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A NOTE ON AN APPROXIMATIVE SCHEME OF FINDING ALMOST HOMOCLINIC SOLUTIONS FOR NEWTONIAN SYSTEMS

ROBERT KRAWCZYK

Faculty of Applied Physics and Mathematics, Gdańsk University of Technology
G. Narutowicza 11/12, 80-233 Gdańsk, Poland
E-mail: rkrawczyk@mif.pg.gda.pl, robertkra@o2.pl

Abstract. In this work we will be concerned with the existence of almost homoclinic solutions for a Newtonian system $\ddot{q}+\nabla_q V(t,q)=f(t)$, where $t\in\mathbb{R}$, $q\in\mathbb{R}^n$. It is assumed that a potential $V:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ is C^1-smooth and its gradient map $\nabla_q V:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$ is bounded with respect to t. Moreover, a forcing term $f:\mathbb{R}\to\mathbb{R}^n$ is continuous, bounded and square integrable. We will show that the approximative scheme due to J. Janczewska (see [J2]) for a time periodic potential extends to our case.

1. Introduction. This paper is devoted to the study of the existence of almost homoclinic solutions for a perturbed Newtonian system

$$\ddot{q}+\nabla_q V(t,q)=f(t), \tag{1}$$

where $t\in\mathbb{R}$ and $q\in\mathbb{R}^n$. If $0\in\mathbb{R}^n$ is not a stationary point of (1) (for example, if $\nabla_q V(t,0)=0$ for all $t\in\mathbb{R}$ and $f\neq 0$, which usually takes place in applications) then (1) does not possess homoclinics to 0 in the classical meaning. Nevertheless we can study the existence of a solution $q:\mathbb{R}\to\mathbb{R}^n$ vanishing at $\pm\infty$. Furthermore, under suitable conditions on V, $\dot{q}(t)\to 0$ as $t\to\pm\infty$ (see [J1, IJ3]). In many papers concerning Newtonian systems, although $0\in\mathbb{R}^n$ is not a stationary point, their solutions satisfying $(q(t),\dot{q}(t))\to(0,0)$ are said to be homoclinic to 0 (see for instance [S, TX2, ZY]).

We introduce the notion of almost homoclinics (to 0), following J. Janczewska (see [J1, J2, IJ3]).

Definition 1.1. A solution $q:\mathbb{R}\to\mathbb{R}^n$ of the system (1) is called *almost homoclinic* if

$$q(\pm\infty)=\lim_{t\to\pm\infty} q(t)=0.$$

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In [J2] J. Janczewska studied the existence of almost homoclinic solutions of the Newtonian system (1) under the following hypotheses:

(C1) $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is C^1-smooth with respect to all variables and T-periodic with respect to t, $T > 0$,

(C2) $f : \mathbb{R} \to \mathbb{R}^n$ is nontrivial, bounded, continuous and square integrable.

In order to briefly sketch her approximative method we set up notation. For simplicity, T will be set equal to 1. From now on, $(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ denotes the standard inner product in $\mathbb{R}^n$ and $|\cdot| : \mathbb{R}^n \to [0,\infty)$ is the induced norm. Let E denote the Sobolev space $W^{1,2}(\mathbb{R},\mathbb{R}^n)$ of functions on $\mathbb{R}$ with values in $\mathbb{R}^n$ equipped with the norm

$$\|q\|_E = \Big(\int_{-\infty}^{\infty} \big(|q(t)|^2 + |\dot{q}(t)|^2\big)\, dt\Big)^{1/2}.$$

For each $k \in \mathbb{N}$, set $E_k = W^{1,2}_{2k}(\mathbb{R},\mathbb{R}^n)$, the Sobolev space of $2k$-periodic functions under the norm

$$\|q\|_{E_k} = \Big(\int_{-k}^{k} \big(|q(t)|^2 + |\dot{q}(t)|^2\big)\, dt\Big)^{1/2}.$$

