CALCULUS OF VARIATIONS AND PDEs
BANACH CENTER PUBLICATIONS, VOLUME 101
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2014

SOME FINE PROPERTIES OF SETS WITH FINITE PERIMETER IN WIENER SPACES

MICHELE MIRANDA JR.

Department of Mathematics and Computer Science, University of Ferrara
via Machiavelli 35, I-44121 Ferrara, Italy
E-mail: michele.miranda@unife.it

Abstract. In this paper we give a brief overview on the state of art of developments of Geometric Measure Theory in infinite-dimensional Banach spaces. The framework is given by an abstract Wiener space, that is a separable Banach space endowed with a centered Gaussian measure. The focus of the paper is on the theory of sets with finite perimeter and on their properties; this choice was motivated by the fact that most of the good properties of functions of bounded variation can be obtained, thanks to coarea formula, from the geometric properties of sets with finite perimeter.

1. Introduction. The aim of this paper is to give an overview of recent developments of some Geometric Measure Theory in infinite-dimensional spaces; a systematic approach to this kind of problems, after the definition given by Malliavin, can be dated from the paper of Fukushima [14], where the author characterized sets with finite perimeter and functions of bounded variation in the Euclidean setting in terms of Dirichlet forms. The idea of this paper was that if we define the Dirichlet form

$$\mathcal{E}(u,v)=\int_{\mathbf{R}^m}\nabla u(x)\cdot\nabla v(x)\varrho(x)\,dx,$$

then the function ϱ has bounded variation if and only if the distorted Brownian path $(X_t^\varrho)_t$ associated to the Dirichlet form admits a decomposition

$$X_t^\varrho-X_0^\varrho=B_t+N_t^\varrho,$$

with B_t the m-dimensional Brownian motion and N_t^ϱ has bounded variation with

$$\lim_{t\to 0}\frac{1}{t}\mathbf{E}^\varrho\left[\int_0^t\chi_K(X_s^\varrho)\,d|N_s|\right]<+\infty$$

for every compact set K.

2010 *Mathematics Subject Classification*: Primary 28C20, 49Q15, 26E15; Secondary 60H07.
Key words and phrases: Wiener Space, Functions of bounded variation, Ornstein–Uhlenbeck semigroup.
The paper is in final form and no version of it will be published elsewhere.

DOI: 10.4064/bc101-0-9

The ideas of this work were then used by the author in Fukushima [15] and in Fukushima, Hino [16] in order to give the definition of functions with bounded variation in the infinite-dimensional setting. Essentially, the idea is that each time some integration by parts formula holds in the space, then functions with bounded variations can be defined in a distributional way. The approach of these papers was essentially based on ideas coming from probability theory.

Subsequently, a different approach has been investigated in Ambrosio et al. [4, 5], much more in the language of the calculus of variations. With this kind of techniques, a fundamental role is played by the theory of sets of finite perimeter, since properties of functions with bounded of variation, particularly on points of discontinuity, can be deduced by using coarea formula. For this reason, in the last years a particular attention has been given in the study of fine properties of sets of finite perimeter; for instance, in Hino [17], it has been proved that the perimeter measure can be characterized as an Hausdorff measure concentrated on the cylindrical essential boundary. Here the notion of Hausdorff measures is based on the work of Feyel, De la Pradelle [13].

In Ambrosio et al. [6], this characterization has been reconsidered, where instead of Hausdorff measure the authors have considered the spherical Hausdorff measure; in addition, a rectifiability result has been proved, in a weaker sense if compared to the Euclidean case.

Finally, recently Ambrosio et al. [3] have considered also the definition of a reduced boundary, proving a blow-up property of sets with finite perimeter.

The study of functions with bounded variations in the infinite-dimensional setting is mainly motivated on stochastic basis; for instance, the theory of Sobolev spaces defined on the whole Banach space is pretty well understood, whether the definition on domains is less clear. In particular, the possibility of studying stochastic differential equations on domains with Dirichlet or Neumann boundary conditions is strictly related on the possibility to define traces on boundaries of regular enough domains; such facts can be achieved if a good notion of reduced boundary has been given. Finally, a probabilistic motivation has recently been given by Pratelli, Trevisan [19], where it has been proved that

$$M_t = \sup_{0 \le s \le t} W_s$$

has second Malliavin derivatives given by finite measures, that is the Malliavin derivatives of M_t have bounded variation.

