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SEMILINEAR HYPERBOLIC FUNCTIONAL EQUATIONS

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Abstract. We consider second order semilinear hyperbolic functional differential equations where the lower order terms contain functional dependence on the unknown function. Existence and uniqueness of solutions for $t \in (0,T)$, existence for $t \in (0,\infty)$ and some qualitative properties of the solutions in $(0,\infty)$ are shown.

1. Introduction. In the present paper we consider weak solutions of initial-boundary value problems of the form

$$u''(t) + \tilde{Q}(u(t)) + \varphi(x)h'(u(t)) + H(t,x;u) + G(t,x;u,u') = F, \quad t > 0, \quad x \in \Omega, \tag{1}$$

$$u(0) = u_0, \quad u'(0) = u_1, \tag{2}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain and we use the notation $u(t) = u(t,x)$, $u' = D_t u$, $u'' = D_t^2 u$, $\tilde{Q}$ may be a linear second order symmetric elliptic differential operator in the variable x; h is a C^1 function having certain polynomial growth, H and G contain nonlinear functional (non-local) dependence on u and u', with some polynomial growth.

There are several papers on semilinear hyperbolic differential equations, see, e.g., [3], [4], [10], [13] and the references therein. Semilinear hyperbolic functional equations were studied, e.g. in [5], [6], [7], with certain non-local terms, generally in the form of particular integral operators containing the unknown function. First order quasilinear evolution equations with non-local terms were considered, e.g., in [12] and [14], second order quasilinear evolution equations with non-local terms were considered in [11], by using the theory of monotone type operators (see [2], [9], [15]).

This paper was motivated by the classical work [9] of J.-L. Lions where the equation (1) was considered in the particular case $\tilde{Q} = -\triangle$, $\varphi = 1$, $h'(\eta) = \eta|\eta|^\lambda$, $H = 0$, $G = 0$

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(semilinear hyperbolic differential equation). The proofs are based on Galerkin's method and imbedding theorems in Sobolev spaces. The aim of this work is to show that the ideas of [9] can be applied to semilinear hyperbolic equations, containing non-local terms of rather general form which may be of different types (integrals with respect to the space or time variable or terms with discrete delay etc.).

In Section 2 the existence of weak solutions will be proved for $t \in (0,T)$ and in Section 3 we shall prove existence and certain properties of solutions for $t \in (0,\infty)$, finally, in Section 4 the uniqueness of the solution will be shown.

2. Existence in $(0,T)$. Denote by $\Omega \subset \mathbb{R}^n$ a bounded domain having the uniform C^1 regularity property (see [1]), $Q_T = (0,T) \times \Omega$. Denote by $W^{1,2}(\Omega)$ the Sobolev space of real valued functions with the norm

$$\|u\| = \Big[\int_\Omega \Big(\sum_{j=1}^n |D_j u|^2 + |u|^2\Big)\,dx\Big]^{1/2}.$$

Further, let $V \subset W^{1,2}(\Omega)$ be a closed linear subspace of $W^{1,2}(\Omega)$ containing $W_0^{1,2}(\Omega)$ (the closure of $C_0^\infty(\Omega)$), $V^\star$ the dual space of V, $H = L^2(\Omega)$, the duality between $V^\star$ and V will be denoted by $\langle\cdot,\cdot\rangle$, the scalar product in H will be denoted by $(\cdot,\cdot)$. Denote by $L^2(0,T;V)$ the Banach space of the set of measurable functions $u : (0,T) \to V$ with the norm

$$\|u\|_{L^2(0,T;V)} = \Big[\int_0^T \|u(t)\|_V^2\,dt\Big]^{1/2}$$

and by $L^\infty(0,T;V)$, $L^\infty(0,T;H)$ the set of measurable functions $u : (0,T) \to V$, $u : (0,T) \to H$, respectively, with the $L^\infty(0,T)$ norm of the functions $t \mapsto \|u(t)\|_V$, $t \mapsto \|u(t)\|_H$, respectively.

Now we formulate the assumptions on the functions in (1).

(A_1) $\tilde{Q} : V \to V^\star$ is a linear continuous operator such that

$$\langle \tilde{Q}\tilde{u}, \tilde{v}\rangle = \langle \tilde{Q}\tilde{v}, \tilde{u}\rangle, \quad \langle \tilde{Q}\tilde{u}, \tilde{u}\rangle \ge c_0\|\tilde{u}\|_V^2$$

for all $\tilde{u}, \tilde{v} \in V$ with some constant $c_0 > 0$. Further we shall use the notation $(Qu)(t) = \tilde{Q}(u(t))$.

(A_2) $\varphi : \Omega \to \mathbb{R}$ is a measurable function satisfying

$$c_1 \le \varphi(x) \le c_2 \text{ for a.a. } x \in \Omega$$

with some positive constants c_1, c_2.

(A_3) $h : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function satisfying

$$h(\eta) \ge 0, \quad |h'(\eta)| \le \text{const}\,|\eta|^\lambda \quad \text{for } |\eta| > 1$$

$$\text{where } 1 < \lambda \le \lambda_0 = \frac{n}{n-2} \text{ if } n \ge 3, \quad 1 < \lambda < \infty \text{ if } n = 2.$$

(A_3') $h : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function satisfying with some positive constants c_3, c_4

$$h(\eta) \geq 0, \quad c_3|\eta|^\lambda \leq |h'(\eta)| \leq c_4|\eta|^\lambda \ \text{ for } |\eta| > 1,\ n \geq 3 \text{ where } \lambda > \lambda_0 = \frac{n}{n-2},$$
$$|h'(\eta)| \leq c_4|\eta|^\lambda \quad \text{for } |\eta| > 1,\ n = 2 \text{ where } 1 < \lambda < \infty.$$

(A_4) $H : Q_T \times L^2(Q_T) \to \mathbb{R}$ is a function for which $(t, x) \mapsto H(t, x; u)$ is measurable for all fixed $u \in L^2(\Omega)$, H has the Volterra property, i.e. for all $t \in [0, T]$, $H(t, x; u)$ depends only on the restriction of u to $(0, t)$; the following inequality holds for all $t \in [0, T]$ and $u \in L^2(\Omega)$:

$$\int_0^t \int_\Omega |H(\tau, x; u)|^2 \, dx \, d\tau \leq \text{const} \int_0^t \int_\Omega h(u(\tau)) \, dx \, d\tau.$$

Further, for any fixed functions $w_1, w_2, \ldots, w_m \in V$ (if (A_3) is satisfied) and $w_1, w_2, \ldots, w_m \in V \cap L^{\lambda+1}(\Omega)$ (if (A_3') holds), respectively, for every $K > 0$ there exists $\psi_K \in L^1(0, T)$ such that for $|(c_1, c_2, \ldots, c_m)| \leq K$

$$\left[\int_\Omega \left| H\Big(t, x; \sum_{k=1}^m c_k w_k\Big)\right|^2 dx\right]^{1/2} \leq \psi_K(t), \quad t \in [0, T].$$

Finally, $(u_k) \to u$ in $L^2(Q_T)$ and $(u_k) \to u$ a.e. in Q_T imply

$$H(t, x; u_k) \to H(t, x; u) \text{ for a.a. } (t, x) \in Q_T.$$

(A_5) $G : Q_T \times L^2(Q_T) \times L^\infty(0, T; H) \to \mathbb{R}$ is a function satisfying: $(t, x) \mapsto G(t, x; u, w)$ is measurable for all fixed $u \in L^2(Q_T)$, $w \in L^\infty(0, T; H)$, G has the Volterra property: for all $t \in [0, T]$, $G(t, x; u, w)$ depends only on the restriction of u, w to $(0, t)$ and

$$|G(t, x; u, w)| \leq c_5|w(t)| + c_6$$

with some constants c_5, c_6.

