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THE EULER–LAGRANGE INCLUSION IN ORLICZ–SOBOLEV SPACES

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Abstract. We establish the Euler–Lagrange inclusion of a nonsmooth integral functional defined on Orlicz–Sobolev spaces. This result is achieved through variational techniques in nonsmooth analysis and an integral representation formula for the Clarke generalized gradient of locally Lipschitz integral functionals defined on Orlicz spaces.

1. Introduction. This paper provides a first-order necessary condition for a general problem in the calculus of variations that involves minimization of a functional defined on Orlicz–Sobolev spaces where the Lagrangian function is Lipschitz continuous in the last two arguments (see (3.1)). This condition has a form of the Euler–Lagrange partial differential inclusion in Orlicz–Sobolev spaces (see Theorem 3.3). The methods used here are based on nonsmooth variational techniques (see, e.g., [2, 6]) together with integral representation formula for the Clarke generalized gradient of locally Lipschitz integral functionals defined on Orlicz spaces (see [14, 15, 17]). Theorem 3.3 is a generalization of the results of Clarke [3] (see also [4, Theorem 4.6.1]) and Papageorgiou–Papageorgiou [16, Theorem 4.1] (see also [8, Theorem V.2.29]). For other results dealing with the Euler–Lagrange inclusions in Sobolev spaces we refer the reader to, e.g., [9, 10, 13, 19].

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2. Preliminaries

2.1. Generalized gradient. Let U be an open subset of a Banach space E. By Clarke [4], if $f : U \to \mathbb{R}$ is Lipschitz continuous, then f has the *generalized directional derivative*

$$f^\circ(x;v) = \limsup_{y\to x,\lambda\to 0^+} \frac{f(y+\lambda v)-f(y)}{\lambda}, \quad v \in E,$$

and the set

$$\partial_C f(x) = \big\{\zeta \in E^* : \langle \zeta, v\rangle \le f^\circ(x;v) \quad \forall v \in E\big\}$$

is called the *generalized gradient of f at x*, where E^* is the dual space of the Banach space E. It is a nonempty convex compact set in the weak star topology $\omega^* = \sigma(E^*, E)$. Furthermore, if f is Fréchet differentiable, i.e., $f \in C^1$, then $\partial_C f(x) = \{f'(x)\}$.

2.2. Orlicz and Orlicz–Sobolev spaces. The terminology from Orlicz and Orlicz–Sobolev spaces follows [5, 11, 12]. A function $\Phi : [0,\infty) \to [0,\infty)$ is called an *N-function* if it is continuous and convex with $\Phi(t) = 0 \Leftrightarrow t = 0$ and $\Phi(t)/t \to 0$ as $t \to 0$, $\Phi(t)/t \to \infty$ as $t \to \infty$. The N-function Φ is said to satisfy the *$\triangle_2$-condition*, denoted by $\Phi \in \triangle_2$, if there exist $k > 0$ and $T > 0$ such that

$$\Phi(2t) \le k\,\Phi(t) \quad \forall t \ge T.$$

The *complementary function* Φ^* to Φ is defined by $\Phi^*(v) = \sup\{uv - \Phi(u) : u \ge 0\}$ for all $v \ge 0$. If Φ is an N-function, then so is Φ^*.

Let Ω be an open subset of $\mathbb{R}^n$ and let Φ be an N-function. The *Orlicz space* $L_\Phi(\Omega,\mathbb{R})$ is defined as the set of (equivalence classes of) real-valued measurable functions u on Ω such that $\int_\Omega \Phi(|u(x)|/\lambda)\,dx < \infty$ for some $\lambda > 0$. Note that $L_\Phi(\Omega,\mathbb{R})$ is a Banach space under the *Luxemburg norm*

$$\|u\|_{L_\Phi} := \inf\Big\{\lambda > 0 : \int_\Omega \Phi(|u(x)|/\lambda)\,dx \le 1\Big\}.$$