Also the possibility of considering functionals with linear growth in infinite-dimensional spaces can be a good motivation for developing the theory.

2. Some functional preliminary. Our framework is given by an infinite-dimensional Banach space X endowed with a centered Gaussian measure γ; by this we mean that on X a Borel measure γ is defined such that for every $x^* \in X^*$, the topological dual of X, the measure $x^*_\# \gamma$ defined by

$$x^*_\# \gamma(A) = \gamma\big(\{x : \langle x, x^* \rangle \in A\}\big),$$

is a centered Gaussian measure on $\mathbf{R}$. Equivalently, a measure γ is Gaussian if its Fourier transform

$$\hat{\gamma}(x^*) = \int_X e^{i\langle x, x^* \rangle} \, d\gamma(x)$$

is Gaussian. The Banach space is not necessarily assumed to be separable, nevertheless the Gaussian measure is concentrated on a separable Banach space contained in X. By a result of Fernique [12], if γ is a Gaussian measure, then there exists $\alpha \in \mathbf{R}$ such that

$$\int_X e^{\alpha \|x\|^2} \, d\gamma(x) < +\infty;$$

this in particular implies that for any $x^* \in X^*$, the map $x \mapsto \langle x, x^* \rangle$ belongs to $L^p(X, \gamma)$ for any $p \in [1, +\infty)$, so we can embed $R^* : X^* \to L^2(X, \gamma)$ and define the space $\mathcal{H}$ to be the closure of $R^* X^*$ in $L^2(X, \gamma)$. Such a space plays a fundamental role in the theory and is called the reproducing kernel of the measure γ. This space can be embedded into X via the map $R : L^2(X, \gamma) \to X$,

$$R\hat{h} := \int_X \hat{h}(x) x \, d\gamma(x), \qquad \hat{h} \in L^2(X, \gamma);$$

such a map is compact. The space $H = R\mathcal{H}$ is a Hilbert space with the inner product

$$[h_1, h_2]_H := \int_X \hat{h}_1(x) \hat{h}_2(x) \, d\gamma(x), \qquad h_i = R\hat{h}_i, \ i = 1, 2.$$

The notation R and R^* is motivated by the fact that

$$[R^* x^*, \hat{h}]_{\mathcal{H}} = \int_X \langle x, x^* \rangle \hat{h}(x) \, d\gamma(x) = \Big\langle \int_X \hat{h}(x) x \, d\gamma(x), x^* \Big\rangle = \langle R\hat{h}, x^* \rangle,$$

that is, R^* is the adjoint of R. In addition, the operator $Q = RR^*$ is the covariance associated to γ since

$$\langle Qx^*, y^* \rangle = \langle RR^* x^*, y^* \rangle = [R^* x^*, R^* y^*]_{\mathcal{H}} = \int_X \langle x, x^* \rangle \langle x, y^* \rangle \, d\gamma(x) \qquad \forall x^*, y^* \in X^*.$$

The non-degeneracy condition of γ is equivalent to the injectivity of the covariance operator Q; the space H is called the Cameron–Martin space. If the measure γ is not degenerate in X, H is dense and separable, the map $R : \mathcal{H} \to X$ is compact and $\gamma(H) = 0$. The importance of this space relies in the integration by parts formula. Indeed, the measure

$$\gamma_v(B) = \gamma(B - v), \qquad v \in X,$$

is absolutely continuous with respect to γ if and only if $v = h \in H$, and the density can be explicitly written, if $h = R\hat{h}$, as

$$\frac{d\gamma_h}{d\gamma}(x) = \exp\Big(-\frac{1}{2}|h|_H^2 + \hat{h}(x)\Big).$$

This density is even differentiable, so we can integrate by parts; if we define, for functions $f \in C_b^1(X)$, the derivative in direction h, as

$$\partial_h f(x) = \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon h) - f(x)}{\varepsilon},$$

we get, for any $g \in C_b^1(X)$,

$$\begin{aligned}\int_X \partial_h f(x) g(x)\, d\gamma(x) &= \frac{d}{d\varepsilon}\int_X f(x+\varepsilon h) g(x)\, d\gamma(x)_{|\varepsilon=0}\\ &= \frac{d}{d\varepsilon}\int_X f(x) g(x-\varepsilon h)\, d\gamma_{\varepsilon h}(x)_{|\varepsilon=0}\\ &= \frac{d}{d\varepsilon}\int_X f(x) g(x-\varepsilon h) \exp\Big(-\frac{\varepsilon^2|h|_H^2}{2}+\varepsilon\hat{h}(x)\Big)\, d\gamma(x)_{|\varepsilon=0}\\ &= -\int_X f(x)\partial_h g(x) d\gamma(x) + \int_X f(x) g(x)\hat{h}(x)\, d\gamma(x).\end{aligned}$$