Further, if

$$(u_k) \to u \text{ in } L^2(Q_T) \text{ and a.e. in } Q_T, \quad (w_k) \to w \text{ weakly in } L^\infty(0, T; H)$$

in the sense that for all fixed $g_1 \in L^1(0, T; H)$

$$\int_0^T \langle g_1(t), w_k(t)\rangle \, dt \to \int_0^T \langle g_1(t), w(t)\rangle \, dt,$$

then

$$G(t, x; u_k, w_k) \to G(t, x; u, w) \text{ weakly in } L^\infty(0, T; H).$$

THEOREM 2.1. *Assume* (A_1), (A_2), (A_3), (A_4), (A_5). *Then for all* $F \in L^2(0, T; H)$, $u_0 \in V$, $u_1 \in H$ *there exists* $u \in L^\infty(0, T; V)$ *such that*

$$u' \in L^\infty(0, T; H), \quad u'' \in L^2(0, T; V^\star),$$

u satisfies (1) *in the sense*: *for a.a.* $t \in [0,T]$, *all* $v \in V$

$$\langle u''(t), v\rangle + \langle \tilde{Q}(u(t)), v\rangle + \int_\Omega \varphi(x) h'(u(t)) v\,dx + \int_\Omega H(t,x;u) v\,dx + \int_\Omega G(t,x;u,u') v\,dx = (F(t), v) \quad (3)$$

and the initial condition (2) *is fulfilled.*

If (A_1), (A_2), (A_3'), (A_4), (A_5) *are satisfied then for all* $F \in L^2(0,T;H)$, $u_0 \in V \cap L^{\lambda+1}(\Omega)$, $u_1 \in H$ *there exists* $u \in L^\infty(0,T; V \cap L^{\lambda+1}(\Omega))$ *such that*

$$u' \in L^\infty(0,T;H),$$

$$u'' \in L^2(0,T;V^\star) + L^\infty(0,T;L^{(\lambda+1)/\lambda}(\Omega)) \subset L^2\big(0,T;[V \cap L^{\lambda+1}(\Omega)]^\star\big)$$

and u satisfies (1) *in the sense*: *for a.a.* $t \in [0,T]$, *all* $v \in V \cap L^{\lambda+1}(\Omega)$ (3) *holds, further, the initial condition* (2) *is fulfilled.*

Remark 2.2. $u'' \in L^2(0,T;V^\star) + L^\infty(0,T;L^{(\lambda+1)/\lambda}(\Omega))$ means that for the distributional derivative $u'' = D_t^2 u$ we have

$$u'' = u_1 + u_2 \text{ where } u_1 \in L^2(0,T;V^\star) \text{ and } u_2 \in L^\infty(0,T;L^{(\lambda+1)/\lambda}(\Omega)).$$

Since in this case

$$(u')' = u'' \in L^2\big(0,T;[V \cap L^{\lambda+1}(\Omega)]^\star\big)$$
$$\text{and } u' \in L^\infty(0,T;L^2(\Omega)) \subset L^2\big(0,T;[V \cap L^{\lambda+1}(\Omega)]^\star\big),$$

by Lemma 1.2 in Chapter 1 of [9]

$$u' \in C([0,T];[V \cap L^{\lambda+1}(\Omega)]^\star),$$

thus the initial condition $u'(0) = u_1 \in H$ makes sense since $H \subset [V \cap L^{\lambda+1}(\Omega)]^\star$.

Similarly, if (A_3) is satisfied, by

$$u'' \in L^2(0,T;V^\star), \quad u' \in L^\infty(0,T;L^2(\Omega)) \subset L^2(0,T;V^\star),$$

we have $u' \in C([0,T];V^\star)$, so the initial condition $u'(0) = u_1 \in H$ makes sense.

Proof. We apply Galerkin's method. Let $w_1, w_2, \dots$ be a linearly independent system in V if (A_3) is satisfied and in $V \cap L^{\lambda+1}(\Omega)$ if (A_3') is satisfied such that the linear combinations are dense in V and $V \cap L^{\lambda+1}(\Omega)$, respectively. We want to find the m-th approximation of u in the form

$$u_m(t) = \sum_{l=1}^m g_{lm}(t) w_l \quad (4)$$

where $g_{lm} \in W^{2,2}(0,T)$ if (A_3) is satisfied and $g_{lm} \in W^{2,2}(0,T) \cap L^\infty(0,T)$ if (A_3') is fulfilled, further, for all $j = 1, \dots, m$

$$\langle u_m''(t), w_j\rangle + \langle \tilde{Q}(u_m(t)), w_j\rangle + \int_\Omega \varphi(x) h'(u_m(t)) w_j\,dx + \int_\Omega H(t,x;u_m) w_j\,dx + \int_\Omega G(t,x;u_m,u_m') w_j\,dx = \langle F(t), w_j\rangle, \quad (5)$$

$$u_m(0) = u_{m0}, \quad u_m'(0) = u_{m1} \quad (6)$$

where u_{m0}, u_{m1} $(m = 1, 2, \dots)$ are linear combinations of $w_1, w_2, \dots, w_m$ satisfying

$$(u_{m0}) \to u_0 \text{ in } V \text{ and } V \cap L^{\lambda+1}(\Omega), \text{ respectively, as } m \to \infty \tag{7}$$

$$\text{and} \quad (u_{m1}) \to u_1 \text{ in } H \text{ as } m \to \infty. \tag{8}$$

It is not difficult to show that all the conditions of the existence theorem for a system of functional differential equations with Carathéodory conditions (see [8]) are satisfied. Indeed, (A_3), (A_3'), (A_4), (A_5), imply that all the terms in (5) containing the coefficients $g_{lm}(t)$ are continuous with respect to $g_{lm}(t)$ and they can be estimated by a Lebesgue integrable function if the variables $g_{lm}(t)$ and $g_{lm}'(t)$ are in a small neighbourhood.

Thus, by using the Volterra property of G and H, we obtain that there exists a solution of (5), (6) in a neighbourhood of 0. Further, the maximal solution of (5), (6) is defined in $[0, T]$. Indeed, multiplying (5) by $g_{lm}'(t)$ and taking the sum with respect to j, we obtain

$$\begin{aligned}\langle u_m''(t), u_m'(t)\rangle + \langle \tilde{Q}(u_m(t)), u_m'(t)\rangle + \int_\Omega \varphi(x) h'(u_m(t)) u_m'(t)\, dx \\ + \int_\Omega H(t, x; u_m) u_m'(t)\, dx + \int_\Omega G(t, x; u_m, u_m') u_m'(t)\, dx = (F(t), u_m'(t)).\end{aligned} \tag{9}$$

Integrating the above equality over $(0, t)$ we find by Young's inequality and by using the formulas

$$\int_0^t \langle \tilde{Q}(u_m(\tau)), u_m'(\tau)\rangle\, d\tau = \frac{1}{2}\langle \tilde{Q}(u_m(t)), u_m(t)\rangle - \frac{1}{2}\langle \tilde{Q}(u_m(0)), u_m(0)\rangle,$$

$$\int_0^t \langle u_m''(\tau), u_m'(\tau)\rangle\, d\tau = \frac{1}{2}\|u_m'(t)\|_H^2 - \frac{1}{2}\|u_m'(0)\|_H^2$$

