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SYSTEMS OF SINGULAR BVPs — EXISTENCE OF SOLUTIONS AND THEIR PROPERTIES

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Abstract. We discuss the existence and properties of solutions for systems of singular second-order ODEs in both sublinear and superlinear cases. Our approach is based on the variational method enriched by some topological ideas. We also investigate the continuous dependence of solutions on functional parameters.

1. Introduction. We investigate the existence of positive solutions for the following system of ODEs with mixed boundary conditions

$$\begin{cases} -(u_i''(t) + \frac{k}{t}u_i'(t)) = f_{u_i}(t, \mathbf{u}(t)) \text{ a.e. in } (0,T), \\ u_i'(0) = 0 \text{ and } u_i(T) = 0, \end{cases} \qquad \text{for all } i = 1, \dots, L, \tag{1}$$

where $T > 0$, $k, L \in N := \{1, 2, \dots\}$, $\mathbf{u} = (u_1, \dots, u_L)$, $f_{u_i}(t, \mathbf{u}) = \frac{\partial}{\partial u_i} f(t, \mathbf{u})$. As a solution of our problem we understand a function $\mathbf{u} : [0,T] \to R^L$ such that $\mathbf{u} \in C^1([0,T])$, $\mathbf{u}'' \in L^2_{\mathrm{loc}}(0,T)$ and $\mathbf{u}$ satisfies (1).

We consider the case when the nonlinearity f satisfies the local conditions:

(f1) $f : [0,T] \times \widetilde{\mathbf{I}} \to R$ is a Carathéodory function, where $\widetilde{\mathbf{I}}$ is an open L-dimensional interval such that $\mathbf{0} \in \widetilde{\mathbf{I}}$ for a.a. $t \in [0,T]$, $f(t, \cdot)$ is convex in $\widetilde{\mathbf{I}}$;

(f2) for each $i \in \{1, \dots, L\}$, f_{u_i} is positive in $(0,T) \times \widetilde{\mathbf{I}}$ and there exists a vector of positive numbers $\mathbf{d} := [d_1, \dots, d_L] \in \widetilde{\mathbf{I}}$ such that $\int_0^T \max_{\mathbf{u} \in \mathbf{I}} f_{u_i}(l, \mathbf{u})\, dl < d_i/T$ and $\max_{\mathbf{u} \in \mathbf{I}} f_{u_i}(\cdot, \mathbf{u}) \in L^2(0,T)$, where $\mathbf{I} := [0, d_1] \times \dots \times [0, d_L]$.

There are a lot of papers devoted to similar problems (see, among others, e.g. [1], [2], or [6] and references therein). In [1] the authors discussed (1) with $L = 1$. Their main tool was the shooting method, but in one part of the proof of the existence result, they use the

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existence of solution for a variational elliptic equation in a ball. D. Bonheure, J. M. Gomes and L. Sanchez ([2]) combine the variational methods with the reasoning employed in [1] to study (1) for one equation. We also have to mention that similar problems appear when we investigate the existence of radial solutions of the elliptic equation

$$-\Delta u(y) = h(y, u(y)) \text{ for } y \in \Omega$$

where $\Omega \subset R^n$ is a ball, an exterior domain, an annulus or R^n, $n > 2$, with various boundary conditions. Such problems lead to (1) with $k = n-1$. (See, among others, e.g. [3], [4], [5], [7], [8] and references therein.)

In the first part of this paper we investigate the existence of positive solutions of (1). Here we employ the reasoning similar to the variational approach presented in [6] describing the case $L = 1$. Thus we treat our system as the Euler–Lagrange system for a certain integral functional J. Let us note that our approach covers both sublinear and superlinear cases and assumptions (f1) and (f2) give only the information about local behavior of the nonlinearity. Hence the classical methods of the calculus of variations cannot be applied in the standard way. First of all assumptions (f1) and (f2) do not guarantee the boundedness of J in its natural domain. Thus we construct the special set U over which we will calculate minimum of J. Next we describe the dual theory based on the Fenchel transform. Finally, we obtain the existence of solution of our problem as the limit of a certain subsequence of a minimizing sequence of J on the set U.

In the last section we consider (1) with the nonlinearity f dependent on a functional parameter. We show that if a sequence of parameters converges a.e. in $(0,T)$ then a sequence of corresponding solutions possesses a subsequence converging to a solution of the limit problem.

Now we introduce some notation. Let A denote the space of absolutely continuous functions $z : [0,T] \to R$. For given $r \in R$, let $L^2_r(0,T)$ be the space consisting of functions $z : [0,T] \to R$ such that $t^{r/2} z \in L^2(0,T)$, and let A^r consist of $z \in A$ such that $z' \in L^2_r(0,T)$.

We treat our problem as the Euler–Lagrange system for the integral functional of the form

$$J(\mathbf{u}) = \int_0^T t^k \Big(-f(t, \mathbf{u}(t)) + \frac{1}{2} \left|\mathbf{u}'(t)\right|^2 \Big)\, dt \tag{2}$$

for $\mathbf{u} \in U$, with U given by (4). Let us note that we have no information concerning the behavior of the nonlinearity outside $\widetilde{\mathbf{I}}$, thus we modify our functional as follows

$$J(\mathbf{u}) = \int_0^T t^k \Big(-\widetilde{f}(t, \mathbf{u}(t)) + \frac{1}{2} \left|\mathbf{u}'(t)\right|^2 \Big)\, dt \tag{3}$$

with

$$\widetilde{f}(t,\mathbf{u}) = \begin{cases} f(t,\mathbf{u}) & \text{if } \mathbf{u} \in \mathbf{I},\ t \in [0,T] \\ +\infty & \text{if } \mathbf{u} \in R^L \setminus \mathbf{I},\ t \in [0,T]. \end{cases}$$