This means that the adjoint of ∂_h in $L^2(X,\gamma)$ is given, up to the sign, by

$$\partial_h^* g(x) = \partial_h g(x) - g(x)\hat{h}(x).$$

The separability of H allows for the definition of a gradient for functions $f \in C_b^1(X)$, by fixing an orthonormal basis $\{h_j\}$ in H and setting

$$\nabla_H f(x) = \sum_{j\in\mathbf{N}} \partial_j f(x) h_j,$$

where we have written ∂_j for ∂_{h_j}; also a divergence is defined for fields $g \in C_b^1(X,H)$ as

$$\mathrm{div_H} g(x) = \sum_{j\in\mathbf{N}} \partial_j^*[g(x),h_j]_H.$$

It can be proved that ∇_H is a closable operator in $L^p(X,\gamma)$ for any $p\in[1,+\infty]$; the domain of its closure is the Sobolev space $W^{1,p}(X,\gamma)$. For $p>1$, $W^{1,p}(X,\gamma)$ embeds continuously in $L^p(X,\gamma)$; the case $p=1$ is particular, since $W^{1,1}(X,\gamma)$ embeds in $L\ln L^{1/2}(X,\gamma)$, the Orlicz space $L^\Phi(X,\gamma)$ defined by the function $\Phi(t)=\ln^{1/2}(1+t)$, where

$$L^\Phi(X,\gamma) = \Big\{u : X\to\mathbf{R} : \exists t \text{ s.t. } \int_X \Phi\Big(\frac{|u(x)|}{t}\Big)\, d\gamma(x) < +\infty\Big\}.$$

3. *BV* functions. Once a gradient and a divergence is defined, it is possible to give the definition of functions with bounded variation. By the formula

$$\int_X u(x)\partial_h^* g(x)\, d\gamma(x) = \int_X u(x)\partial_h g(x)\, d\gamma(x) - \int_X u(x) g(x)\hat{h}(x)\, d\gamma(x),$$

it is clear that the last integral is not well defined if we only require u to belong to $L^1(X,\gamma)$ and $g\in C_b^1(X)$; this is because $\hat{h}(x)$ has a linear growth in x. This motivates why to give the definition of functions with bounded variation we have to start from $L\ln L^{1/2}(X,\gamma)$.

Definition 3.1. Let $u \in L\ln L^{1/2}(X,\gamma)$; u is said to *have bounded variation*, $u\in BV(X,\gamma)$, if

$$|D_\gamma u|(X) = \sup\Big\{\int_X u(x)\,\mathrm{div}_H g(x)\, d\gamma(x) : g\in C_b^1(X,H),\ |g(x)|_H\le 1\Big\} < +\infty.$$

A set $E\subset X$ is said to have finite perimeter if $u=\chi_E\in BV(X,\gamma)$; in this case the variation $|D_\gamma\chi_E|(X) = P_\gamma(E,X)$ is called the perimeter of E in X.

An important link between the theory of BV functions and sets with finite perimeter is given by the coarea formula; it states that a function u has bounded variation if and only if almost every level set $\{u > t\}$ has finite perimeter and

$$|D_\gamma u|(X) = \int_{\mathbf{R}} P_\gamma\big(\{u > t\}, X\big)\, dt < +\infty.$$

A large amount of properties of BV functions can be derived using coarea formula starting from fine properties of sets with finite perimeter; especially the properties of the jump part of u are related to properties of sets with finite perimeter.

We briefly give some example of sets with finite perimeter.

Example 3.2.