Given a separable Banach space E and an Orlicz space $L_\Phi(\Omega,\mathbb{R})$, the *Orlicz–Bochner space* $L_\Phi(\Omega, E)$ is defined (see, e.g., [18]) as the normed space of (equivalence classes of) strongly measurable E-valued functions u on Ω such that the function $\Omega \ni x \mapsto \|u(x)\|_E$ belongs to $L_\Phi(\Omega,\mathbb{R})$ with the norm

$$\|u\|_{L_\Phi(\Omega,E)} := \big\|\|u(\cdot)\|_E\big\|_{L_\Phi}.$$

Recall that a function $u : \Omega \to E$ is said to be a strongly measurable function if there exists a sequence (u_n) of simple measurable functions such that $\lim_{n\to\infty} \|u_n(x) - u(x)\|_E = 0$ for almost all $x \in \Omega$.

The *Orlicz–Sobolev* space $W^1L_\Phi(\Omega,\mathbb{R}^d)$ is the space of all $u \in L_\Phi(\Omega,\mathbb{R}^d)$ such that $Du \in L_\Phi(\Omega,\mathbb{M}^{n\times d})$, where Du is a vector function whose all components are distributional partial derivatives of u and $\mathbb{M}^{n\times d}$ is the space of real $n \times d$ matrices. The space $W^1L_\Phi(\Omega,\mathbb{R}^d)$ is a Banach space under the norm

$$\|u\|_{W^1L_\Phi} = \|u\|_{L_\Phi} + \|Du\|_{L_\Phi}.$$

Let $\Phi \in \triangle_2$. Recall that $\mathring{W}^1L_\Phi(\Omega,\mathbb{R}^d)$ is defined as the norm-closure of $C_0^\infty(\Omega,\mathbb{R}^d)$ in $W^1L_\Phi(\Omega,\mathbb{R}^d)$ and $W^{-1}L_\Phi(\Omega,\mathbb{R}^d) := (\mathring{W}^1L_\Phi(\Omega,\mathbb{R}^d))^*$.

3. The Euler–Lagrange condition. Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain and $\hat{u}$ be an element of $W^1L_\Phi(\Omega,\mathbb{R}^d)$. Consider

$$\inf\Big\{G(w) := \int_\Omega g(x,w(x),Dw(x))\,dx : w \in \hat{u} + \mathring{W}^1L_\Phi(\Omega,\mathbb{R}^d)\Big\}, \tag{3.1}$$

where $g : \Omega \times \mathbb{R}^d \times \mathbb{M}^{n\times d} \to \mathbb{R}$ is a Carathéodory function such that $g(x,\cdot,\cdot)$ is locally Lipschitz continuous for almost all $x \in \Omega$. We will assume that one of the following conditions is satisfied.

Conditions 3.1. Let $\Phi, \Phi^* : [0,\infty) \to [0,\infty)$ be a pair of complementary N-functions.

(EL1) $\Phi \in \triangle_2$ and for some $R > 0$ there exist positive constants b_R, d_R and a function $a_R \in L^1(\Omega,[0,\infty))$ such that

$$u^* \in \partial_C g(x,u) \Rightarrow \Phi^*\left(\|u^*\|_{\mathbb{R}^d\times\mathbb{M}^{n\times d}}/d_R\right) \le a_R(x) + b_R\Phi\left(\|u\|_{\mathbb{R}^d\times\mathbb{M}^{n\times d}}/R\right)$$

for almost all $x \in \Omega$ and for all $u \in \mathbb{R}^d \times \mathbb{M}^{n\times d}$.

