some optimal control problem connected with the fractional Roesser model.** Let us consider the following fractional optimal control problem

$$\begin{cases}\begin{cases}\big(D^{\alpha_1}_{a_1+,t_1}x_1\big)(t) = A_1(t)x_1(t) + A_2(t)x_2(t) + B_1(t)u(t)\\ \big(D^{\alpha_2}_{a_2+,t_2}x_2\big)(t) = A_3(t)x_1(t) + A_4(t)x_2(t) + B_2(t)u(t), \qquad (8.1)\\ t = (t_1,t_2) \in P = [a_1,b_1]\times[a_2,b_2] \text{ a.e.}\end{cases}\\ \begin{cases}\big(I^{1-\alpha_1}_{a_1+,t_1}x_1\big)(a_1,t_2) = 0, \quad t_2 \in [a_2,b_2] \text{ a.e.}\\ \big(I^{1-\alpha_2}_{a_2+,t_2}x_2\big)(t_1,a_2) = 0, \quad t_1 \in [a_1,b_1] \text{ a.e.,}\end{cases} \qquad (8.2)\\ u(t) \in M \subset \mathbb{R}^m, \quad t \in P,\\ J(x_1,x_2,u) = \int_P f_0(t, x_1(t), x_2(t), u(t))\,dt,\end{cases} \tag{8}$$

where $A_j : P \to \mathbb{R}^{n\times n}$, $j = 1,\dots,4$, $B_k : P \to \mathbb{R}^{n\times m}$, $k = 1,2$, $f_0 : P\times\mathbb{R}^n\times\mathbb{R}^n\times M \to \mathbb{R}$, $x = (x_1,x_2)$ and $\alpha_i \in (0,1)$ for $i = 1,2$.

Let us fix $p \in (1,\infty)$ and let $\mathcal{U}_M := \big\{u \in L^p(P,\mathbb{R}^m) : u(t) \in M,\ t \in P\big\}$.

Definition 4.1. We say that a pair $(x^*, u^*) \in I^{\alpha}_{a+}(L^p)(t_1, t_2) \times \mathcal{U}_M$ is an *optimal solution of problem* (8), if $x^*$ is a solution of system (8.1)–(8.2) corresponding to control $u^*$ and

$$J(x^*, u^*) \leqslant J(x, u)$$

for all pairs $(x, u) \in I^{\alpha}_{a+}(L^p)(t_1, t_2) \times \mathcal{U}_M$ satisfying (8.1), (8.2).

Remark 4.2. From Theorem 3.1 it follows that, if the functions $A_j$, $B_k$, $j = 1, \dots, 4$, $k = 1, 2$, are essentially bounded, then there exists a unique solution $x^u$ to system (8.1)–(8.2) corresponding to any control $u$.

In the next theorem, we shall use the following lemma (cf. [Maw]).

Lemma 4.3. *Let $\mathcal{U}$ be a convex, closed and bounded subset of a reflexive Banach space. If the functional $F : \mathcal{U} \longrightarrow \mathbb{R}$ is convex and lower semicontinuous on $\mathcal{U}$, then there exists an element $u_* \in \mathcal{U}$ such that*

$$F(u_*) \leqslant F(u)$$

*for any $u \in \mathcal{U}$.*

Now, we shall prove a theorem on the existence of an optimal solution to problem (8).

Theorem 4.4. *Let assume that*

1. *the set $M$ is convex and compact,*
2. *$f_0(\cdot, x_1, x_2, u)$ is measurable on $P$ for all $x_1, x_2 \in \mathbb{R}^n$, $u \in M$, $f_0(t, \cdot, \cdot, \cdot)$ is continuous on $\mathbb{R}^n \times \mathbb{R}^n \times M$ for a.e. $t \in P$,*
3. *$f_0(t, \cdot, \cdot, \cdot)$ is convex on $\mathbb{R}^n \times \mathbb{R}^n \times M$ for a.e. $t \in P$,*
4. *the functions $A_j$, $B_k$, $j = 1, \dots, 4$, $k = 1, 2$, are essentially bounded,*
5. *there exist a function $a \in L^1(P, \mathbb{R})$ and a constant $\gamma_1 \geqslant 0$ such that*

   $$|f_0(t, x_1, x_2, u)| \leqslant a(t) + \gamma_1 \big(|x_1|^p + |x_2|^p\big)$$

   *for a.e. $t \in P$ and all $x_1, x_2 \in \mathbb{R}^n$, $u \in M$.*

*Then problem* (8) *possesses an optimal solution $(x^*, u^*) \in I^{\alpha}_{a+}(L^p)(t_1, t_2) \times \mathcal{U}_M$.*

*Proof.* Using the same arguments as in the proof of Theorem 3 in the paper [I1], we assert that the existence of an optimal solution to problem (8) in the space $I^{\alpha}_{a+}(L^p)(t_1, t_2) \times \mathcal{U}_M$ is equivalent to the existence of an optimal solution to system (8.1)–(8.2) with the cost functional

$$\tilde{J} : \mathcal{U}_M \ni u \mapsto \int_P f_0\big(t, x^u_1(t), x^u_2, u(t)\big)\, dt,$$

in the space $\mathcal{U}_M$.