Finally, let $C^m_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^n)$, where $m \in \mathbb{N} \cup \{0\}$, denote the space of C^m functions under the topology of almost uniformly convergence of functions and all derivatives up to the order m. Let us consider the sequence of periodic boundary value problems

$$\begin{cases} \ddot{q}(t) + \nabla_q V(t,q(t)) = f_k(t) \\ q(-k) - q(k) = \dot{q}(-k) - \dot{q}(k) = 0, \end{cases} \tag{2}$$

where for each $k \in \mathbb{N}$, $f_k : \mathbb{R} \to \mathbb{R}^n$ is a $2k$-periodic extension of the restriction of f to the interval $[-k,k)$.

THEOREM 1.2. *Let V and f satisfy* (C1) *and* (C2). *Assume also that for each $k \in \mathbb{N}$ the boundary value problem* (2) *has a solution $q_k \in E_k$ and $\{\|q_k\|_{E_k}\}_{k\in\mathbb{N}}$ is a bounded sequence in $\mathbb{R}$. Then there exists a subsequence $\{q_{k_j}\}_{j\in\mathbb{N}}$ converging in the topology of $C^2_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^n)$ to a function $q \in E$ which is an almost homoclinic solution of the Newtonian system* (1).

Theorem 1.2 provides the approximative method of finding almost homoclinics for (1). The original system (1) is approximated by time periodic ones (2), with larger and larger time periods. The proof relies mainly on the Ascoli-Arzelà lemma. The idea to get a homoclinic solution as a limit of periodic ones goes back at least as far as [R], where Paul H. Rabinowitz studied unperturbed Newtonian systems

$$\ddot{q} + \nabla_q V(t,q) = 0,$$

$t \in \mathbb{R}$, $q \in \mathbb{R}^n$, with periodic potentials of the form

$$V(t,q) = -\tfrac{1}{2}(L(t)q,q) + W(t,q).$$

Besides time periodicity condition on L and W, he assumed that L is a continuous matrix valued function such that $L(t)$ is positive definite and symmetric for each $t \in \mathbb{R}$, W is C^1-smooth, satisfies the superquadratic growth condition by Ambrosetti and Rabinowitz, and $\nabla_q W(t,q) = o(|q|)$ as $|q| \to 0$ uniformly in t. In [IJ1] M. Izydorek and J. Janczewska extended his result to perturbed Newtonian systems with a nonperiodic forcing term f and V of the form

$$V(t,q) = -K(t,q) + W(t,q),$$

where W as above and K is a C^1-smooth time periodic potential satisfying the so-called pinching condition. In [IJ2] they applied the same approximative scheme to another class of potentials (generally speaking, coercive ones). This gave a motivation to formulate and prove an abstract result which is Theorem 1.2. Up till now this method has been applied many times (see for instance [TX1, TX2, ZY] and citations for [IJ1] and [IJ2] in MathSciNet). Moreover, the method was also adapted to second order discrete Hamiltonian systems (see [TLX]), Liénard type systems (see [Z]), second order differential systems with p-Laplacian operator (see [L, TX3]) and mixed type functional differential equations (see [J3]).

Our aim is to extend Theorem 1.2 to a wider class of potentials. Namely, we show that (C1) may be replaced by

(C3) V is C^1-smooth and $\nabla_q V$ is bounded with respect to a time variable,

which needs only a slight change in the proof. We will consider the sequence of periodic boundary value problems

$$\begin{cases} \ddot{q}(t) + \nabla_q V_k(t, q(t)) = f_k(t) \\ q(-k) - q(k) = \dot{q}(-k) - \dot{q}(k) = 0, \end{cases} \tag{3}$$

where for each $k \in \mathbb{N}$, f_k is as above and $V_k : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a $2k$-periodic extension of $V : [-k, k) \times \mathbb{R}^n \to \mathbb{R}$.

We will prove the following theorem.