(see [15]):

$$\begin{aligned}&\frac{1}{2}\|u_m'(t)\|_H^2 + \frac{1}{2}\langle \tilde{Q}(u_m(t)), u_m(t)\rangle + \int_\Omega \varphi(x) h(u_m(t))\, dx \\ &\quad + \int_0^t \Big[\int_\Omega H(\tau, x; u_m) u_m'(\tau)\, dx\Big] d\tau + \int_0^t \Big[\int_\Omega G(\tau, x; u_m, u_m') u_m'(\tau)\, dx\Big] d\tau \\ &= \int_0^t (F(\tau), u_m'(\tau))\, d\tau + \frac{1}{2}\|u_m'(0)\|_H^2 + \frac{1}{2}\langle (Qu_m)(0), u_m(0)\rangle + \int_\Omega \varphi(x) h(u_m(0))\, dx \\ &\leq \frac{1}{2}\int_0^T \|F(\tau)\|_H^2\, d\tau + \frac{1}{2}\int_0^t \|u_m'(\tau)\|_H^2\, d\tau + \text{const}\end{aligned} \tag{10}$$

where the constant is not depending on m and t. Indeed, by (6)–(8), $(u_m(0))$ is bounded in V and $V \cap L^{\lambda+1}(\Omega)$, respectively, and $(u_m'(0))$ is bounded in H; $(Qu_m)(0)$ is bounded in $V^\star$ by (A_1). Further, $(h(u_m(0)))$ is bounded in $L^1(\Omega)$ since by (A_3)

$$\begin{aligned}\int_\Omega h(u_m(0))\, dx &\leq \text{const} \int_\Omega \big[1 + (u_m(0))^{\lambda+1}\big]\, dx \\ &\leq \text{const} \int_\Omega \big[1 + (u_m(0))^{(2n-2)/(n-2)}\big]\, dx \leq \text{const} \int_\Omega \big[1 + (u_m(0))^{2n/(n-2)}\big]\, dx\end{aligned}$$

and by Sobolev's imbedding theorem $W^{1,2}(\Omega)$ is continuously imbedded into $L^{2n/(n-2)}(\Omega)$

and if (A_3') is satisfied then

$$\int_\Omega h(u_m(0))\,dx \le \text{const} \int_\Omega \left[1 + (u_m(0))^{\lambda+1}\right] dx \le \text{const}$$

because $(u_m(0))$ is bounded in $L^{\lambda+1}(\Omega)$.

By using (A_2), (A_4), (A_5) and the Cauchy–Schwarz inequality, we obtain from (10)

$$\begin{aligned}
&\frac{1}{2}\,\|u_m'(t)\|_H^2 + \frac{1}{2}\,\langle \tilde{Q}(u_m(t)), u_m(t)\rangle + c_1 \int_\Omega h(u_m(t))\,dx \\
&\le \frac{1}{2}\int_0^T \|F(\tau)\|_H^2\,d\tau + \text{const}\int_0^t \|u_m'(\tau)\|_H^2\,d\tau + \text{const}\int_0^t \left[\int_\Omega h(u_m(\tau))\,dx\right] d\tau + \text{const} \\
&= \text{const}\int_0^t \left[\|u_m'(\tau)\|_H^2 + \int_\Omega h(u_m(\tau))dx\right] d\tau + \text{const}. \qquad (11)
\end{aligned}$$

Consequently,

$$\|u_m'(t)\|_H^2 + \int_\Omega h(u_m(t))\,dx \le \text{const}\left\{1 + \int_0^t \left[\|u_m'(\tau)\|_H^2 + \int_\Omega h(u_m(\tau))\,dx\right] d\tau\right\}$$

where the constant is not depending on t and m. Thus by Gronwall's lemma

$$\|u_m'(t)\|_H^2 + \int_\Omega h(u_m(t))\,dx \le \text{const}. \qquad (12)$$

Hence by (11) and (A_1) we obtain in a neighbourhood of 0

$$\|u_m(t)\|_V \le \text{const} \qquad (13)$$

and the constant is not depending on t which implies that the maximal solution of (5), (6) is defined in $[0, T]$. Further, the estimates (12), (13) hold for all $t \in [0, T]$ and in the case $\lambda > \lambda_0$, $n \ge 3$

$$\|u_m(t)\|_{V \cap L^{\lambda+1}(\Omega)} \le \text{const}, \qquad (14)$$

thus

$$\|u_m\|_{L^\infty(0,T;V\cap L^{\lambda+1}(\Omega))} \le \text{const}. \qquad (15)$$

By (12), (13), if (A_3) is satisfied, there exist a subsequence of (u_m), again denoted by (u_m) and $u \in L^\infty(0, T; V)$ such that

$$(u_m) \to u \text{ weakly in } L^\infty(0, T; V), \qquad (16)$$
$$(u_m') \to u' \text{ weakly in } L^\infty(0, T; H) \qquad (17)$$

in the following sense: for any fixed $g \in L^1(0, T; V^\star)$ and $g_1 \in L^1(0, T; H)$

$$\int_0^T \langle g(t), u_m(t)\rangle\,dt \to \int_0^T \langle g(t), u(t)\rangle\,dt,$$
$$\int_0^T (g_1(t), u_m'(t))\,dt \to \int_0^T (g_1(t), u'(t))\,dt.$$

Similarly, in the case $\lambda > \lambda_0$, $n \ge 3$, there exist a subsequence of (u_m) and a function $u \in L^\infty(0, T; V) \cap L^{\lambda+1}(\Omega))$ such that

$$(u_m) \to u \text{ weakly in } L^\infty(0, T; V \cap L^{\lambda+1}(\Omega)), \qquad (18)$$

which means: for any fixed $g \in L^1(0,T;(V \cap L^{\lambda+1}(\Omega))^\star)$

$$\int_0^T \langle g(t), u_m(t)\rangle \, dt \to \int_0^T \langle g(t), u(t)\rangle \, dt.$$

Since the imbedding $W^{1,2}(\Omega)$ into $L^2(\Omega)$ is compact, by (16)–(18) we have for a subsequence

$$(u_m) \to u \text{ in } L^2(0,T;H) = L^2(Q_T) \text{ and a.e. in } Q_T. \tag{19}$$

As $\tilde{Q} : V \to V^\star$ is a linear and continuous operator, by (16) for all $v \in V$ and $v \in V \cap L^{\lambda+1}(\Omega)$, respectively, we have

$$\langle (Qu_m)(t), v\rangle \to \langle (Qu)(t), v\rangle \text{ weakly in } L^\infty(0,T) \tag{20}$$

and by (17)

$$\langle u_m''(t), v\rangle = \frac{d}{dt}\langle u_m'(t), v\rangle \to \langle u''(t), v\rangle \tag{21}$$

with respect to the weak convergence of the space of distributions $D'(0,T)$.

Further, by (19) and the continuity of h'

$$\varphi(x)h'(u_m(t)) \to \varphi(x)h'(u(t)) \text{ for a.e. } (t,x) \in Q_T.$$

Now we show that for any fixed

$$v \in L^2(0,T;V), \quad v \in L^2(0,T;V) \cap L^1(0,T;L^{\lambda+1}(\Omega)),$$

respectively, the sequence of functions

$$\varphi(x)h'(u_m(t))v \tag{22}$$

is equiintegrable in Q_T. Indeed, if (A_3) is satisfied then by Sobolev's imbedding theorem and (13) for all $t \in [0,T]$

$$\begin{aligned}\|\varphi(x)h'(u_m(t))\|^2_{L^2(\Omega)} &\le \text{const}\, \|h'(u_m(t))\|^2_{L^2(\Omega)} \\ &\le \text{const}\left[1 + \int_\Omega |u_m(t)|^{2\lambda_0}\, dx\right] \le \text{const}\left[1 + \|u_m(t)\|_V^{2\lambda_0}\right] \le \text{const},\end{aligned}$$

thus the Cauchy–Schwarz inequality implies that the sequence of functions (22) is equiintegrable in Q_T.