Now we are ready to define the set of arguments of J in the following way:

$$\begin{aligned} U := \big\{ \mathbf{u} = (u_1, \ldots, u_L) \in C^1([0,T]) : 0 \le u_i(t) \le d_i \text{ in } [0,T],\ u_i(T) = 0, \\ t^k u_i' \in A^{-k},\ u_i'(t) \le 0 \text{ in } [0,T],\ u_i'' \in L^2_{\mathrm{loc}}((0,T]) \big\}. \end{aligned} \tag{4}$$

We develop the duality theory which relates the infimum on U of the action functional associated with our problem to the infimum of the dual functional on a corresponding set U^d. The links between minimizers of both functionals give a numerical version of the variational principle. As a consequence of this result we obtain the existence of solution of our boundary value problem. (Since we are going to restrict our investigation to the set U, we will not change the notation for the functional J containing f or $\widetilde{f}$, which is necessary only for the purpose of the duality equalities.)

LEMMA 1.1. *If* (f1)–(f2) *hold then for each* $u = (u_1, \dots, u_L) \in U$ *there exists* $\widetilde{\mathbf{u}} = (\widetilde{u}_1, \dots, \widetilde{u}_L)$ *such that for all* $i = 1, 2, \dots, L$,

$$-(t^k \widetilde{u}_i'(t))' = t^k f_{u_i}(t, \mathbf{u}(t)) \ \text{a.e. in } (0,T). \tag{5}$$

Proof. Fix $\mathbf{u} \in U$. Let us note that each of L equations of the above system can be solved independently. Thus we define $\widetilde{\mathbf{u}} = (\widetilde{u}_1, \dots, \widetilde{u}_L)$ as follows

$$\widetilde{u}_i(t) = \int_t^T \frac{1}{s^k} \int_0^s l^k f_{u_i}(l, \mathbf{u}(l))\, dl\, ds.$$

After easy calculations one sees that $\widetilde{u}_i$ satisfies (5), $u_i(T) = 0$ and for each $t \in [0,T]$

$$0 \le \widetilde{u}_i(t) \le \int_0^T \int_0^s f_{u_i}(l, \mathbf{u}(l))\, dl\, ds \le T \int_0^T f_{u_i}(l, \mathbf{d})\, dl \le d_i.$$

Moreover, the function $t \mapsto t^k \widetilde{u}_i'(t) = -\int_0^t l^k f_{u_i}(l, \mathbf{u}(l))\, dl$ belongs to A^{-k}, $\widetilde{u}_i'(t) \le 0$ for $t \in (0,T]$ and $\widetilde{u}_i'' \in L^2_{\mathrm{loc}}(0,T)$. Our task is now to prove that $\widetilde{u}_i'$ is bounded and continuous also in $[0,T]$. We have the following chain of inequalities

$$\begin{aligned}\left|t^k \widetilde{u}_i'(t)\right| &\le \Big(\int_0^t l^{2k}\, dl\Big)^{1/2} \Big(\int_0^t \big(f_{u_i}(l, \mathbf{u}(l))\big)^2\, dl\Big)^{1/2} \\ &\le \Big(\frac{1}{2k+1}\Big)^{1/2} \Big(\int_0^T \max_{\mathbf{u}\in\mathbf{I}} (f_{u_i}(l, \mathbf{u}))^2\, dl\Big)^{1/2} t^{k+1/2}\end{aligned}$$

for all $t \in [0,T]$, which gives

$$|\widetilde{u}_i'(t)| \le c_i t^{1/2} \text{ for all } t \in (0,T], \tag{6}$$

where $c_i = (\frac{1}{2k+1})^{1/2} \big(\int_0^T \max_{\mathbf{u}\in\mathbf{I}} (f_{u_i}(l, \mathbf{u}))^2\, dl\big)^{1/2}$, consequently $\lim_{t\to 0^+} \widetilde{u}_i'(t) = 0$ and $\widetilde{u}_i'(0) = 0$. Finally, each $\widetilde{u}_i \in C^1([0,T])$ and further $\widetilde{\mathbf{u}} \in U$. ∎

It is worth noting that Lemma 1.1 shows topological roots of the definition of U. It is a well known fact that the first step of methods based on the fixed point theorems is to find an invariant set for a certain operator. In this paper we define the action functional on the set U satisfying property (5). If we consider the integral operator

$$\mathbf{A}:\ \mathbf{u} \mapsto \int_t^T \frac{1}{s^k} \int_0^s l^k f_{u_i}(l, \mathbf{u}(l))\, dl\, ds,$$

Lemma 1.1 can be rewritten as $\mathbf{A}U \subset U$. In our approach we will not investigate the operator $\mathbf{A}$. We concentrate only on property (5) which plays the crucial role in the duality.

2. Existence results

2.1. Duality. Now we define the dual functional $J_D : U^d \longrightarrow R$

$$J_D(\mathbf{p}) = -\int_0^T \frac{1}{2t^k} |\mathbf{p}(t)|^2 \, dt + \int_0^T t^k f^*\Big(t, -\frac{1}{t^k}\,\mathbf{p}'(t)\Big)\, dt, \tag{7}$$

where

$$U^d := \big\{\mathbf{p} = (p_1, \dots, p_L) \in A^{-k} : \text{there exists } \mathbf{u} \in U \text{ such that } \mathbf{p}(t) = t^k \mathbf{u}'(t) \text{ in } [0, T]\big\}.$$

Taking into account conditions (f1)–(f2) and Lemma 1.1 we can prove useful properties of both sets.