Main Theorem 1.3. *Let f and V satisfy* (C2) *and* (C3). *Assume also that for each $k \in \mathbb{N}$ the boundary value problem* (3) *possesses a periodic solution $q_k \in E_k$ and $\{\|q_k\|_{E_k}\}_{k\in\mathbb{N}}$ is a bounded sequence in $\mathbb{R}$. Then there exists a subsequence $\{q_{k_j}\}_{j\in\mathbb{N}}$ going in the topology of $C^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n)$ to a function $q \in E$ which is an almost homoclinic solution of the Newtonian system* (1).

2. The proof of Main Theorem. This section will be devoted to the proof of Theorem 1.3. At the beginning we review two standard inequalities.

Let $L^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n)$ denote the space of functions on $\mathbb{R}$ with values in $\mathbb{R}^n$ locally square integrable.

Fact 2.1 (see [IJ1, Fact 2.8]). *Let $q : \mathbb{R} \to \mathbb{R}^n$ be a continuous mapping such that $\dot{q} \in L^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n)$. Then for each $t \in \mathbb{R}$,*

$$|q(t)| \leq \sqrt{2}\left(\int_{t-1/2}^{t+1/2} \left(|q(s)|^2 + |\dot{q}(s)|^2\right) ds\right)^{1/2}. \tag{4}$$

Let $L^\infty_{2k}(\mathbb{R}, \mathbb{R}^n)$ be the space of $2k$-periodic essentially bounded measurable functions on $\mathbb{R}$ with values in $\mathbb{R}^n$ under the norm

$$\|q\|_{L^\infty_{2k}} = \operatorname{ess\,sup}\{|q(t)| : t \in [-k, k]\}.$$

From the estimate (4) we conclude that for each $k \in \mathbb{N}$ and $q \in E_k$,

$$\|q\|_{L^\infty_{2k}} \le \sqrt{2}\,\|q\|_{E_k} \tag{5}$$

(see [IJ1, Proposition 1.1]).

We have divided the proof of Theorem 1.3 into two lemmas.

Lemma 2.2. *Let f and V satisfy* (C2) *and* (C3). *Assume that for each $k \in \mathbb{N}$ the boundary value problem* (3) *has a solution $q_k \in E_k$. If $\{\|q_k\|_{E_k}\}_{k\in\mathbb{N}}$ is a bounded sequence in $\mathbb{R}$ then there exist a subsequence $\{q_{k_j}\}_{j\in\mathbb{N}}$ and a function $q \in E$ such that*

$$q_{k_j} \to q \quad as \quad j \to \infty$$

in $C^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n)$.

Proof. Our first goal is to show that $\{q_k\}_{k\in\mathbb{N}}$, $\{\dot q_k\}_{k\in\mathbb{N}}$ and $\{\ddot q_k\}_{k\in\mathbb{N}}$ are uniformly bounded sequences in $L^\infty_{2k}(\mathbb{R}, \mathbb{R}^n)$. By assumption, there is $M > 0$ such that for all $k \in \mathbb{N}$ we have

$$\|q_k\|_{E_k} \le M. \tag{6}$$

Combining (5) with (6) we get

$$\|q_k\|_{L^\infty_{2k}} \le \sqrt{2}\,M \equiv M_1. \tag{7}$$

By (C2) and (C3), it may be concluded that there is $M_2 > 0$ such that for each $k \in \mathbb{N}$ and $t \in [-k, k)$,

$$|\ddot q_k(t)| \le |\nabla_q V_k(t, q_k(t))| + |f_k(t)| = |\nabla_q V(t, q_k(t))| + |f(t)| \le M_2,$$

and, in consequence, for each $k \in \mathbb{N}$,

$$\|\ddot q_k\|_{L^\infty_{2k}} \le M_2. \tag{8}$$

Let q^l_k $(l = 1, 2, \ldots, n)$ stand for the l-th coordinate of the mapping q_k. Using the Mean Value Theorem, for each $k \in \mathbb{N}$, $l = 1, 2, \ldots, n$, and $t \in \mathbb{R}$ there is $t^l_k \in [t-1, t]$ such that

$$\dot q^l_k(t^l_k) = \int_{t-1}^{t} \dot q^l_k(s)\,ds = q^l_k(t) - q^l_k(t-1).$$