If (A_3') is satisfied then for all $t \in [0,T]$

$$\int_\Omega |\varphi(x)h'(u_m(t))|^{(\lambda+1)/\lambda}\, dx \le \text{const} \int_\Omega [h(u_m(t)) + 1]\, dx \le \text{const}$$

thus Hölder's inequality implies that the sequence (22) is equiintegrable in Q_T. Consequently, by Vitali's theorem we obtain that for any fixed

$$v \in L^2(0,T;V), \quad v \in L^2(0,T;V) \cap L^1(0,T;L^{\lambda+1}(\Omega)),$$

respectively,

$$\lim_{m\to\infty} \int_{Q_T} \varphi(x)h'(u_m(t))v\, dt\, dx = \int_{Q_T} \varphi(x)h'(u(t))v\, dt\, dx \tag{23}$$

and

$$\varphi(x)h'(u(t)) \in L^2(0,T;V^\star), \quad \varphi(x)h'(u(t)) \in L^\infty(0,T;L^{(\lambda+1)/\lambda}(\Omega)) \tag{24}$$

if (A_3), (A_3') holds, respectively.

Further, by (19) and (A_4)

$$H(t,x;u_m) \to H(t,x;u) \text{ a.e. in } Q_T \tag{25}$$

and by (12)

$$\int_{Q_T} |H(t,x;u_m)|^2\,dx\,dt \le \text{const} \int_{Q_T} h(u_m(t))\,dx\,dt \le \text{const},$$

hence, by the Cauchy–Schwarz inequality, for any fixed $v \in L^2(0,T;V)$, the sequence of functions $H(t,x;u_m)v$ is equiintegrable in Q_T, thus by (25) and Vitali's theorem

$$\lim_{m\to\infty} \int_{Q_T} H(t,x;u_m)v\,dt\,dx = \int_{Q_T} H(t,x;u)v\,dt\,dx \tag{26}$$

and

$$H(t,x;u) \in L^2(0,T;V^\star).$$

Similarly, (17), (19) and (A_5) imply

$$G(t,x;u_m,u_m') \to G(t,x;u,u') \text{ weakly in } L^\infty(0,T;H) \tag{27}$$

and for arbitrary $v \in L^2(Q_T)$ and, consequently, for all $v \in L^2(0,T;V)$ by (27)

$$\lim_{m\to\infty} \int_{Q_T} G(t,x;u_m,u_m')v\,dt\,dx = \int_{Q_T} G(t,x;u,u')v\,dt\,dx \tag{28}$$

and

$$G(t,x;u,u') \in L^2(Q_T) \subset L^2(0,T;V^\star).$$

Now let

$$v \in V \text{ and } \psi \in C_0^\infty(0,T)$$

be arbitrary functions. Further, let $z_N = \sum_{j=1}^N b_j w_j$, $b_j \in \mathbb{R}$, be a sequence of functions such that

$$(z_N) \to v \text{ in } V \text{ and } V \cap L^{\lambda+1}(\Omega), \tag{29}$$

respectively. Further, by (5) we have for all $m \ge N$

$$\begin{aligned}\int_0^T \langle -u_m'(t), z_N\rangle \psi'(t)\,dt + \int_0^T \langle \tilde{Q}(u_m(t)), z_N\rangle \psi(t)\,dt \\ + \int_0^T \int_\Omega \varphi(x)h'(u_m(t))z_N\psi(t)\,dt\,dx + \int_0^T\int_\Omega H(t,x;u_m)z_N\psi(t)\,dt\,dx \\ + \int_0^T\int_\Omega G(t,x;u_m,u_m')z_N\psi(t)\,dt\,dx = \int_0^T \langle F(t), z_N\rangle\psi(t)\,dt.\end{aligned} \tag{30}$$

By (17), (20), (23), (26), (28) we obtain from (30) as $m \to \infty$

$$\begin{aligned}-\int_0^T \langle u'(t), z_N\rangle \psi'(t)\,dt + \int_0^T \langle \tilde{Q}(u(t)), z_N\rangle \psi(t)\,dt \\ + \int_0^T \int_\Omega \varphi(x)h'(u(t))z_N\psi(t)\,dt\,dx + \int_0^T\int_\Omega H(t,x;u)z_N\psi(t)\,dt\,dx \\ + \int_0^T\int_\Omega G(t,x;u,u')z_N\psi(t)\,dt\,dx = \int_0^T \langle F(t), z_N\rangle\psi(t)\,dt.\end{aligned}$$

From equality (30) we obtain as $N \to \infty$

$$-\int_0^T \langle u'(t), v\rangle \psi'(t)\,dt + \int_0^T \langle \tilde{Q}(u(t)), v\rangle \psi(t)\,dt$$
$$+ \int_0^T \int_\Omega \varphi(x) h'(u(t)) v\psi(t)\,dt\,dx + \int_0^T \int_\Omega H(t,x;u) v\psi(t)\,dt\,dx$$
$$+ \int_0^T \int_\Omega G(t,x;u,u') v\psi(t)\,dt\,dx = \int_0^T \langle F(t), \rangle \psi(t)\,dt. \quad (31)$$

Since $v \in V$ and $\psi \in C_0^\infty(0,T)$ are arbitrary functions, (31) means that

$$u'' \in L^2(0,T;V^\star) \text{ and } u'' \in L^2(0,T;(V \cap L^{\lambda+1}(\Omega))^\star), \quad (32)$$

respectively (see, e.g. [15]), and for a.a. $t \in [0,T]$

$$u'' + Qu + \varphi(x)h'(u) + H(t,x;u) + G(t,x;u,u') = F, \quad (33)$$

i.e. we proved (1).

Now we show that the initial condition (2) holds. Since $u \in L^\infty(0,T;V)$, $u' \in L^\infty(0,T;H)$, we have $u \in C([0,T];H)$ and for arbitrary $\psi \in C^\infty[0,T]$ with the properties $\psi(0) = 1$, $\psi(T) = 0$, and all j

$$\int_0^T \langle u'(t), w_j\rangle \psi(t)\,dt = -(u(0), w_j)_H - \int_0^T \langle u(t), w_j\rangle \psi'(t)\,dt,$$
$$\int_0^T \langle u_m'(t), w_j\rangle \psi(t)\,dt = -(u_m(0), w_j)_H - \int_0^T \langle u_m(t), w_j\rangle \psi'(t)\,dt.$$

Hence by (6), (7), (16), (17), we obtain as $m \to \infty$

$$(u_0, w_j)_H = \lim_{m\to\infty} (u_{m0}, w_j)_H = \lim_{m\to\infty} (u_m(0), w_j)_H = (u(0), w_j)_H$$

for all j which implies $u(0) = u_0$.