Remark 2.1. For each $\mathbf{u} = (u_1, \dots, u_L) \in U$ there exists $\mathbf{p} = (p_1, \dots, p_L) \in U^d$ such that

$$-\mathbf{p}'(t) \in t^k \partial_u f(t, \mathbf{u}(t)) \text{ a.e. in } (0, T),$$

where $\partial_u f$ is the subdifferential of $\widetilde{f}$ with respect to the second variable. We can rewrite this assertion as follows

$$\int_0^T \langle -\mathbf{p}'(t), \mathbf{u}(t)\rangle \, dt - \int_0^T t^k f^*\Big(t, -\frac{1}{t^k}\,\mathbf{p}'(t)\Big)\, dt = \int_0^T t^k \widetilde{f}(t, \mathbf{u}(t))\, dt,$$

where f^* is a Fenchel conjugate of $\widetilde{f}$ with respect to the second variable.

To prove this fact we fix $\mathbf{u} = (u_1, \dots, u_L) \in U$. Now it suffices to take $\mathbf{p} = (p_1, \dots, p_L)$ such that $p_i(t) = t^k \widetilde{u}_i'(t)$, where $\widetilde{\mathbf{u}} \in U$ and satisfies (5).

Remark 2.2. By the definition of U^d we infer that for all $\mathbf{p} \in U^d$ there exists $\mathbf{u} \in U$ such that

$$\int_0^T \langle \mathbf{u}'(t), \mathbf{p}(t)\rangle \, dt - \int_0^T \frac{1}{2} t^k |\mathbf{u}'(t)|^2 \, dt = \int_0^T \frac{1}{2t^k} |\mathbf{p}(t)|^2 \, dt$$

for all $i = 1, \dots, L$.

Theorem 2.3. *The following equality holds*

$$\inf_{\mathbf{u}\in U} J(\mathbf{u}) = \inf_{\mathbf{p}\in U^d} J_D(\mathbf{p}).$$

Proof. We start the proof with the observation that for each $\mathbf{u} \in U$,

$$\sup_{\mathbf{p}\in U^d}\Big\{\int_0^T \langle -\mathbf{p}'(t), \mathbf{u}(t)\rangle \, dt - \int_0^T t^k f^*\Big(t, -\frac{1}{t^k}\,\mathbf{p}'(t)\Big)\, dt\Big\} = \int_0^T t^k \widetilde{f}(t, \mathbf{u}(t))\, dt. \tag{8}$$

Indeed, for given $\mathbf{u} \in U$, Remark 2.1 leads to the existence of $\mathbf{p_u} \in U^d$ such that

$$\int_0^T \langle -\mathbf{p}_{\mathbf{u}}'(t), \mathbf{u}(t)\rangle \, dt - \int_0^T t^k f^*\Big(t, -\frac{1}{t^k}\,\mathbf{p}_{\mathbf{u}}'(t)\Big)\, dt = \int_0^T t^k \widetilde{f}(t, \mathbf{u}(t))\, dt.$$

Therefore we get

$$\begin{aligned}\int_0^T t^k \widetilde{f}(t,\mathbf{u}(t))\,dt &= \int_0^T \langle -\mathbf{p}'_{\mathbf{u}}(t),\mathbf{u}(t)\rangle\,dt - \int_0^T t^k f^*\Big(t,-\frac{1}{t^k}\,\mathbf{p}'_{\mathbf{u}}(t)\Big)\,dt\\ &\le \sup_{\mathbf{p}\in U^d}\left\{\int_0^T \langle -\mathbf{p}'(t),\mathbf{u}(t)\rangle\,dt - \int_0^T t^k f^*\Big(t,-\frac{1}{t^k}\,\mathbf{p}'(t)\Big)\,dt\right\}\\ &\le \int_0^T t^k f^{**}(t,\mathbf{u}(t))\,dt = \int_0^T t^k \widetilde{f}(t,\mathbf{u}(t))\,dt,\end{aligned}$$

where $f^{**} = (f^*)^*$ and the last equality is due to the convexity of $\widetilde{f}(t,\cdot)$. Thus all inequalities are actually equalities and (8) holds.

Now we show that for all $\mathbf{p} = (p_1,\dots,p_L) \in U^d$

$$\sup_{\mathbf{u}\in U}\left\{\int_0^T \langle \mathbf{p}(t),\mathbf{u}'(t)\rangle\,dt - \int_0^T \frac{1}{2}\,t^k\,|\mathbf{u}'(t)|^2\,dt\right\} = \int_0^T \frac{1}{2t^k}\,|\mathbf{p}(t)|^2\,dt. \tag{9}$$

To this effect we fix $\mathbf{p} = (p_1,\dots,p_L) \in U^d$. By Remark 2.2 we infer the existence of $\mathbf{u}_{\mathbf{p}} \in U$ such that

$$\int_0^T \langle \mathbf{p}(t),\mathbf{u}'_{\mathbf{p}}(t)\rangle\,dt - \int_0^T \frac{1}{2}\,t^k\,\left|\mathbf{u}'_{\mathbf{p}}(t)\right|^2\,dt = \int_0^T \frac{1}{2t^k}\,|\mathbf{p}(t)|^2\,dt.$$

Applying the above assertion one can see that

$$\begin{aligned}\int_0^T \frac{1}{2t^k}\,|\mathbf{p}(t)|^2\,dt &= \int_0^T \langle \mathbf{p}(t),\mathbf{u}'_{\mathbf{p}}(t)\rangle\,dt - \int_0^T \frac{1}{2}\,t^k\,|\mathbf{u}_{\mathbf{p}}(t)|^2\,dt\\ &\le \sup_{\mathbf{u}\in U}\left\{\int_0^T \langle \mathbf{p}(t),\mathbf{u}'(t)\rangle\,dt - \int_0^T \frac{1}{2}\,t^k\,|\mathbf{u}'(t)|^2\,dt\right\}\\ &\le \sup_{\mathbf{w}\in L^2(0,T)}\left\{\int_0^T \langle \mathbf{p}(t),\mathbf{w}(t)\rangle\,dt - \int_0^T \frac{1}{2}\,t^k\,|\mathbf{w}(t)|^2\,dt\right\} = \int_0^T \frac{1}{2t^k}\,|\mathbf{p}(t)|^2\,dt\end{aligned}$$

Finally, we state again that all inequalities are actually equalities and (9) takes place.