1. As already stated, almost every level set of a function with bounded variation has finite perimeter; examples of functions with bounded variation are Sobolev functions in $W^{1,1}(X,\gamma)$ and, in particular, Lipschitz functions can be used to construct sets with finite perimeter. There are two important classes of Lipschitz functions; the first class is given by the functions that are Lipschitz with respect to the norm of X, the second and wider class is the class of H-Lipschitz functions, that is functions $f : X \to \mathbf{R}$ such that there exists $c > 0$ with

$$|f(x+h) - f(x)| \le c|h|_H \qquad \forall h \in H \ \forall x \in X.$$

   The compact embedding of H into X implies the existence of a constant $c > 0$ such that

$$\|h\|_X \le c|h|_H \qquad \forall h \in H;$$

   then any Lipschitz function is H-Lipschitz. The definition of H-Lipschitz functions give a control only in directions belonging to H, so such functions can also be discontinuous as functions on X; nevertheless, H-Lipschitz functions belong to $W^{1,p}(X,\gamma)$ for any $p \in [1,+\infty]$.
2. If we fix a point $x_0 \in X$, we can use the map $f(x) = \|x - x_0\|_X$ to conclude that almost every ball $B_r(x_0)$ has finite perimeter. To prove that every ball has finite perimeter is a delicate matter; it is true, but is based on a Brunn–Minkowski argument and was proved by Caselles et al. [7]. In that paper it is proved that any convex set with non-empty interior has finite perimeter, so any open ball has finite perimeter; in the same paper it is proved that the condition on non-empty interior is important, since an example of a convex set with empty interior and infinite perimeter is constructed. Finally, we point out that it makes no sense to use the distance induced by the norm of H; indeed, the map $f(x) = |x - x_0|_H$ is γ-a.e. equal to infinity. On the other hand, since $\gamma(H) = 0$, the balls in the Cameron–Martin space have null measure, then also perimeter equals to 0.
3. Hino and Uchida [18] gave an example of a set with finite perimeter in classical Wiener space using the reflecting Brownian motion. They proved that if $\Omega \subset \mathbf{R}^d$ is an open set satisfying the exterior ball condition and

$$X = \big\{w \in C([0,+\infty), \mathbf{R}^d) : w(0) = 0\big\}$$

is the Wiener space endowed with the Wiener measure γ, then the set

$$E_\Omega = \big\{w \in X : w(t) \in \overline{\Omega}\ \forall t \geq 0\big\},$$

has finite perimeter in X.

4. Airault and Malliavin [1] constructed a surface measure on boundaries of regular level sets; more precisely, given a function

$$f \in W^\infty(X,\gamma) = \bigcap_{p>1, r\in\mathbf{N}} W^{r,p}(X,\gamma),$$

where $W^{r,p}(X,\gamma)$ is the Sobolev space of order r with p-integrability, such that

$$\frac{1}{|\nabla_H f|_H} \in \bigcap_{p\geq 1} L^p(X,\gamma),$$

then the set $\{f > 0\}$ has finite perimeter. In details, Airault and Malliavin constructed for such functions a surface measure on the set $\{f = 0\}$; in [7] it is proved that such a measure, under the additional assumption that f is continuous, coincides with the perimeter measure of $\{f > 0\}$.

There is another way to construct example of BV functions and sets with finite perimeter, and is based on the finite-dimensional case by a cylindrical construction.

If we fix a finite-dimensional subspace $F \leq QX^*$, $\dim F = m < +\infty$, we may consider the orthogonal decomposition $H = F \otimes F^\perp$. Assuming F generated by the m-vectors $h_1, \dots h_m$ with $h_j = Qx_j^*$, it is possible to define the projection $\Pi_F : X \to F$

$$\Pi_F(x) = \sum_{j=1}^m \langle x, x_j^*\rangle h_j;$$

it is then possible to decompose $X = F \oplus \mathrm{Ker}(\Pi_F)$. The spaces F and $\mathrm{Ker}(\Pi_F)$ are still Gaussian spaces with Gaussian measures given by γ_F and $\gamma_F^\perp$ respectively; such measures have F and $F^\perp$ as Cameron–Martin spaces and the original measure is decomposed as $\gamma = \gamma_F \otimes \gamma_F^\perp$. So, if $v \in BV(F,\gamma_F)$ is a finite-dimensional function with bounded variation, the map $u = v \circ \Pi_F \in BV(X,\gamma)$. We shall use this factorization property of the measure to characterize sets of finite perimeter. In the next section we give a short overview of the finite-dimensional theory of sets with finite perimeter, recalling some basic properties.