Similarly, since

$$u' \in L^\infty(0,T;H) \text{ and } u'' \in L^2(0,T;V^\star) + L^\infty(0,T;L^{(\lambda+1)/\lambda}(\Omega))$$

if (A$_3'$) holds, we obtain by Remark 2.2 with a function $\psi \in C^\infty[0,T]$ with the properties $\psi(0) = 1$, $\psi(T) = 0$

$$\int_0^T \langle u''(t), w_j\rangle \psi(t)\,dt = \int_0^T \frac{d}{dt}\langle u'(t), w_j\rangle \psi(t)\,dt$$
$$= -(u'(0), w_j)_H - \int_0^T \langle u'(t), w_j\rangle \psi'(t)\,dt,$$
$$\int_0^T \langle u_m''(t), w_j\rangle \psi(t)\,dt = -(u_m'(0), w_j)_H - \int_0^T \langle u_m'(t), w_j\rangle \psi'(t)\,dt$$

whence by (6), (8), (17), (32), we obtain as $m \to \infty$

$$(u_1, w_j)_H = \lim_{m\to\infty} (u_{m1}, w_j)_H = \lim_{m\to\infty} (u_m'(0), w_j)_H = (u'(0), w_j)_H$$

for all j which implies $u'(0) = u_1$. The case where (A$_3$) holds is similar. ■

EXAMPLE 2.3. Let the operator $\tilde{Q}$ be defined by

$$\langle \tilde{Q}\tilde{u}, \tilde{v}\rangle = \int_\Omega \Big[\sum_{j,l=1}^n a_{jl}(x)(D_l\tilde{u})(D_j\tilde{v}) + d(x)\tilde{u}\tilde{v}\Big]\,dx$$
$$+\sum_{j=1}^n \int_\Omega \Big[D_j\tilde{v}(x)\int_\Omega K_j(x,y)D_j\tilde{u}(y)\,dy\Big]\,dx + \int_\Omega \Big[\tilde{v}(x)\int_\Omega K_0(x,y)\tilde{u}(y)\,dy\Big]\,dx$$

where $a_{jl}, d \in L^\infty(\Omega)$, $a_{jl} = a_{lj}$, $\sum_{j,l=1}^n a_{jl}(x)\xi_j\xi_l \geq c_0|\xi|^2$, $d \geq c_0$ with some positive constant c_0 and the functions $K_j \in L^2(\Omega\times\Omega)$ satisfy

$$K_j(x,y) = K_j(y,x) \text{ for a.a. } x,y \in \Omega \text{ and } \int_{\Omega\times\Omega} K_j(x,y)w(x)w(y)\,dx\,dy \geq 0$$

for all $w \in L^2(\Omega)$. (The last assumption means that the integral operators defined by the kernels K_j are selfadjoint and positive.) Then, clearly, assumption (A_1) is satisfied.

If h is a C^1 function such that $h(\eta) = |\eta|^{\lambda+1}$ if $|\eta| > 1$ then (A_3), (A_3'), respectively, are satisfied.

Further, let $\tilde{h} : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying

$$\text{const}\,|\eta|^{(\lambda+1)/2} \leq |\tilde{h}(\eta)| \leq \text{const}\,|\eta|^{(\lambda+1)/2} \text{ for } |\eta| > 1$$

with some positive constants. It is not difficult to show that the operators H defined by one of the formulas

$$H(t,x;u) = \chi(t,x)\tilde{h}\Big(\int_{Q_t} u(\tau,\xi)\,d\tau\,d\xi\Big),$$
$$H(t,x;u) = \chi(t,x)\tilde{h}\Big(\int_0^t u(\tau,x)\,d\tau\Big),$$
$$H(t,x;u) = \chi(t,x)\tilde{h}\Big(\int_\Omega u(t,\xi)\,d\xi\Big),$$
$$H(t,x;u) = \chi(t,x)\tilde{h}\big(u(\tau(t),x)\big)$$
$$\text{where}\quad \tau \in C^1,\quad 0 \leq \tau(t) \leq t,\quad \tau'(t) \geq c_1 > 0,$$

satisfy (A_4) if $\chi \in L^\infty(Q_T)$.

The operator G may have the form

$$G(t,x;u,w) = \psi_1(t,x;u)w(t) + \psi_2(t,x;u)$$

where the values of the operators (of Volterra type) $\psi_1, \psi_2 : Q_T \times L^2(Q_T) \to \mathbb{R}$ are bounded,

$$(u_k) \to u \text{ in } L^2(Q_T) \text{ and a.e. in } Q_T$$
$$\text{imply}\quad \psi_j(t,x;u_k) \to \psi_j(t,x;u) \text{ for a.a. } (t,x) \in Q_T \quad (j=1,2).$$

Then (A_5) is fulfilled. The operators ψ_1, ψ_2 may have form similar to the above forms of H with bounded continuous functions $\tilde{h}$.

REMARK 2.4. Instead of $\int_{Q_t} u(\tau,\xi)\,d\tau\,d\xi$ one may consider $\int_{Q_t} K(t,x;\tau,\xi)u(\tau,\xi)\,d\tau\,d\xi$ with "sufficiently good" kernel K. Similar generalizations of $\int_0^t u(\tau,x)\,d\tau$ and $\int_\Omega u(t,\xi)\,d\xi$ can be considered.

3. Solutions in $(0,\infty)$. Now we formulate and prove existence of solutions for $t \in (0,\infty)$. Denote by $L^p_{\mathrm{loc}}(0,\infty;V)$ the set of functions $u:(0,\infty)\to V$ such that for each fixed finite $T>0$, their restrictions to $(0,T)$ satisfy $u|_{(0,T)} \in L^p(0,T;V)$ and let $Q_\infty = (0,\infty)\times\Omega$, $L^\alpha_{\mathrm{loc}}(Q_\infty)$ the set of functions $u: Q_\infty \to \mathbb{R}$ such that $u|_{Q_T} \in L^\alpha(Q_T)$ for any finite T.

Now we formulate assumptions on H and G.

(B_4) The function $H: Q_\infty \times L^2_{\mathrm{loc}}(Q_\infty) \to \mathbb{R}$ is such that for all fixed $u \in L^2_{\mathrm{loc}}(Q_\infty)$ the function $(t,x)\mapsto H(t,x;u)$ is measurable, H has the Volterra property (see (A_4)) and for each fixed finite $T>0$, the restriction H_T of H to $Q_T \times L^2(Q_T)$ satisfies (A_4).

Remark 3.1. Since H has the Volterra property, the restriction H_T is well defined by the formula

$$H_T(t,x;\tilde{u}) = H(t,x;u), \quad (t,x)\in Q_T, \quad \tilde{u}\in L^2(Q_T)$$

where $u \in L^2_{\mathrm{loc}}(Q_\infty)$ may be any function satisfying $u(t,x)=\tilde{u}(t,x)$ for $(t,x)\in Q_T$.

(B_5) The operator

$$G: Q_\infty \times L^2_{\mathrm{loc}}(Q_\infty) \times L^\infty_{\mathrm{loc}}(0,\infty;H) \to \mathbb{R}$$

is such that for all fixed $u \in L^2_{\mathrm{loc}}(Q_\infty)$, $w \in L^\infty_{\mathrm{loc}}(0,\infty;H)$ the function $(t,x)\mapsto G(t,x;u,w)$ is measurable, G has the Volterra property and for each fixed finite $T>0$, the restriction G_T of G to $Q_T \times L^2(Q_T) \times L^\infty(0,T;H)$ satisfies (A_5).

Theorem 3.2. *Assume* (A_1)–(A_3), (B_4), (B_5). *Then for all* $F \in L^2_{\mathrm{loc}}(0,\infty;H)$, $u_0 \in V$, $u_1 \in H$ *there exists*

$$u \in L^\infty_{\mathrm{loc}}(0,\infty;V) \text{ such that } u' \in L^\infty_{\mathrm{loc}}(0,\infty;H), \quad u'' \in L^2_{\mathrm{loc}}(0,\infty;V^\star),$$

u satisfies (1) *for a.a.* $t\in(0,\infty)$ (*in the sense formulated in Theorem* 2.1) *and the initial condition* (2).