In the last step we combine both assertions (8) and (9) and calculate

$$\begin{aligned}&\inf_{\mathbf{u}\in U} J(\mathbf{u})\\ &= \inf_{\mathbf{u}\in U}\int_0^T t^k\Big(-\widetilde{f}(t,\mathbf{u}(t)) + \frac{1}{2}|\mathbf{u}'(t)|^2\Big)\,dt\\ &= \inf_{\mathbf{u}\in U}\left(\int_0^T \frac{1}{2}\,t^k|\mathbf{u}'(t)|^2\,dt - \sup_{\mathbf{p}\in U^d}\left\{\int_0^T \langle -\mathbf{p}'(t),\mathbf{u}(t)\rangle\,dt - \int_0^T t^k f^*\Big(t,-\frac{1}{t^k}\,\mathbf{p}'(t)\Big)\,dt\right\}\right)\\ &= \inf_{\mathbf{u}\in U}\inf_{\mathbf{p}\in U^d}\left(\int_0^T \frac{1}{2}\,t^k\,|\mathbf{u}'(t)|^2\,dt - \int_0^T \langle \mathbf{p}(t),\mathbf{u}'(t)\rangle\,dt + \int_0^T t^k f^*\Big(t,-\frac{1}{t^k}\,\mathbf{p}'(t)\Big)\,dt\right)\\ &= \inf_{\mathbf{p}\in U^d}\left(\int_0^T t^k f^*\Big(t,-\frac{1}{t^k}\,\mathbf{p}'(t)\Big)\,dt - \sup_{\mathbf{u}\in U}\left[\int_0^T \langle \mathbf{p}(t),\mathbf{u}'(t)\rangle\,dt - \int_0^T \frac{1}{2}\,t^k\,|\mathbf{u}'(t)|^2\,dt\right]\right)\\ &= \inf_{\mathbf{p}\in U^d}\left(\int_0^T t^k f^*\Big(t,-\frac{1}{t^k}\,\mathbf{p}'(t)\Big)\,dt - \int_0^T \frac{1}{2t^k}\,|\mathbf{p}(t)|^2\,dt\right) = \inf_{\mathbf{p}\in U^d} J_D(\mathbf{p}),\end{aligned}$$

as we have claimed. ■

2.2. Minimizing sequences. In the sequel we assume that (f1) and (f2) hold. Our task is now to describe some properties of minimizing sequences of both functionals. This result plays the major role in the proof of the existence of at least one positive solution for system (1).

THEOREM 2.4. *If* $(\mathbf{u}_m)_{m\in N}$, *with* $\mathbf{u}_m = (u_{m1}, \ldots, u_{mL}) \in U$, $m = 1, 2, \ldots$, *is a minimizing sequence of* $J : U \to R$ *then there exists a sequence* $(\mathbf{p}_m)_{m\in N} \subset U^d$, *with* $\mathbf{p}_m = (p_{m1}, \ldots, p_{mL})$, *minimizing* $J_D : U^d \to R$ *such that*

$$-p'_{im}(t) = t^k f_{u_i}(t, \mathbf{u}_m(t)) \text{ a.e. on } (0,T) \tag{10}$$

for all $m = 1, 2, \ldots$, $i = 1, \ldots, L$, *and*

$$\lim_{m\to\infty} \int_0^T \Big(\frac{1}{2t^k} |p_{im}(t)|^2 + \frac{1}{2} t^k |u'_{im}(t)|^2 - p_{im}(t)u'_{im}(t)\Big)\, dt = 0 \tag{11}$$

for all $i = 1, \ldots, L$.

Proof. By the definition of U we obtain

$$J(\mathbf{u}) = \int_0^T t^k\Big(-\widetilde{f}(t, \mathbf{u}(t)) + \frac{1}{2} |\mathbf{u}'(t)|^2\Big)\, dt$$
$$\geq \sup_{\mathbf{u}\in \mathbf{I}} \int_0^T t^k\big(-\widetilde{f}(t, \mathbf{u}(t))\big)\, dt \geq -T^{k-1} \min_{i\in\{1,\ldots,L\}} d_i, \tag{12}$$

which implies the boundedness of J from below. Let $a := \inf_{\mathbf{u}\in U} J(\mathbf{u})$. Thus for given $\varepsilon > 0$ there exists $m_0 \in N$ such that $J(\mathbf{u}_m) - a < \varepsilon$, for all $m \geq m_0$. For each $m \in N$ Lemma 1.1 leads to the existence of $\mathbf{p}_m = (p_1, \ldots, p_L) \in U^d$ such that

$$-\mathbf{p}'_m(t) \in t^k \partial_u \widetilde{f}(t, \mathbf{u}_m(t)) \text{ a.e. in } (0,T).$$

Thus we get (10). Moreover, this inclusion gives the Fenchel equality for the functional $L^2(0,T) \ni \mathbf{u} \mapsto \int_0^T t^k \widetilde{f}(t, \mathbf{u}(t))\, dt$, which can be rewritten as follows

$$\int_0^T t^k \widetilde{f}(t, \mathbf{u}_m(t))\, dt = \int_0^T -t^k f^*\Big(t, -\frac{1}{t^k}\, \mathbf{p}'_m(t)\Big)\, dt - \int_0^T \langle \mathbf{p}'_m(t), \mathbf{u}_m(t)\rangle\, dt. \tag{13}$$