4. Finite-dimensional sets with finite perimeter. For sets with finite perimeter the topological boundary is not the right notion in order to obtain a representation of the perimeter measure; as a classical example, one may consider the enlarged rational numbers

$$E = \bigcup_{j\in\mathbf{N}} B_{\varrho_j}(x_j),$$

where $\{x_j\}_{j\in\mathbf{N}}$ is a dense set in $\mathbf{R}^m$ and $(\varrho_j)_{j\in\mathbf{N}}$ is a sequence of positive numbers such that

$$\sum_{j\in\mathbf{N}} \varrho_j^{m-1} < +\infty.$$

E is an example of a set with finite perimeter but with infinite full measure of the boundary $\mathcal{L}^m(\partial E) = +\infty$.

For this reason, a different notion of boundary have been introduced; for instance, De Giorgi [9] (see also [10]) defined the reduced boundary of E as the set

$$\mathcal{F}E = \Big\{x \in \operatorname{spt}|D\chi_E| : \exists \lim_{\varrho\to 0} \frac{D\chi_E(B_\varrho(x))}{|D\chi_E|(B_\varrho(x))} = \nu_E(x),\ |\nu_E(x)| = 1\Big\};$$

De Giorgi proved that such a set is $\mathcal{H}^{m-1}$-rectifiable, in the sense that it is contained, up to $\mathcal{H}^{m-1}$-negligible sets, into a countable union of Lipschitz graph,

$$\mathcal{H}^{m-1}\Big(\mathcal{F}E \setminus \bigcup_{j\in\mathbf{N}} \Gamma_j\Big) = 0,$$

with $\Gamma_j = f_j(\mathbf{R}^{m-1})$, $f_j : \mathbf{R}^{m-1} \to \mathbf{R}^m$ Lipschitz. In addition, the following representation for the perimeter measure is true

$$D\chi_E = \nu_E \mathcal{H}^{m-1} \llcorner \mathcal{F}E, \tag{1}$$

in the sense that for any $B \subset \mathbf{R}^m$ Borel set

$$D\chi_E(B) = \int_{B\cap\mathcal{F}E} \nu_E(x)\, d\mathcal{H}^{m-1}(x).$$

It is also possible to define the points of density for E as

$$E^{(\alpha)} = \Big\{x \in \mathbf{R}^m : \exists \lim_{\varrho\to 0} \frac{\mathcal{L}^m(E\cap B_\varrho(x))}{\mathcal{L}^m(B_\varrho(x))} = \alpha\Big\}$$

and define the essential boundary of E as

$$\partial^* E = \mathbf{R}^m \setminus (E^{(0)} \cup E^{(1)}).$$

The following inclusions are immediate

$$\mathcal{F}E \subset E^{(1/2)} \subset \partial^* E;$$

in addition, Federer [11] proved that

$$\mathcal{H}^{m-1}(\partial^* E \setminus \mathcal{F}E) = 0,$$

and so the fact that the perimeter measure is concentrated either on the essential boundary, or on the reduced boundary or again on the set of points with density $1/2$.

It can be pointed out that, in the Euclidean case, points of density may also be defined using the heat semigroup; indeed, if $(H_t)_{t\geq 0}$ denotes the heat semigroup defined pointwise by formula

$$H_t u(x) = \frac{1}{(2\pi t)^{m/2}} \int_{\mathbf{R}^m} u(y) e^{-|x-y|^2/(2t)}\, dy,$$

then, for points $x \in E^{(\alpha)}$ of density α, we have that

$$\exists \lim_{t\searrow 0^+} H_t \chi_E(x) = \alpha.$$

Such a property goes back to the original definition of functions with bounded variation given by De Giorgi; indeed, in [8] (see also [10]) it was shown that for a given function $u \in L^1(\mathbf{R}^m)$ the map

$$t \mapsto \int_{\mathbf{R}^m} |\nabla H_t u(x)|\, dx$$

is monotone decreasing and that u has bounded variation if and only if

$$\mathcal{I}(u) := \sup_{t\geq 0} \int_{\mathbf{R}^m} |\nabla H_t u(x)|\, dx < +\infty$$

and the quantity $\mathcal{I}(u)$ coincides with the total variation $|Du|(\mathbf{R}^m)$.