If (A_1), (A_2), (A_3'), (B_4), (B_5) *are fulfilled then for all* $F \in L^2_{\mathrm{loc}}(0,\infty;H)$, $u_0 \in V\cap L^{\lambda+1}(\Omega)$, $u_1\in H$ *there exists*

$$u \in L^\infty_{\mathrm{loc}}(0,\infty;V\cap L^{\lambda+1}(\Omega)) \text{ such that } u' \in L^\infty_{\mathrm{loc}}(0,\infty;H),$$
$$u'' \in L^2_{\mathrm{loc}}(0,\infty;V^\star) + L^\infty_{\mathrm{loc}}(0,\infty;L^{(\lambda+1)/\lambda}(\Omega)) \subset L^2_{\mathrm{loc}}(0,\infty;[V\cap L^{\lambda+1}(\Omega)]^\star),$$

u satisfies (1) *for a.a.* $t\in(0,\infty)$ (*in the sense formulated in Theorem* 2.1) *and the initial condition* (2).

If there exists a finite $T_0>0$ *such that*

$$\text{for a.a. } t>T_0, \quad F(t)=0, \quad G(t,x;u,w)=0, \tag{34}$$
$$\text{for a.a. } t>T_0, \quad H(t,x;u)=0 \tag{35}$$

then for the above solution u we have

$$u\in L^\infty(0,\infty;V), \quad u\in L^\infty(0,\infty;V\cap L^{\lambda+1}(\Omega)), \text{ respectively,} \tag{36}$$
$$\text{and} \quad u' \in L^\infty(0,\infty;H). \tag{37}$$

Further, if instead of (34) *the condition*

$$F - F_\infty \in L^2(0,\infty;H) \text{ and } G(t,x;u,u')u'(t) \ge \tilde{c}u'(t)^2 \tag{38}$$

holds with some constant $\tilde{c} > 0$ *and with some* $F_\infty \in H$ *such that there exists* $u_\infty \in V$ *satisfying* $\tilde{Q}u_\infty = F_\infty$ *then*

$$\|u'(t)\|_H \le \text{const}\, e^{-\tilde{c}t}, \quad t \in (0,\infty) \tag{39}$$

and there exists $w_0 \in H$ *such that*

$$u(T) \to w_0 \text{ in } H \text{ as } T \to \infty, \quad \|u(T) - w_0\|_H \le \text{const}\, e^{-\tilde{c}T}. \tag{40}$$

Proof. Similarly to the proof of Theorem 2.1, we apply Galerkin's method and we want to find the m-th approximation of solution u for $t \in (0,\infty)$ in the form (see (4)).

$$u_m(t) = \sum_{l=1}^{m} g_{lm}(t) w_l$$

where $g_{lm} \in W^{2,2}_{\text{loc}}(0,\infty)$ if (A_3) is satisfied and $g_{lm} \in W^{2,2}_{\text{loc}}(0,\infty) \cap L^\infty_{\text{loc}}(0,\infty)$ if (A_3') is satisfied. Here $W^{2,2}_{\text{loc}}(0,\infty)$ and $L^\infty_{\text{loc}}(0,\infty)$ denote the set of functions $g : (0,\infty) \to \mathbb{R}$ such that the restriction of g to $(0,T)$ belongs to $W^{2,2}(0,T)$, $L^\infty(0,T)$, respectively.

According to the arguments in the proof of Theorem 2.1, there exists a solution of (5), (6) in a neighbourhood of $t = 0$. Further, we obtain estimates (12)–(13) and (14)–(15), respectively, for $t \in [0,T]$ with sufficiently small T where on the right hand side are finite constants (depending on T). Consequently, the maximal solutions of (5), (6) are defined in $(0,\infty)$ and the estimates (12)–(15) hold for all finite $T > 0$ (if $t \in [0,T]$), the constants on the right hand sides are depending only on T.

Let $(T_k)_{k\in\mathbb{N}}$ be a monotone increasing sequence, converging to $+\infty$. According to the arguments in the proof of Theorem 2.1, there is a subsequence (u_{m1}) of (u_m) for which (16), (17) and (18) hold, respectively, with $T = T_1$. Further, there is a subsequence (u_{m2}) of (u_{m1}) for which (16), (17) and (18) hold, respectively, with $T = T_2$, etc. By a diagonal process we obtain a sequence $(u_{mm})_{m\in\mathbb{N}}$ such that (16), (17), (18) hold for every fixed $T > 0$; further,

$$\begin{gathered} u \in L^\infty_{\text{loc}}(0,\infty;V), \quad u' \in L^\infty_{\text{loc}}(0,\infty;H), \quad u'' \in L^2_{\text{loc}}(0,\infty;V^\star) \\ \text{and} \quad u \in L^\infty_{\text{loc}}(0,\infty;V \cap L^{\lambda+1}(\Omega)), \quad u' \in L^\infty_{\text{loc}}(0,\infty;H), \\ u'' \in L^2_{\text{loc}}(0,\infty;V^\star) + L^\infty_{\text{loc}}(0,\infty;L^{(\lambda+1)/\lambda}(\Omega)), \end{gathered}$$

respectively.

Now we consider the case where (34) holds. Then by (10) we obtain for all $t > 0$

$$\begin{aligned} &\frac{1}{2}\|u_m'(t)\|_H^2 + \frac{1}{2}\langle (Qu_m)(t), u_m(t)\rangle + c_1 \int_\Omega h(u_m(t))\,dx \\ &\qquad \le \|F\|_{L^2(0,T_0;H)} \|u_m'\|_{L^2(0,T_0;H)} + \frac{1}{2}\|u_m'(0)\|_H^2 \\ &\qquad\quad + \frac{1}{2}\langle (Qu_m)(0), u_m(0)\rangle + c_2 \int_\Omega h(u_m(0))\,dx \\ &\qquad\quad + \text{const} \int_0^{T_0}\int_\Omega h(u_m(\tau))\,dx\,d\tau + \text{const}\left[\|u_m'\|^2_{L^2(0,T_0;H)} + 1\right]. \end{aligned}$$

Since the right hand side of this inequality can be estimated by a constant not depending on m and $t > 0$, we obtain (36) and (37).

If (38) holds instead of (34), we find (39) from (10) in a similar way. By (9), $\tilde{Q}u_\infty = F_\infty$, (38) we obtain for $w_m = u_m - u_\infty$ (since $w_m' = u_m'$):

$$\langle w_m''(t) w_m'(t)\rangle + \langle (Qw_m)(t), w_m'(t)\rangle + \int_\Omega \varphi(x) h'(u_m) u_m'(t)\, dx$$
$$+ \int_\Omega H(t,x;u_m) w_m'(t)\, dx + \int_\Omega G(t,x;u_m,u_m') w_m'(t)\, dx = \langle F(t) - F_\infty, w_m'(t)\rangle. \quad (41)$$

Integrating over $[0,t]$ we find (similarly to (10))

$$\frac{1}{2}\|w_m'(t)\|_H^2 + \frac{1}{2}\langle \tilde{Q}(w_m(t)), w_m(t)\rangle + \int_\Omega \varphi(x) h(u_m(t))\, dx + \tilde{c}\int_0^t \Big[\int_\Omega |w_m'(\tau)|^2\, dx\Big]\, d\tau$$
$$\le \varepsilon \int_0^t \Big[\int_\Omega |w_m'(\tau)|^2\, dx\Big]\, d\tau + C(\varepsilon)\int_0^t \|F(\tau) - F_\infty\|_H^2\, d\tau$$
$$+ \frac{1}{2}\|u_m'(0)\|_H^2 + \frac{1}{2}\langle (Qu_m)(0), u_m(0)\rangle + c_2 \int_\Omega h(u_m(0))\, dx$$
$$+ \text{const}\left\{\int_0^{T_0}\Big[\int_\Omega h(u_m(\tau))\, dx\Big]\, d\tau\right\}^{1/2} \|w_m'\|_{L^2(0,T_0;H)}. \quad (42)$$

Choosing $\varepsilon = \tilde{c}/2$ we obtain

$$\int_0^t \Big[\int_\Omega |w_m'(\tau)|^2\, dx\Big]\, d\tau \le \text{const} \quad (43)$$

for all $t > 0$, m which implies $u' \in L^2(0,\infty;H)$ because for every finite $T > 0$

$$w_m' = u_m' \to u' \text{ weakly in } L^\infty(0,T;H).$$

Further, from (42), (43) we obtain

$$\|u_m'(t)\|_H^2 + \tilde{c}\int_0^t \|u_m'(\tau)\|_H^2\, d\tau \le c^\star$$

with some positive constant $c^\star$ not depending on m and t. Thus by Gronwall's lemma we find

$$\|u_m'(t)\|_H^2 = \|w_m'(t)\|_H^2 \le c^\star \mathrm{e}^{-\tilde{c}t}, \quad t > 0,$$

which implies (39).