Therefore we get for all $m \geq m_0$,

$$a + \varepsilon > J(\mathbf{u}_m) = \int_0^T t^k f^*\Big(t, -\frac{1}{t^k}\, \mathbf{p}'_m(t)\Big)\, dt + \int_0^T \langle \mathbf{p}'_m(t), \mathbf{u}_m(t)\rangle\, dt + \int_0^T \frac{t^k}{2} |\mathbf{u}'_m(t)|^2\, dt$$
$$\geq \inf_{\mathbf{u}\in \mathbf{U}} \Big(\int_0^T t^k f^*\Big(t, -\frac{1}{t^k}\, \mathbf{p}'_m(t)\Big)\, dt - \int_0^T \langle \mathbf{p}_m(t), \mathbf{u}'(t)\rangle\, dt + \int_0^T \frac{t^k}{2} |\mathbf{u}'(t)|^2\, dt\Big)$$
$$= -\sup_{\mathbf{u}\in U} \bigg(\int_0^T \langle \mathbf{p}_m(t), \mathbf{u}'(t)\rangle\, dt - \int_0^T \frac{t^k}{2} |\mathbf{u}'(t)|^2)\, dt\bigg) + \int_0^T t^k f^*\Big(t, -\frac{1}{t^k}\, \mathbf{p}'_m(t)\Big)\, dt$$
$$= \int_0^T t^k f^*\Big(t, -\frac{1}{t^k}\, \mathbf{p}'_m(t)\Big)\, dt - \int_0^T \frac{1}{2t^k} |\mathbf{p}_m(t)|^2\, dt = J_D(\mathbf{p}_m),$$

where the last equality follows from (9). Moreover, by Theorem 2.3, we have $a := \inf_{\mathbf{u}\in U} J(\mathbf{u}) = \inf_{\mathbf{p}\in U^d} J_D(\mathbf{p})$. Finally one infers that $(\mathbf{p}_m)_{m\in N}$ in a minimizing sequence of J_D, namely $a := \inf_{m\in N} J_D(\mathbf{p}_m)$. Summarizing, we have proved that for given $\varepsilon > 0$

there exists $m_0 \in N$ such that for all $m > m_0$

$$-\varepsilon \leq J(\mathbf{u}_m) - J_D(\mathbf{p}_m) \leq \varepsilon.$$

Now, taking into account the definition of the Fenchel transform for $\mathbf{u} \mapsto \int_0^T \frac{1}{2} t^k \left|\mathbf{u}'(t)\right|^2 \, dt$ and (13), we get

$$\begin{aligned}
0 &\leq -\int_0^T \langle \mathbf{p}_m(t), \mathbf{u}_m'(t)\rangle \, dt + \int_0^T \frac{1}{2}\, t^k \left|\mathbf{u}_m'(t)\right|^2 dt + \int_0^T \frac{1}{2t^k} \left|\mathbf{p}_m(t)\right|^2 \, dt \\
&= -\int_0^T \langle \mathbf{p}_m(t), \mathbf{u}_m'(t)\rangle \, dt + \int_0^T \frac{1}{2}\, t^k \left|\mathbf{u}_m'(t)\right|^2 \, dt + \int_0^T \frac{1}{2t^k} \left|\mathbf{p}_m(t)\right|^2 \, dt \\
&\quad - \int_0^T t^k \widetilde{f}(t, \mathbf{u}_m(t)) \, dt - \int_0^T t^k f^*\Big(t, -\frac{1}{t^k}\, \mathbf{p}_m'(t)\Big) \, dt - \int_0^T \langle \mathbf{p}_m'(t), \mathbf{u}_m(t)\rangle \, dt \\
&= J(\mathbf{u}_m) - J_D(\mathbf{p}_m) \leq \varepsilon,
\end{aligned}$$

which implies (11). ∎

2.3. The existence of positive solutions for the BVP. In this section we prove that there exists at least one positive solution of our problem in U. We also describe an example of the application of our theory.

THEOREM 2.5. *If the conditions* (f1)–(f2) *are satisfied then there exists at least one solution* $\overline{\mathbf{u}} = (\overline{u}_1, \dots, \overline{u}_L) \in U$ *of* (1) *such that each coordinate* $\overline{u}_i$ *is positive and decreasing in* $(0, T)$. *Moreover,*

$$\inf_{\mathbf{u} \in U} J(\mathbf{u}) = J(\overline{\mathbf{u}}). \tag{14}$$

Proof. We start the proof with the observation that it is possible to choose a minimizing sequence $(\mathbf{u}_m)_{m\in N} = (u_{1m}, \dots, u_{Lm})_{m\in N}$ for J from the sets $S_a := \{u \in U\colon J(u) \leq a\}$, where $a \in R$ is sufficiently large to make S_a nonempty. Hence we state the boundedness of $(t^{k/2} u_{im}')_{m\in N}$, for each $i = 1, \dots, L$, in $L^2(0, T)$, consequently, by the definition of U, we infer the boundedness of $((t^k u_{im})')_{m\in N}$ in $L^2(0, T)$. Therefore $(t^k u_{im})_{m\in N}$ is uniformly convergent (up to a subsequence) to a certain $z_i \in W_0^{1,2}(0, T)$. Thus we can consider z_i as a continuous function in $[0, T]$ and state that $0 \leq z_i(t) \leq t^k d_i$ in $[0, T]$. On the other hand, $(u_{im})_{m\in N}$ is also weakly convergent (up to a subsequence) in $L^2(0, T)$ to certain $\overline{u}_i \in L^2(0, T)$. It is clear that $z_i(t) = t^k \overline{u}_i(t)$ a.e. in $(0, T)$ and $0 \leq \overline{u}_i(t) \leq d_i$. Finally, we can consider $\overline{u}_i$ as the element of $C((0, T])$. Our task is now to show that $\overline{\mathbf{u}} = (\overline{u}_1, \dots, \overline{u}_L) \in C^1([0, T])$. To this effect we apply Theorem 2.4 which leads to the existence of $(\mathbf{p}_m)_{m\in N} = (p_{1m}, \dots, p_{Lm})_{m\in N} \subset U^d$ such that for all $i \in \{1, \dots, L\}$,