Moreover, we have

$$\dot q^l_k(t) = \int_{t^l_k}^{t} \ddot q^l_k(s)\,ds + \dot q^l_k(t^l_k).$$

Applying (7) and (8), we obtain

$$|\dot q^l_k(t)| \le \int_{t-1}^{t} |\ddot q^l_k(s)|\,ds + |q^l_k(t) - q^l_k(t-1)| \le M_2 + 2M_1,$$

and finally,

$$\|\dot q_k\|_{L^\infty_{2k}} \le \sqrt{n}\,(M_2 + 2M_1) \equiv M_3. \tag{9}$$

To finish the proof, it is sufficient to show that $\{q_k\}_{k\in\mathbb{N}}$ and $\{\dot q_k\}_{k\in\mathbb{N}}$ are equicontinuous.

Fix $k \in N$ and $t_1, t_2 \in \mathbb{R}$. The estimates (9) and (8) now lead to

$$|q_k(t_2) - q_k(t_1)| = \left|\int_{t_1}^{t_2} \dot q_k(s)\,ds\right| \le M_3|t_2 - t_1|$$

and

$$|\dot{q}_k(t_2) - \dot{q}_k(t_1)| = \left|\int_{t_1}^{t_2} \ddot{q}_k(s)\,ds\right| \leq M_2|t_2 - t_1|.$$

Note that we have actually proved that $\{q_k\}_{k\in\mathbb{N}}$ and $\{\dot{q}_k\}_{k\in\mathbb{N}}$ satisfy the Lipschitz condition with constants independent of k. Hence they are equicontinuous. Using the Arzeli–Ascolà lemma we receive the claim. ■

Lemma 2.3. *Let f and V satisfy* (C2) *and* (C3)*. The function q given by Lemma* 2.2 *is an almost homoclinic solution of the Newtonian system* (1) *and $q_{k_j} \to q$ as $j \to \infty$ in the topology of $C^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n)$.*

The proof is the same as that of Lemma 2.5 in [J2], therefore we omit it.

3. Applications. One can ask how strong the assumption of the existence of an approximative sequence $\{q_k\}_{k\in\mathbb{N}}$ both in Theorem 1.2 and Theorem 1.3 is.

It is evident that to apply Theorem 1.2 and Theorem 1.3 we need suitable assumptions on the potential and the external force besides (C1)–(C2) or (C2)–(C3), respectively. By suitable we mean assumptions implying the existence of an approximative sequence $\{q_k\}_{k\in\mathbb{N}}$. For a treatment of perturbed Newtonian systems with time periodic potentials, possessing approximative sequences we refer the reader to [IJ1, IJ2, IJ3, J2, TX1, TX2, ZY]. It is worth pointing out that in all these papers $\{q_k\}_{k\in\mathbb{N}}$ is obtained by variational methods (the Mountain Pass Lemma or standard minimizing arguments).

In this section we present an example of a class of perturbed Newtonian systems without periodic potentials, having approximative sequences.

Consider the Newtonian system

$$\ddot{q} - \nabla_q V(t, q) = f(t), \tag{10}$$

where $t \in \mathbb{R}$, $q \in \mathbb{R}^n$, and $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}^n$ satisfy the hypotheses (C2)–(C3). Moreover,

(C4) $V(t, q) \geq b(t)|q|^2$ for all $t \in \mathbb{R}$ and $q \in \mathbb{R}^n$, where $b : \mathbb{R} \to (0, \infty)$ is a continuous function that achieves a minimum on $\mathbb{R}$,

(C5) $V(t, 0) = 0$ for each $t \in \mathbb{R}$.