5. The infinite-dimensional case. In the infinite-dimensional case things do not work the same; Preiss [20] gave an example of a Gaussian measure γ on a Hilbert space X and of a measurable set $E \subset X$ with $0 < \gamma(E) < 1$ such that

$$\lim_{\varrho\to 0} \frac{\gamma(E\cap B_\varrho(x))}{\gamma(B_\varrho(x))} = 1, \qquad \forall x \in X.$$

So the notion of points of density is not good in this setting. Also the characterization (1) is intrinsically finite-dimensional, since based on the Besicovitch Theorem on derivation of measures.

In the infinite-dimensional setting, the idea is to use the factorization of $\gamma = \gamma_F \otimes \gamma_F^\perp$ and finite-dimensional slicing; for more details on this part, we refer to Hino [17] and to Ambrosio et al. [6]. By writing $X = F \oplus \mathrm{Ker}(\Pi_F)$ and $x = y + z$, $y \in F$, $z \in \mathrm{Ker}(\Pi_F)$, we may define the finite-dimensional slice of E at z as

$$E_z = \{y \in F : y + z \in E\} \subset F;$$

so it is possible to define the essential boundary of E relative to F as

$$\partial_F^* E = \{x = y + z : y \in \partial^*(E_z)\}.$$

In the same spirit, we can also define the codimension one spherical Hausdorff measure relative to F as

$$\mathcal{S}_F^{\infty-1}(B) = \int_{\mathrm{Ker}(\Pi_F)} d\gamma_F^\perp(z) \int_{B_z} \frac{1}{(2\pi)^{m/2}}\, e^{-|y|^2/2}\, d\mathcal{S}^{m-1}(y).$$

Here the measure is the spherical Hausdorff measure, i.e. the Hausdorff measure defined using coverings done only by balls

$$\mathcal{S}^k(B) = \lim_{\delta\to 0} \mathcal{S}_\delta^k(B)$$

with

$$\mathcal{S}_\delta^k(B) = \inf\Big\{\sum_{i\in\mathbf{N}} \omega_k r_i^k : B \subset \bigcup_{i\in\mathbf{N}} B_{r_i}(x_i), r_i < \delta\Big\}.$$

It is important to use spherical rather than classical Hausdorff measure for the following relevant monotonicity formula

$$\mathcal{S}_F^{\infty-1}(B) \leq \mathcal{S}_G^{\infty-1}(B) \qquad \forall B \in \mathcal{B}(X),$$

whenever $F \leq G \leq H$ are finite-dimensional subspaces of H. In this way the codimension one spherical Hausdorff measure is well defined:

$$\mathcal{S}^{\infty-1}(B) = \sup_{F\leq H} \mathcal{S}_F^{\infty-1}(B),$$

where the supremum is taken among all finite-dimensional subspaces F of H.

Another important fact is the following almost monotonicity

$$\mathcal{S}_G^{\infty-1}(\partial_G^* E \setminus \partial_F^* E) = 0,$$

that is monotonicity up to negligible sets. So, if we fix a countable family $\mathcal{F}$ of finite-dimensional subspaces $F_j \le H$ such that

$$H = \overline{\bigcup_{j\in\mathbf{N}} F_j},$$

we can define the cylindrical essential boundary relative to $\mathcal{F}$ as

$$\partial^*_{\mathcal{F}} E = \liminf_{j\to+\infty} \partial^*_{F_j} E = \bigcup_{h\in\mathbf{N}} \bigcap_{j\ge h} \partial^*_{F_j} E.$$

If we define the cylindrical spherical Hausdorff measure

$$\mathcal{S}^{\infty-1}_{\mathcal{F}}(B) = \sup_{F_j\in\mathcal{F}} \mathcal{S}^{\infty-1}_{F_j}(B),$$

in Hino [17] and with a simplified proof in Ambrosio et al. [6] it is proved that

$$|D_\gamma \chi_E| = \mathcal{S}^{\infty-1}_{\mathcal{F}} \llcorner \partial^*_{\mathcal{F}} E.$$

In [6] it is also showed that $\partial^*_{\mathcal{F}} E$ is Sobolev rectifiable, in the sense that it is contained, up to negligible sets, in a countable union of graphs of Sobolev functions $f_j \in W^{1,1}(X',\gamma')$, where X' is a codimension one Banach sub-space in X.