Further, for arbitrary $T_1 < T_2$

$$\|u(T_2) - u(T_1)\|_H^2 = (u(T_2), u(T_2) - u(T_1))_H - (u(T_1), u(T_2) - u(T_1))_H$$
$$= \int_{T_1}^{T_2} \langle u'(t), u(T_2) - u(T_1)\rangle\, dt = \int_{T_1}^{T_2} (u'(t), u(T_2) - u(T_1))_H\, dt$$
$$\le \|u(T_2) - u(T_1)\|_H \int_{T_1}^{T_2} \|u'(t)\|_H\, dt$$

which implies

$$\|u(T_2) - u(T_1)\|_H \le \int_{T_1}^{T_2} \|u'(t)\|_H\, dt. \quad (44)$$

Hence by (39)

$$\|u(T_2) - u(T_1)\|_H \to 0 \text{ as } T_1, T_2 \to \infty$$

which implies (40) and by (44), (39) we obtain

$$\|u(T) - w_0\|_H \le \int_T^\infty \|u'(t)\|_H \, dt \le \text{const}\, e^{-\tilde{c}T}. \ \blacksquare$$

4. Uniqueness of the solution

THEOREM 4.1. *Assume that the conditions* (A_1)–(A_5) *are fulfilled such that*

$$G(t, x; u, u') = \tilde{\psi}(x)u'(t)$$

where $\tilde{\psi}$ *is measurable and*

$$0 \le \tilde{\psi}(x) \le \text{const}, \tag{45}$$

h is twice continuously differentiable and

$$|h''(\eta)| \le \text{const}\, |\eta|^{\lambda-1} \text{ for } |\eta| > 1. \tag{46}$$

Further, for all $t \in [0, T]$

$$\int_0^t \left[\int_\Omega |H(\tau, x; u_1) - H(\tau, x; u_2)|^2 \, dx\right] d\tau \le M(K) \int_0^t \left[\int_\Omega |u_1 - u_2|^2 \, dx\right] d\tau$$
$$\text{if } u_j \in L^\infty(0, T; V) \text{ and } \|u_j\|_{L^\infty(0,T;V)} \le K, \tag{47}$$

where $M(K)$ *is a constant depending on* K.

Then the solution of (1), (2) (*formulated in Theorem* 2.1) *is unique. Further, if* u_j *is a solution of* (1), (2) *with* $F = F_j$, $u_0 = u_0^j$, $u_1 = u_1^j$ $(j = 1, 2)$ *then for*

$$w = u_1 - u_2 \text{ and } w_1(s) = \int_0^s [u_1(\tau) - u_2(\tau)] \, d\tau$$

we have

$$\|w(s)\|_H^2 + \|w_1(s)\|_V^2 \le \chi_0(F_j, u_0^j, u_1^j) e^s \left[\|f_1 - f_2\|_{L^2(Q_s)}^2 + \|u_0^1 - u_0^2\|_V^2 + \|u_1^1 - u_1^2\|_H^2\right] \tag{48}$$

where χ_0 *is a function whose values are bounded if* $\|F_j\|_{L^2(Q_T)}$, $\|u_0^j\|_V$, $u_1^j\|_H$ *are bounded and*

$$f_j(t) = \int_0^t F_j(\tau) \, d\tau.$$

Proof. Assume that u_j is a solution of (1), (2) with $F = F_j$, $u_0 = u_0^j$, $u_1 = u_1^j$ $(j = 1, 2)$. Let $s \in [0, T]$ be an arbitrary fixed number and apply (3) (with u_j) to v defined by

$$v(t) = \int_t^s [u_1(\tau) - u_2(\tau)] \, d\tau \text{ if } 0 \le t \le s \quad \text{and} \quad v(t) = 0 \text{ if } s < t \le T.$$

It is not difficult to show that $v \in L^2(0, T; V)$ thus we may apply (3) to v, further,

$$v \in C(0, T; V), \quad v' \in L^\infty(0, T; V), \tag{49}$$

$$v'(t) = -w(t) = u_2(t) - u_1(t) \text{ if } t < s \text{ and } v'(t) = 0 \text{ if } s < t$$

and thus

$$\langle w''(t), v(t)\rangle + \langle Qw(t), v(t)\rangle + \int_\Omega \varphi(x)[h'(u_1) - h'(u_2)], v(t) \, dx$$
$$+ \int_\Omega [H(t, x; u_1) - H(t, x; u_2)]v(t) \, dx + \int_\Omega \tilde{\psi}(x)w'(t)v(t) \, dx = \langle F_1(t) - F_2(t), v(t)\rangle.$$

Integrating over $(0, s)$, by (49) we obtain

$$\int_0^s \langle w''(t), v(t)\rangle \, dt + \int_0^s \langle Qw(t), v(t)\rangle \, dt + \int_0^s \Big[\int_\Omega \tilde{\psi}(x) w'(t) v(t) \, dx\Big] \, dt$$
$$+ \int_0^s \langle F_1(t) - F_2(t), v(t)\rangle \, dt - \int_0^s \Big[\int_\Omega \varphi(x)[h'(u_1) - h'(u_2)], v(t) \, dx\Big] \, dt$$
$$- \int_0^s \Big[\int_\Omega [H(t,x;u_1) - H(t,x;u_2)] v(t) \, dx\Big] \, dt. \quad (50)$$

Since

$$w \in L^\infty(0,T;V), \quad w' \in L^\infty(0,T;H), \quad w'' \in L^2(0,T;V^\star), \quad (51)$$

by (49) and Remark 2.2 we obtain

$$\int_0^s \langle w''(t), v(t)\rangle \, dt = \int_0^s \langle w'(t), w(t)\rangle \, dt - \langle w'(0), v(0)\rangle$$
$$= \frac{1}{2} \|w(s)\|_H^2 - \frac{1}{2} \|w(0)\|_H^2 - \langle w'(0), v(0)\rangle. \quad (52)$$

It is not difficult to show (see, e.g. [15], [12]) that by (A_1)

$$\int_0^s \langle Qw(t), v(t)\rangle \, dt = - \int_0^s \langle Qv'(t), v(t)\rangle \, dt = -\frac{1}{2} \langle Qv(s), v(s)\rangle + \frac{1}{2} \langle Qv(0), v(0)\rangle. \quad (53)$$