Theorem 3.1. *Let V and f satisfy* (C2)–(C5)*. Then the Newtonian system* (10) *possesses an almost homoclinic solution.*

We will prove Theorem 3.1 by using Theorem 1.3. The approximative sequence of periodic boundary value problems for the Newtonian system (10) takes the form

$$\begin{cases} \ddot{q}(t) - \nabla_q V_k(t, q(t)) = f_k(t) \\ q(-k) - q(k) = \dot{q}(-k) - \dot{q}(k) = 0, \end{cases} \tag{11}$$

where for all $k \in \mathbb{N}$, $f_k : \mathbb{R} \to \mathbb{R}^n$ and $V_k : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are $2k$-periodic extensions of $f : [-k, k) \to \mathbb{R}^n$ and $V : [-k, k) \times \mathbb{R}^n \to \mathbb{R}$, respectively.

For all $k \in \mathbb{N}$, let the functional $I_k : E_k \to \mathbb{R}$ be given by

$$I_k(q) = \int_{-k}^{k} \Big(\frac{1}{2}|\dot{q}(t)|^2 + V_k(t, q(t)) + (f_k(t), q(t))\Big)\,dt.$$

One knows that for a fixed $k \in \mathbb{N}$ critical points of the functional I_k are $2k$-periodic solutions of (11). To prove that I_k achieves a minimum on E_k we apply a classical result of the calculus of variations.

THEOREM 3.2 (see [MW, Theorem 1.1]). *If $\varphi : X \to \mathbb{R}$ is a weakly lower semicontinuous functional on a reflexive Banach space X and has a bounded minimizing sequence, then φ has a minimum on X.*

Proof of Theorem 3.1. Let $B = \min_{t\in\mathbb{R}} b(t)$, $A = \min\{\frac{1}{2}, B\}$ and $L = \|f\|_{L^2(\mathbb{R},\mathbb{R}^n)}$. Applying (C4) and the Schwarz inequality we get

$$I_k(q) \geq \int_{-k}^{k} \Big(\frac{1}{2}|\dot{q}(t)|^2 + b(t)|q(t)|^2 + (f_k(t), q(t))\Big)\, dt \geq A\|q\|^2_{E_k} - L\|q\|_{E_k}.$$

Thus I_k is bounded from below and coercive.

Let $L^2_{2k}(\mathbb{R},\mathbb{R}^n)$ be the space of $2k$-periodic square integrable functions on $\mathbb{R}$ with values in $\mathbb{R}^n$ under the norm

$$\|q\|_{L^2_{2k}} = \Big(\int_{-k}^{k} |q(t)|^2\, dt\Big)^{1/2}.$$

Assume that $q_m \rightharpoonup q$ in E_k and, in consequence, $\dot{q}_m \rightharpoonup \dot{q}$ in $L^2_{2k}(\mathbb{R},\mathbb{R}^n)$. As the square of norm in a Hilbert space is weakly lower semicontinuous, we deduce that the functional $\varphi_k : E_k \to \mathbb{R}$ given by

$$\varphi_k(q) = \int_{-k}^{k} \frac{1}{2}|\dot{q}(t)|^2\, dt = \frac{1}{2}\|\dot{q}\|^2_{L^2_{2k}}$$

is also weakly lower semicontinuous. Furthermore, $q_m \to q$ almost uniformly on $\mathbb{R}$, and hence

$$\int_{-k}^{k} \big(V_k(t, q_m(t)) + (f_k(t), q_m(t))\big)\, dt \to \int_{-k}^{k} \big(V_k(t, q(t)) + (f_k(t), q(t))\big)\, dt$$

as $m \to \infty$, which means that the functional $\psi_k : E_k \to \mathbb{R}$ given by

$$\psi_k(q) = \int_{-k}^{k} \big(V_k(t, q(t)) + (f_k(t), q(t))\big)\, dt$$

is weakly continuous. By the above, I_k is weakly lower semicontinuous and by Theorem 3.2, I_k achieves a minimum on E_k, i.e. for all $k \in \mathbb{N}$ there is $q_k \in E_k$ such that

$$I_k(q_k) = \min_{q\in E_k} I_k(q) \qquad \text{and} \qquad I'_k(q_k) = 0.$$

By (C5), for all $k \in \mathbb{N}$, we have $I_k(0) = 0$. Choosing $\delta = L/A$, we see that for all $k \in \mathbb{N}$, if $\|q\|_{E_k} > \delta$ then $I_k(q) > 0$. Hence $\|q_k\|_{E_k} \leq \delta$ for all $k \in \mathbb{N}$. By the use of Theorem 1.3 we get the claim. ∎