Moreover, analogously as in the mentioned paper, we obtain that $\mathcal{U}_M$ is the convex, closed and bounded subset of the reflexive Banach space $L^p(P, \mathbb{R}^m)$ and the functional $\tilde{J}$ is convex.

Finally, we shall prove that the functional $\tilde{J}$ is continuous on $\mathcal{U}_M$. Indeed, let $u^k \underset{k\to\infty}{\longrightarrow} u^0$ in $\mathcal{U}_M$. Theorem 3.2 implies that $x^k \underset{k\to\infty}{\longrightarrow} x^0$ in $I^{\alpha}_{a+}(L^p)(t_1, t_2)$. From Lemma 2.18 it follows that $x^k \underset{k\to\infty}{\longrightarrow} x^0$ in $L^p(P, \mathbb{R}^{2n})$. To show the convergence

$$f_0\big(\cdot, x^k(\cdot), u^k(\cdot)\big) \underset{k\to\infty}{\longrightarrow} f_0\big(\cdot, x^0(\cdot), u^0(\cdot)\big)$$

it suffices to use [IR, Theorem 2] with the spaces $L_1 = L^p(P, \mathbb{R}^{2n})$, $L_2 = \mathcal{U}_M$, $L = L_1 \times L_2$ and the functional $f_0$ [1]. From [B, Theorem.4.9] and by assumption 5 it follows that $f_0$ satisfies assumptions of Theorem 2 from paper [IR]. So, from this theorem it follows that

$$f_0\big(\cdot, x^k(\cdot), u^k(\cdot)\big) \underset{k\to\infty}{\longrightarrow} f_0\big(\cdot, x^0(\cdot), u^0(\cdot)\big).$$

Consequently, from Lemma 4.3 we get the existence of an optimal solution $(x^*, u^*) \in I^{\alpha}_{a+}(L^p)(t_1, t_2) \times \mathcal{U}_M$ to problem (8). ∎

## 5. Example

EXAMPLE 5.1. Let us consider problem (8) with the following data:

$$A_1 = A_2 = A_4 = 0,\ A_3 = 1,\ B_1 = 1,\ B_2 = -1,$$

$$\begin{aligned} &f_0(t, x, u) = f_0(t_1, t_2, x_1, x_2, u) \\ &= x_1 - 2x_2 + \left(\frac{2}{\Gamma(7/4)\Gamma(3/2)}(1-t_2)^{3/4}(1-t_1)^{1/2} - \frac{2}{\Gamma(7/4)}(1-t_2)^{3/4} - \frac{1}{\Gamma(1/2)}\right)u, \end{aligned}$$

$$M = [0,1], \quad P = [0,1]\times[0,1], \quad \alpha_1 = \frac{1}{2}, \quad \alpha_2 = \frac{3}{4}, \quad p = 2.$$

Let

$$Z = \big\{t = (t_1, t_2) : \tfrac{3}{4} \leqslant t_1 \leqslant 1 \text{ and } t_2 \in [0,1]\big\}.$$

One can show (see [K1]) that $(x^*, u^*)$, where

$$u^*(t) = \begin{cases} 1, & t \in Z \\ 0, & t \in P \setminus Z, \end{cases}$$

$$x^*(t) = \begin{bmatrix} x_1^*(t) \\ x_2^*(t) \end{bmatrix} = \begin{cases} \begin{bmatrix} \frac{1}{\Gamma(3/2)}\left(t_1 - \frac{3}{4}\right)^{1/2} \\ \frac{1}{\Gamma(3/2)\Gamma(7/4)}\left(t_1 - \frac{3}{4}\right)^{1/2} t_2^{3/4} - \frac{1}{\Gamma(7/4)} t_2^{3/4} \end{bmatrix}, & t \in Z \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, & t \in P \setminus Z. \end{cases}$$

for a.e. $t \in P$, is the only pair, which can be an optimal solution to problem (8). It is easy to check that all assumptions of Theorem 4.4 are satisfied. Consequently, $(x^*, u^*)$ is an optimal solution to problem (8).

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[1] In the cited theorem, the Carathéodory function $f : \Omega \times \mathbb{R}^{l_1} \times \ldots \times \mathbb{R}^{l_k} \to \mathbb{R}^m$ is considered. This theorem is also true, if we consider a function $f : \Omega \times M_1 \times \ldots \times M_k \to \mathbb{R}^m$, where $M_i \subset \mathbb{R}^{l_i}$, $i = 1, \ldots, k$ (in our proof we apply the case $k = 2$).

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