$$-p_{im}'(t) = t^k f_{u_i}(t, \mathbf{u}_m(t)) \text{ for a.e. } t \in (0, T), \tag{15}$$

and

$$\lim_{m\to\infty} \int_0^T \Big(\frac{1}{2t^k} \left|p_{im}(t)\right|^2 + \frac{1}{2}\, t^k \left|u_{im}'(t)\right|^2 + p_{im}'(t) u_{im}(t)\Big) \, dt = 0. \tag{16}$$

Taking into account (15) and (f2) we infer the boundedness of the sequences $(p_{im}')_{m\in N}$ and $(p_{im}'/t^{k/2})_{m\in N}$ in $L^2(0, T)$, and consequently, going if necessary to a subsequence, we state that $(p_{im}')_{m\in N}$ and $(p_{im}'/t^{k/2})_{m\in N}$ tend weakly in $L^2(0, T)$. Moreover, by (16), we infer the boundedness of $(p_{im})_{m\in N}$ in $L^2(0, T)$. Now, passing if necessary to a subsequence, we state the weak convergence of $(p_{im})_{m\in N}$ in $W^{1,2}(0, T)$. Finally, there exists

$\overline{p}_i \in W^{1,2}(0,T)$ such that $(p_{im})_{m\in N}$ tends uniformly to $\overline{p}_i$. Thus we obtain the continuity of $\overline{p}_i$. Now we claim that for all $i=1,\ldots,L$ the following assertion holds

$$\overline{p}_i'(t) = -t^k f_{u_i}(t,\overline{\mathbf{u}}(t)) \text{ a.e. in } (0,T). \tag{17}$$

Applying assumption (f1), equality (15) and properties of $(\mathbf{u}_m)_{m\in N}$ and $(\mathbf{p}_m')_{m\in N}$ we have

$$\begin{aligned} 0 &= \int_0^T \langle \mathbf{p}_m'(t), \mathbf{u}_m(t)\rangle\, dt + \int_0^T t^k f^*\Big(t, -\frac{\mathbf{p}_m'(t)}{t^k}\Big)\, dt + \int_0^T t^k f(t,\mathbf{u}_m(t))\, dt \\ &\geq \int_0^T \langle \overline{\mathbf{p}}'(t), \overline{\mathbf{u}}(t)\rangle\, dt + \int_0^T t^k f^*\Big(t, -\frac{\overline{\mathbf{p}}'(t)}{t^k}\Big)\, dt + \int_0^T t^k f(t,\overline{\mathbf{u}}(t))\, dt \geq 0. \end{aligned}$$

Now the properties of the Fenchel conjugate imply

$$\big\langle \overline{\mathbf{p}}'(t), \overline{\mathbf{u}}(t)\big\rangle + t^k f^*\Big(t, -\frac{\overline{\mathbf{p}}'(t)}{t^k}\Big) + t^k f(t,\overline{\mathbf{u}}(t)) = 0$$

and further $-\overline{\mathbf{p}}'(t) \in t^k \partial_{\mathbf{u}} f(t,\overline{\mathbf{u}}(t))$ a.e. in $(0,T)$, which gives (17).

Taking into account (16), the uniform convergence of $(p_{im})_{m\in N}$ and the weak convergence of $(u_{im}')_{m\in N}$ in $L^2(0,T)$, we calculate

$$\begin{aligned} 0 &\geq \lim_{m\to\infty} \int_0^T \Big(\frac{1}{2t^k}\, |p_{im}(t)|^2 + \frac{1}{2}\, t^k\, |u_{im}'(t)|^2 - p_{im}(t) u_{im}'(t)\Big)\, dt \\ &\geq \int_0^T \Big(\frac{1}{2t^k}\, |\overline{p}_i(t)|^2 + \frac{1}{2}\, t^k\, |\overline{u}_i'(t)|^2 - \overline{p}_i(t)\overline{u}_i'(t)\Big)\, dt \geq 0, \end{aligned}$$

where the last inequality is due to the properties of the Fenchel transform. Thus we get for all $i=1,\ldots,L$, $\overline{p}_i(t) = t^k \overline{u}_i'(t)$ a.e. in $(0,T)$. Combining the previous assertion with (17), we obtain

$$(t^k \overline{u}_i'(t))' = -t^k f_{u_i}(t,\overline{\mathbf{u}}(t)) \text{ for a.a. } t\in(0,T)$$

and further we infer that $\overline{\mathbf{u}}$ satisfies (1).