EXAMPLE 3.3. Let $V : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be given by $V(t, q) = (e^{-t^2} + 1)q^2$ and $f(t) = e^{-t^2}$. The Newtonian system is as follows

$$\ddot{q}(t) - 2(e^{-t^2} + 1)q(t) = e^{-t^2}.$$

It is immediate that V and f satisfy the hypotheses (C2)–(C5) of Theorem 3.1, and so the system above has an almost homoclinic solution.

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References

[IJ1] M. Izydorek, J. Janczewska, *Homoclinic solutions for a class of the second order Hamiltonian systems*, J. Differential Equations 219 (2005), 375–389.

[IJ2] M. Izydorek, J. Janczewska, *Homoclinic solutions for nonautonomous second order Hamiltonian systems with a coercive potential*, J. Math. Anal. Appl. 335 (2007), 1119–1127.

[IJ3] M. Izydorek, J. Janczewska, *Approximative sequences and almost homoclinic solutions for a class of second order perturbed Hamiltonian systems*, this volume, 87–92.

[J1] J. Janczewska, *Almost homoclinic solutions for the second order Hamiltonian systems*, Topol. Methods Nonlinear Anal. 32 (2008), 131–137.

[J2] J. Janczewska, *An approximative scheme of finding almost homoclinic solutions for a class of Newtonian systems*, Topol. Methods Nonlinear Anal. 33 (2009), 169–177.

[J3] J. Janczewska, *Almost homoclinic solutions for a certain class of mixed type functional differential equations*, Ann. Polon. Math. 100 (2011), 13–24.

[L] S. Lu, *Homoclinic solutions for a nonlinear second order differential system with p-Laplacian operator*, Nonlinear Anal. Real World Appl. 12 (2011), 525Ű-534.

[MW] J. Mawhin, M. Willem, *Critical Point Theory and Hamiltonian Systems*, Appl. Math. Sci. 74, Springer, New York, 1989.

[R] P. H. Rabinowitz, *Homoclinic orbits for a class of Hamiltonian systems*, Proc. Roy. Soc. Edinburgh Sect. A 114 (1990), 33–38.

[S] A. Salvatore, *Multiple homoclinic orbits for a class of second order perturbed Hamiltonian systems*, Discrete Contin. Dyn. Syst. (2003), suppl., 778–787.

[TLX] X. H. Tang, X. Lin, L. Xiao, *Homoclinic solutions for a class of second order discrete Hamiltonian systems*, J. Difference Equ. Appl. 16 (2010), 1257Ű-1273.

[TX1] X. H. Tang, L. Xiao, *Homoclinic solutions for nonautonomous second-order Hamiltonian systems with a coercive potential*, J. Math. Anal. Appl. 351 (2009), 586–594.

[TX2] X. H. Tang, L. Xiao, *Homoclinic solutions for a class of second-order Hamiltonian systems*, Nonlinear Anal. 71 (2009), 1140–1152.

[TX3] X. H. Tang, L. Xiao, *Homoclinic solutions for ordinary p-Laplacian systems with a coercive potential*, Nonlinear Anal. 71 (2009), 1124–1132.

[Z] Y. Zhang, *Homoclinic solutions for a forced Liénard type system*, Results Math. 57 (2010), 69–78.

[ZY] Z. Zhang, R. Yuan, *Homoclinic solutions for some second order non-autonomous Hamiltonian systems with the globally superquadratic condition*, Nonlinear Anal. 72 (2010), 1809–1819.