Ambrosio and Figalli [2] used the Ornstein–Uhlenbeck semigroup defined by the Mehler formula

$$T_t u(x) = \int_X u\big(e^{-t}x + \sqrt{1-e^{-2t}}\, y\big)\, d\gamma(y)$$

to define points with density. More precisely, they proved that if E has finite perimeter, then

$$\lim_{t\to 0} \int_X \Big| T_t \chi_E(x) - \frac{1}{2}\Big|^2 d|D_\gamma \chi_E|(x) = 0;$$

in particular, there exists a sequence of times $t_j \searrow 0^+$ for which

$$\lim_{j\to+\infty} T_{t_j}\chi_E(x) = \frac{1}{2}\,, \qquad |D_\gamma\chi_E|\text{-a.e. } x\in X.$$

In some sense the points of density are defined in terms of the Ornstein–Uhlenbeck semigroup as

$$E^{(\alpha)} = \Big\{ x\in X : \exists t_j \searrow 0^+ \text{ such that } \lim_{j\to+\infty} T_{t_j}\chi_E(x) = \alpha \Big\}.$$

The sequence t_j in the Ambrosio and Figalli result depends of the set E itself, and so at the moment it is not possible to give a clean definition of points with density.

Concerning the definition of reduced boundary, miming the construction done for the essential boundary, Ambrosio, Figalli and Runa [3] have defined the reduced boundary relative to F as

$$\mathcal{F}_F E = \{ x = y+z : y \in \mathcal{F}(E_z)\}$$

and for a fixed family of countable many finite-dimensional subspaces $\mathcal{F}$, the cylindrical reduced boundary

$$\mathcal{F}_{\mathcal{F}} E = \liminf_{j\to+\infty} \mathcal{F}_{F_j} E = \bigcup_{h\in\mathbf{N}} \bigcap_{j\ge h} \mathcal{F}_{F_j} E.$$

If for $x\in\mathcal{F}_{\mathcal{F}}E$, recalling that $\nu_E(x) = R\widehat{\nu_E(x)}$ with $\widehat{\nu_E(x)} \in \mathcal{H}$, we define

$$S(x) = S_{\nu_E(x)} = \{ y\in X : \widehat{\nu_E(x)}(y) > 0\},$$

then in Ambrosio, Figalli, Runa it is proved that

$$\lim_{t\to 0}\int_X d|D_\gamma\chi_E|(x)\int_X |\chi_E(e^{-t}x+\sqrt{1-e^{-2t}}\,y)-\chi_{S(x)}(y)|\,d\gamma(y)=0,$$

that is, since

$$\chi_E(e^{-t}x+\sqrt{1-e^{-2t}}\,y)=\chi_{E_{x,t}}(y),\qquad E_{x,t}=\frac{E-e^{-t}x}{\sqrt{1-e^{-2t}}}\,,$$

the fact that $\chi_{E_{x,t}}\to\chi_{S(x)}$ in $L^1(X,\gamma)$ for $|D_\gamma\chi_E|$-a.e. $x\in X$.

6. Open problems. We list here some open problems in the theory.

1. To pass from Sobolev rectifiability to Lipschitz rectifiability of $\partial^*_{\mathcal{F}}E$ is a big problem; indeed, also prove a Lusin type result is an open question, that is, it is not known if a Sobolev map in X coincides in a set with positive measure with a Lipschitz map.
2. Give a definition of points with density or, elsewhere, prove that, without passing to subsequences,
$$\exists\lim_{t\to 0}T_t\chi_E(x)=\frac{1}{2}\,,\qquad |D_\gamma\chi_E|\text{-a.e. } x\in X.$$
3. Referring to Example 3.2.3, Hino and Uchida in [18] proved that the perimeter measure of E_Ω concentrates on the set
$$\partial' E_\Omega=\big\{w\in X: w(t)\in\overline{\Omega}\text{ and there exists one and only one } t>0 \text{ such that } w(t)\in\partial\Omega\big\};$$
it would be interesting to compare this set with $E_\Omega^{(1/2)}$.
4. Show the following decomposition, up to $\mathcal{S}^{\infty-1}$ negligible sets:
$$X=E^{(1)}\cup E^{(1/2)}\cup E^{(0)}.$$
5. Give an intrinsic notion of reduced and essential boundary; for the latter, once points of density are well defined, it could be simply defined as in the Euclidean case as
$$X\setminus(E^{(1)}\cup E^{(0)}).$$
For the reduced boundary, a possible definition could be given in terms of the Ornstein–Uhlenbeck semigroup as the set of points for which
$$\exists\lim_{t\to 0}T_t\Big(\frac{dT_t^*D_\gamma\chi_E}{dT_t^*|D_\gamma\chi_E|}\Big)(x)=\nu_E(x)$$
and $|\nu_E(x)|_H=1$, where T_t^* is the dual semigroup of T_t defined on the set of finite measures.
6. Study fine properties of BV functions, together with functional properties (traces, jump set, etc.).