Now we investigate properties of $\overline{u}_i$. The above reasoning implies that $\overline{u}_i \in C^1((0,T])$, $\overline{u}_i'' \in L^2_{\mathrm{loc}}(0,T)$, and $0 \leq \overline{u}_i(t) \leq d$ in $[0,T]$. Applying the same scheme as in the proof of Lemma 1.1 we get $\lim_{t\to 0^+} \overline{u}_i'(t) = 0$. So we can consider $\overline{u}_i$ as the element of $C^1([0,T])$ with $\overline{u}_i'(0)=0$. Moreover, one can note that it is decreasing, namely $\overline{u}_i' < 0$ in $(0,T]$. If we suppose otherwise, we can infer the existence of a positive number $s_0 < T$ such that $\overline{u}_i'(s_0)=0$, and further

$$\int_0^{s_0} l^k f_{u_i}(l,\overline{\mathbf{u}}(l))\, dl = -\int_0^{s_0} (t^k \overline{u}_i'(t))'\, dt = -s_0^k \overline{u}_i'(s_0) = 0,$$

which implies $f_{u_i}(l,\overline{\mathbf{u}}(l)) = 0$ for a.a. $l\in(0,s_0)$. This is a contradiction with the positivity of f_{u_i} in $(0,T)\times I$. Thus one can also infer that $\overline{u}_i(t) > 0$ for all $t\in[0,T)$. Finally $\overline{\mathbf{u}} \in U$.

Coming to the last part of the proof, we have to note that

$$\inf_{\mathbf{u}\in U} J(\mathbf{u}) = \liminf_{m\to\infty} \int_0^T t^k \Big(-f(t,\mathbf{u}_m(t)) + \frac{1}{2}\, |\mathbf{u}_m'(t)|^2\Big)\, dt \geq J(\overline{\mathbf{u}}). \ \blacksquare$$

EXAMPLE 2.6. Let us consider the problem

$$\begin{cases} -(u_1''(t)+\frac{k}{t}u_1'(t)) = \frac{1}{8}\big(a_1(t)e^{u_1(t)} + u_1(t) + u_2(t)\big) & \text{a.e. in } (0,1), \\ -(u_2''(t)+\frac{k}{t}u_2'(t)) = \frac{1}{8}\big(a_2(t)e^{u_2(t)} + \frac{u_2(t)+1}{(3-u_2(t))(5-u_2(t))} + u_1(t)+u_2(t)\big) & \text{a.e. in } (0,1), \\ u_1'(0)=u_2'(0)=0 \text{ and } u_1(1)=u_2(1)=0, \end{cases} \tag{18}$$

where $k > 1$ and $a_i \in L^\infty(0,1)$ for $i = 1,2$. If we assume that $a_i(t) > 0$ a.e. in $(0,1)$ and $\operatorname{ess\,sup} a_i(t) \leq 11/e^2$, then there exists at least one positive solution $\overline{\mathbf{u}} = (\overline{u}_1, \overline{u}_2) \in C^1([0,1])$ of (18) such that each u_i, $i = 1,2$, is decreasing.

Proof. Let us note that in our example we have

$$f(t,u_1,u_2) = \frac{1}{8}\Big[a_1(t)e^{u_1} + a_2(t)e^{u_2} + 3\ln(u_2-5) - 2\ln(u_2-3) + \frac{1}{2}(u_1+u_2)^2\Big].$$

It is clear that f satisfies (f1) for $\widetilde{\mathbf{I}} = (-\frac{1}{2},3)\times(-\frac{1}{2},3)$. Now we show that (f2) holds for $d_1 = d_2 = 2$. Let $\mathbf{I} = [0,2]\times[0,2]$. After easy calculations we state that for $i \in \{1,2\}$, $\sup_{\mathbf{u}\in\mathbf{I}} f_{u_i}(t,\mathbf{u}) \in L^2(0,T)$ and

$$\int_0^1 \sup_{\mathbf{u}\in\mathbf{I}} f_{u_1}(t,\mathbf{u})\,dt \leq \frac{e^2}{8}\int_0^1 a_1(t)\,dt + \frac{1}{2} \leq 2,$$
$$\int_0^1 \sup_{\mathbf{u}\in\mathbf{I}} f_{u_2}(t,\mathbf{u})\,dt \leq \frac{e^2}{8}\int_0^1 a_2(t)\,dt + \frac{5}{8} \leq 2.$$

Theorem 2.5 yields the existence of at least one solution $\overline{\mathbf{u}} = (\overline{u}_1,\overline{u}_2) \in C^1([0,1])$ for (18) such that for each $i = 1,2$, $\overline{u}_i$ is positive, decreasing and bounded by 2. ■

3. Systems of BVPs with parameters. In this section we investigate the continuous dependence of solutions on functional parameters for the system

$$\begin{cases} -(u_i''(t)+\frac{k}{t}u_i'(t)) = f_{u_i}(t,\mathbf{u}(t),\mathbf{w}(t)) \text{ a.e. in } (0,T), \\ u_i'(0)=0 \text{ and } u_i(T)=0, \end{cases} \quad \text{for all } i=1,\ldots,L \tag{19}$$

where $\mathbf{w} : (0,T) \to R^q$, $\mathbf{w} \in W \subset L^p(0,T)$, with $p > 1$, $k,q \in N$, $f_{u_i}(t,\mathbf{u},\mathbf{w}) = \frac{\partial}{\partial u_i} f(t,\mathbf{u},\mathbf{w})$. We shall prove that a sequence of positive solutions $(\mathbf{u}_m)_{m\in N}$ of (19) (corresponding to a sequence of parameters $(\mathbf{w}_m)_{m\in N}$) possesses a subsequence, still denoted by $(\mathbf{u}_m)_{m\in N}$, tending uniformly to an element $\mathbf{u}_0$ in $[0,T]$ (where $\mathbf{u}_0$ is a solution of (19) with $\mathbf{w} = \mathbf{w}_0$), provided that the sequence of parameters $(\mathbf{w}_m)_{m\in N}$ tends to $\mathbf{w}_0$ a.e. in $(0,T)$. We start with the assumptions which guarantee that for each parameter $\mathbf{w} \in W$ there exists at least one positive solution $\mathbf{u}_{\mathbf{w}} \in U$:

(f1w) $f : [0,T]\times\widetilde{\mathbf{I}}\times R^q \to R$ is the Carathéodory function, where $\widetilde{\mathbf{I}}$ is an open L-dimensional interval such that $\mathbf{0} \in \widetilde{\mathbf{I}}$; for a.a. $t \in [0,T]$ and all $\mathbf{w} \in R^q$, $f(t,\cdot,\mathbf{w})$ is convex in $\widetilde{\mathbf{I}}$ and for each $i = 1,\ldots,L$, f_{u_i} is positive in $(0,T)\times\widetilde{\mathbf{I}}\times R^q$;

(f2w) there exists a vector of positive numbers $\mathbf{d} := [d_1,\ldots,d_L] \in \widetilde{\mathbf{I}}$ such that for all $\mathbf{w} \in W$ and $i \in \{1,\ldots,L\}$,

$$\sup_{\mathbf{u}\in\mathbf{I}} f_{u_i}(\cdot,\mathbf{u},\mathbf{w}) \in L^2(0,T) \quad\text{and}\quad \int_0^T \sup_{\mathbf{u}\in\mathbf{I}} f_{u_i}(l,\mathbf{u},\mathbf{w}(l))\,dl < d_i/T,$$

where $\mathbf{I} := [0,d_1]\times\ldots\times[0,d_L]$.

THEOREM 3.1. *Suppose that conditions* (f1w) *and* (f2w) *hold and consider a sequence of parameters* $(\mathbf{w}_m)_{m\in N} \subset W$ *converging a.e. in* $(0,T)$ *to a certain* $\mathbf{w}_0 \in W$. *For each* $m \in N$, *denote by* $\mathbf{u}_m \in U$ *a solution of* (19) *corresponding to the parameter* $\mathbf{w}_m$. *Then there exists a subsequence, still denoted by* $(\mathbf{u}_m)_{m\in N}$, *which converges to* $\mathbf{u}_0$ *being a solution of* (19) *with parameter* $\mathbf{w}_0$.

Proof. Since $(\mathbf{u}_m)_{m\in N} \subset U$ we have

$$0 \le u_{mi}(t) \le d_i \text{ in } [0,T]$$

for all $i = 1,\dots,L$. Moreover, we get

$$\begin{aligned}\int_0^T (t^{k/2}u'_{im}(t))^2\,dt &= -\int_0^T (t^k u'_{im}(t))' u_{im}(t)\,dt \\ &= \int_0^T f_{u_i}(t,\mathbf{u}_m(t),\mathbf{w}_m(t))u_{im}(t)\,dt \le d_i \int_0^T \sup_{\mathbf{u}\in\mathbf{I}} f_{u_i}(l,\mathbf{u},\mathbf{w}_m(l))\,dl < d_i^2/T,\end{aligned}$$

which implies the boundedness of $(t^{k/2}u'_{im})_{m\in N}$ in $L^2(0,T)$ and further the boundedness of $\big((t^k u_{im})'\big)_{m\in N}$ in $L^2(0,T)$. Summarizing, we state the existence of a subsequence still denoted by $((t^k u_{im})')_{m\in N}$ which is weakly convergent in $W_0^{1,2}(0,T)$ to a certain $z_i \in W_0^{1,2}(0,T)$. Thus $((t^k u_{im})')_{m\in N}$ is uniformly convergent to z_i in $[0,T]$. On the other hand, $(u_{im})_{m\in N}$ is also bounded in $L^2(0,T)$. Therefore (up to a subsequence) $(u_{im})_{m\in N}$ tends weakly to a certain $u_{i0} \in L^2(0,T)$. Thus $z_i(t) = t^k u_{i0}(t)$ a.e. in $[0,T]$ and consequently $u_{i0} \in C((0,T])$. Our task is now to show that $\mathbf{u}_0 := (u_{10},\dots,u_{L0})$ is a solution of (19) corresponding to parameter $\mathbf{w}_0$. To this effect we consider the sequence $\{\mathbf{p}_m\}_{m\in N}$, given by $\mathbf{p}_m := (p_{1m},\dots,p_{Lm})$, where

$$p_{im}(t) := t^k u'_{im}(t) \text{ in } (0,T) \tag{20}$$

for each $i \in \{1,\dots,L\}$. Since for each $m \in \mathbf{N}$, $\mathbf{u}_m$ is the solution of (19) corresponding to the parameter $\mathbf{w}_m$ we get

$$p'_{im}(t) = -f_{u_i}\big(t,\mathbf{u}_m(t),\mathbf{w}_m(t)\big) \text{ a.e. in } (0,T). \tag{21}$$

Combining both assertions (20) and (21) with the boundedness of $(u'_{im})_{m=1}^{\infty}$ in $L^2(0,T)$, we derive the existence of a subsequence still denoted by $(p_{im})_{m\in N}$ weakly convergent to a certain p_{i0} in $W^{1,2}(0,T)$. Thus $(p_m)_{m\in N}$ is uniformly convergent to p_{i0}, and further p_{i0} is continuous. Taking into account (20) and (21) and applying the reasoning as in the proof of Theorem 2.5 we obtain for each $i \in \{1,\dots,k\}$

$$p'_{i0}(t) = -f_{u_i}(t,\mathbf{u}_0(t),\mathbf{w}_0(t)) \text{ a.e. in } (0,T), \tag{22}$$

$$p_{i0}(t) := t^k u'_{i0}(t) \text{ in } (0,T), \tag{23}$$

and $\mathbf{u}_0 \in U$. Substituting (23) into (22) one infers that $\mathbf{u}_0$ is a solution of (19) corresponding to the parameter $\mathbf{w}_0$. ■

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