(EL2) $\Phi \in \triangle_2$ and for some $R > 0$ there exist positive constants b_R, d_R and functions $a_R \in L^1(\Omega,[0,\infty))$, $h_R : \Omega \times [0,\infty) \to [0,\infty)$ such that

$$|g(x,u) - g(x,v)| \le h_R(x, \|u\|_{\mathbb{R}^d\times\mathbb{M}^{n\times d}} + \|v\|_{\mathbb{R}^d\times\mathbb{M}^{n\times d}})\|u - v\|_{\mathbb{R}^d\times\mathbb{M}^{n\times d}}$$

for almost all $x \in \Omega$ and for all $u, v \in \mathbb{R}^d \times \mathbb{M}^{n\times d}$, and

$$\Phi^*(h_R(x,\alpha)/d_R) \le a_R(x) + b_R\Phi(\alpha/R)$$

for almost all $x \in \Omega$ and for all $\alpha \in [0,\infty)$.

(EL3) $\Phi \in \triangle_2$ and there exists $c > 0$ such that

$$u^* \in \partial_C g(s,u) \Longrightarrow \|u^*\|_{\mathbb{R}^d\times\mathbb{M}^{n\times d}} \le c\big(1 + \varphi(\|u\|_{\mathbb{R}^d\times\mathbb{M}^{n\times d}})\big)$$

for all $u \in \mathbb{R}^d \times \mathbb{M}^{n\times d}$, where φ is the right derivative of Φ.

Remark 3.2. Note that (EL2) implies (EL1), by [4, Proposition 2.1.2/(a)]. The condition (EL3) follows Płuciennik–Tian–Wang [17], but the conditions (EL1), (EL2) follow [14].

Theorem 3.3. *Suppose that either one of the conditions* (EL1), (EL2) *or* (EL3) *holds and $g(\cdot,0,0) \in L^1(\Omega,\mathbb{R})$. If $0 \in \partial_C G(w)$ then there exists $v^* \in L_{\Phi^*}(\Omega,\mathbb{M}^{n\times d})$ such that $\operatorname{div} v^* \in L_{\Phi^*}(\Omega,\mathbb{R}^d)$ and*

$$\big(\operatorname{div} v^*(x), v^*(x)\big) \in \partial_C g(x,w(x),Dw(x)) \;\; a.e. \tag{3.2}$$

Furthermore, if $w \in \hat{u} + \mathring{W}^1L_\Phi(\Omega,\mathbb{R}^d)$ solves (3.1), *then* (3.2) *holds for some v^* from $L_{\Phi^*}(\Omega,\mathbb{M}^{n\times d})$.*

Proof. Let $A : \mathring{W}^1L_\Phi(\Omega,\mathbb{R}^d) \to L_\Phi(\Omega,\mathbb{R}^d) \times L_\Phi(\Omega,\mathbb{M}^{n\times d})$ be defined by

$$A(u) = (u, Du). \tag{3.3}$$

Note that A is a continuous linear operator.

We show that $A^* : L_{\Phi^*}(\Omega,\mathbb{R}^d) \times L_{\Phi^*}(\Omega,\mathbb{M}^{n\times d}) \to W^{-1}L_\Phi(\Omega,\mathbb{R}^d)$ has the form

$$A^*(u^*,v^*) = u^* - \operatorname{div} v^*. \tag{3.4}$$

In fact, by $\Phi \in \triangle_2$ we have $(L_\Phi(\Omega, \mathbb{R}^d) \times L_\Phi(\Omega, \mathbb{M}^{n\times d}))^* = L_{\Phi^*}(\Omega, \mathbb{R}^d) \times L_{\Phi^*}(\Omega, \mathbb{M}^{n\times d})$. Hence for $y^* = (u^*, v^*)$ with $u^* \in L_{\Phi^*}(\Omega, \mathbb{R}^d)$ and $v^* \in L_{\Phi^*}(\Omega, \mathbb{M}^{n\times d})$ we deduce that

$$\begin{aligned}[A^*(u^*, v^*)](u) &= (u^*, v^*)[Au] = \big\langle (u^*, v^*), (u, Du)\big\rangle \\ &= u^*(u) + v^*(Du) = u^*(u) - \operatorname{div} v^*(u) = (u^* - \operatorname{div} v^*)(u),\end{aligned}$$