Acknowledgments. This research was partially supported by the GNAMPA 2012 "Problemi di evoluzione e Teoria Geometrica della Misura in spazi metrici".

References

[1] H. Airault, P. Malliavin, *Intégration géométrique sur l'espace de Wiener*, Bull. Sci. Math. (2) 112 (1988), 3–52.
[2] L. Ambrosio, A. Figalli, *Surface measures and convergence of the Ornstein–Uhlenbeck semigroup in Wiener spaces*, Ann. Fac. Sci. Toulouse Math. (6) 20 (2011), 407–438.
[3] L. Ambrosio, A. Figalli, E. Runa, *On sets of finite perimeter in Wiener spaces: reduced boundary and convergence to halfspaces*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Math. Appl. 24 (2013), 111–122.
[4] L. Ambrosio, M. Miranda, Jr., S. Maniglia, D. Pallara, *BV functions in abstract Wiener spaces*, J. Funct. Anal. 258 (2010), 785–813.
[5] L. Ambrosio, M. Miranda Jr., S. Maniglia, D. Pallara, *Towards a theory of BV functions in abstract Wiener spaces*, Phys. D. 239 (2010), 1458–1469.
[6] L. Ambrosio, M. Miranda, Jr., D. Pallara, *Sets with finite perimeter in Wiener spaces, perimeter measure and boundary rectifiability*, Discrete Contin. Dyn. Syst. 28 (2010), 591–606.
[7] V. Caselles, A. Lunardi, M. Miranda Jr., M. Novaga, *Perimeter of sublevel sets in infinite dimensional spaces* Adv. Calc. Var. 5 (2012), 59–76.
[8] E. De Giorgi, *Su una teoria generale della misura* $(r-1)$*–dimensionale in uno spazio ad* r *dimensioni*, Ann. Mat. Pura Appl. 36 (1954), 191–213.
[9] E. De Giorgi, *Nuovi teoremi relativi alle misure* $(r-1)$*–dimensionali in uno spazio ad* r *dimensioni*, Ricerche Mat. 4 (1955), 95–113.
[10] E. De Giorgi, *Selected Papers*, Springer, Berlin, 2006.
[11] H. Federer, *Geometric Measure Theory*, Grundlehren Math. Wiss. 153, Springer, New York, 1969.
[12] X. Fernique, *Intégrabilitè des vecteurs gaussiens*, C. R. Acad. Sci. Paris Sér. A-B 270 (1970), A1698–A1699.
[13] D. Feyel, A. de La Pradelle, *Hausdorff measures on the Wiener space*, Potential Anal. 1 (1992), 177–189.
[14] M. Fukushima, *On semi-martingale characterizations of functionals of symmetric Markov processes*, Electron. J. Prob. 4 (1999), no. 18.
[15] M. Fukushima, *BV functions and distorted Ornstein–Uhlenbeck processes over the abstract Wiener space*, J. Funct. Anal. 174 (2000), 227–249.
[16] M. Fukushima, M. Hino, *On the space of BV functions and a related stochastic calculus in infinite dimensions*, J. Funct. Anal. 183 (2001), 245–268.
[17] M. Hino, *Sets of finite perimeter and the Hausdorff–Gauss measure on the Wiener space*, J. Funct. Anal. 258 (2010), 1656–1681.
[18] M. Hino, H. Uchida, *Reflecting Ornstein–Uhlenbeck processes on pinned path spaces*, in: Proceedings of RIMS Workshop on Stochastic Analysis and Applications, RIMS Kôkyûroku Bessatsu B6, Res. Inst. Math. Sci. (RIMS), Kyoto, 2008, 111–128.
[19] M. Pratelli, D. Trevisan, *Functions of bounded variation on the classical Wiener space and an extended Ocone–Karatzas formula*, Stochastic Process. Appl. 122 (2012), 2383–2399.
[20] D. Preiss, *Gaussian measures and the density theorem*, Comment. Math. Univ. Carolin. 22 (1981), 181–193.