since $C_0^\infty(\Omega)$ is dense in $L_{\Phi^*}(\Omega)$ (see [5, Lemma 2.1]) and

$$\begin{aligned}v^*(Du) &= \int_\Omega \sum_{i=1}^n \Big\langle \frac{\partial}{\partial x_i} u, v_i^*\Big\rangle\, dx = -\int_\Omega \sum_{i=1}^n \Big\langle u, \frac{\partial}{\partial x_i} v_i^*\Big\rangle\, dx \\ &= -\int_\Omega \Big\langle u, \sum_{i=1}^n \frac{\partial}{\partial x_i} v_i^*\Big\rangle\, dx = -\int_\Omega \langle u, \operatorname{div} v^*\rangle\, dx = -\operatorname{div} v^*(u).\end{aligned}$$

Now, by (3.3) one can rewrite the functional G in the form

$$G(w) = (\mathcal{F} \circ A)(w), \tag{3.5}$$

where $\mathcal{F} : L_\Phi(\Omega, \mathbb{R}^d) \times L_\Phi(\Omega, \mathbb{M}^{n\times d}) \to \mathbb{R}$ is defined by

$$\mathcal{F}(w, z) := \int_\Omega g(x, w(x), z(x))\, dx.$$

By [14, Theorem 4.3] and Płuciennik–Tian–Wang [17, Theorem 2], the functional $\mathcal{F}$ is Lipschitz continuous on each ball of $L_\Phi(\Omega, \mathbb{R}^d) \times L_\Phi(\Omega, \mathbb{M}^{n\times d})$. Hence, by (3.5), Aubin [1, Proposition 2, p. 216] and Clarke [4, Theorem 2.3.10, Remark 2.3.11], we obtain

$$\partial_C(\mathcal{F} \circ A)(w) \subset A^* \partial_C(\mathcal{F})[A(w)].$$

So, if $0 \in \partial_C G(w)$ then there exists $(w^*, v^*) \in L_{\Phi^*}(\Omega, \mathbb{R}^d) \times L_{\Phi^*}(\Omega, \mathbb{M}^{n\times d})$ such that

$$(w^*, v^*) \in \partial_C(\mathcal{F})[A(w)] \quad \text{and} \quad A^*(w^*, v^*) = 0. \tag{3.6}$$

By (3.4), we obtain $w^* = \operatorname{div} v^* \in L_{\Phi^*}(\Omega, \mathbb{R}^d)$ for $v^* \in L_{\Phi^*}(\Omega, \mathbb{M}^{n\times d})$.

Next, either by [14, Theorem 4.3] under one of the conditions (EL1), (EL2), or by Płuciennik–Tian–Wang [17, Theorem 2] under (EL3) for the functional $\mathcal{F}$ defined on $L_\Phi(\Omega, \mathbb{R}^d \times \mathbb{M}^{n\times d})$, we infer that $\partial_C(\mathcal{F})[A(w)]$ contains measurable selections of the multifunction $x \mapsto \partial_C g(x, w(x), Dw(x))$. By (3.6), it follows that $(\operatorname{div} v^*(x), v^*(x)) \in \partial_C g(x, w(x), Dw(x))$ a.e., and (3.2) is proved.

Now suppose that w is a local minimizer for G. By Clarke [4, Proposition 2.3.2], we obtain $0 \in \partial_C G(w)$, and so w satisfies (3.2). ∎

Remark 3.4. If $g(x, \cdot) : \mathbb{R}^d \times \mathbb{M}^{n\times d} \to \mathbb{R}$ is a C^1-class function, then $\partial_C g(x, u) = \{g'_u(x, u)\}$ due to Clarke [4, Proposition 2.3.6]. Hence (3.2) implies the Euler–Lagrange equation in Orlicz–Sobolev spaces (see Gossez and Manásevich [7]).

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