Consequently, since $v(s) = 0$, integrating by parts, from (50), (52), (53) we get

$$\frac{1}{2} \|w(s)\|_H^2 + \frac{1}{2} \langle Qv(0), v(0)\rangle + \int_0^s \Big[\int_\Omega \tilde{\psi}(x) w^2(t) \, dx\Big] \, dt$$
$$= \int_0^s \langle F_1(t) - F_2(t), v(t)\rangle \, dt + \int_\Omega w'(0) v(0) \, dx + \int_\Omega \tilde{\psi}(x) w(0) v(0) \, dx$$
$$+ \frac{1}{2} \|w(0)\|_H^2 - \int_0^s \Big[\int_\Omega \varphi(x)[h'(u_1) - h'(u_2)] v(t) \, dx\Big] \, dt$$
$$- \int_0^s \Big[\int_\Omega [H(t,x;u_1) - H(t,x;u_2)] v(t) \, dx\Big] \, dt. \quad (54)$$

By using the definition of v, w and the notation $w_1(s) = \int_0^s w(\tau) \, d\tau$ we have

$$v(0) = \int_0^s w(\tau) \, d\tau = w_1(s) \quad (55)$$

and by (A_1)

$$\langle Qv(0), v(0)\rangle \geq c_0 \|v(0)\|_V^2 = c_0 \|w_1(s)\|_V^2. \quad (56)$$

Further, by using the notation $f_j(t) = \int_0^t F_j(\tau) \, d\tau$, integrating by parts, we obtain by Young's inequality

$$\Big|\int_0^s \langle F_1(t) - F_2(t), v(t)\rangle \, dt\Big| = \Big|\int_\Omega \Big\{\int_0^s [f_1'(t) - f_2'(t)] v(t) \, dt\Big\} \, dx\Big|$$
$$= \Big|\int_\Omega \Big\{\int_0^s [f_1(t) - f_2(t)] w(t) \, dt\Big\} \, dx\Big| \leq \frac{1}{2} \int_0^s \|w(t)\|_H^2 \, dt + \frac{1}{2} \|f_1 - f_2\|_{L^2(Q_s)}^2. \quad (57)$$

Similarly, by (55)

$$\Big|\int_\Omega w'(0) v(0) \, dx\Big| \leq \varepsilon \|w_1(s)\|_V^2 + C_1(\varepsilon) \|w'(0)\|_H^2 \quad (58)$$

and by (45)

$$\left| \int_\Omega \tilde{\psi}(x) w(0) v(0)\, dx \right| \le \varepsilon \|w_1(s)\|_V^2 + C_2(\varepsilon)\|w(0)\|_H^2. \tag{59}$$

($C_j(\varepsilon)$ denote constants depending on ε.)

The first nonlinear term on the right hand side of (54) can be estimated as follows: by (A_2) and (46)

$$\begin{aligned} &\left| \int_0^s \Big[\int_\Omega \varphi(x)[h'(u_1) - h'(u_2)] v(t)\, dx \Big]\, dt \right| \\ &\qquad \le \text{const} \left| \int_0^s \Big[\int_\Omega \sup\{|h''(\eta)| : \eta \in (a,b)\} |u_1(t) - u_2(t)||v(t)|\, dx \Big]\, dt \right| \\ &\qquad \le \text{const} \int_0^s \Big[\int_\Omega \big(|u_1(t)|^{\lambda_0 - 1} + |u_2(t)|^{\lambda_0 - 1} + 1 \big) |u_1(t) - u_2(t)||v(t)|\, dx \Big]\, dt \end{aligned} \tag{60}$$

where

$$a = \min\{u_1(t), u_2(t)\}, \quad b = \max\{u_1(t), u_2(t)\}$$

since

$$|h''(\eta)| \le \text{const}\, |\eta|^{\lambda_0 - 1} = \text{const}\, |\eta|^{2/(n-2)} \text{ if } |\eta| > 1$$

(for $n = 2$, λ_0 may be any positive number).

Since V is continuously imbedded into $L^q(\Omega)$ where $q = \frac{2n}{n-2} = n(\lambda_0 - 1)$, we may apply Hölder's inequality by $\frac{1}{n} + \frac{1}{2} + \frac{1}{q} = 1$:

$$\begin{aligned} &\int_0^s \Big[\int_\Omega \big(|u_1(t)|^{\lambda_0 - 1} + |u_2(t)|^{\lambda_0 - 1} + 1 \big) |u_1(t) - u_2(t)||v(t)|\, dx \Big]\, dt \\ &\quad \le \text{const} \int_0^s \big[\| |u_1(t)|^{\lambda_0 - 1} \|_{L^n(\Omega)} + \| |u_2(t)|^{\lambda_0 - 1} \|_{L^n(\Omega)} + 1 \big] \|w(t)\|_H \|v(t)\|_{L^q(\Omega)}\, dt \\ &\qquad \le \text{const} \int_0^s \big[\|u_1(t)\|_V^{\lambda_0 - 1} + \|u_2(t)\|_V^{\lambda_0 - 1} + 1 \big] \|w(t)\|_H \|v(t)\|_V\, dt. \end{aligned} \tag{61}$$

Since u_1, u_2 are solutions of (1), (2), by using arguments in the proof of Theorem 2.1, one can show that the $L^\infty(0,T;V)$ norm of u_j can be estimated by a function of $\|F_j\|_{L^2(Q_T)}$, $\|u_0^j\|_V$, $\|u_1^j\|_H$, the values of which are bounded if $\|F_j\|_{L^2(Q_T)}$, $\|u_0^j\|_V$, $\|u_1^j\|_H$ are bounded. (See the proof of (12)–(15).) Therefore, since

$$v(t) = w_1(s) - w_1(t) \text{ for } t \le s,$$

we obtain from (60), (61)

$$\begin{aligned} &\left| \int_0^s \Big[\int_\Omega \varphi(x)[h'(u_1) - h'(u_2)] v(t)\, dx \Big]\, dt \right| \le \chi(F_j, u_0^j, u_1^j) \int_0^s \|w(t)\|_H \|v(t)\|_V\, dt \\ &\qquad \le \chi(F_j, u_0^j, u_1^j) \int_0^s \|w(t)\|_H [\|w_1(t)\|_V + \|w_1(s)\|_V]\, dt \\ &\qquad \le \chi(F_j, u_0^j, u_1^j) \big[\varepsilon \|w_1(s)\|_V^2 + C(\varepsilon) \int_0^s \big(\|w(t)\|_H^2 + \|w_1(t)\|_V^2 \big)\, dt \big] \end{aligned} \tag{62}$$

where $\chi(F_j, u_0^j, u_1^j)$ is bounded if $\|F_j\|_{L^2(Q_T)}$, $\|u_0^j\|_V$, $\|u_1^j\|_H$ are bounded.

For the last term on the right hand side of (54) we have, by using the notation

$$\chi_j(t) = \int_0^t H(\tau, x; u_j)\, d\tau, \quad j = 1, 2,$$

$$\begin{aligned}
&\left| \int_0^s \Big[\int_\Omega (H(t,x;u_1) - H(t,x;u_2)) v(t)\, dx \Big] dt \right| \\
&= \left| \int_\Omega \Big[\int_0^s (\chi_1'(t) - \chi_2'(t)) v(t)\, dt \Big] dx \right| = \left| \int_\Omega \Big[\int_0^s (\chi_1(t) - \chi_2(t)) w(t)\, dt \Big] dx \right| \\
&\qquad \le \left\{ \int_\Omega \Big[\int_0^s |\chi_1(t) - \chi_2(t)|^2\, dt \Big] dx \right\}^{1/2} \left\{ \int_0^s \|w(t)\|_H^2\, dt \right\}^{1/2}. \qquad (63)
\end{aligned}$$

The assumption (47) implies

$$\begin{aligned}
&\int_\Omega \Big[\int_0^s |\chi_1(t) - \chi_2(t)|^2\, dt \Big] dx \\
&\qquad = \int_\Omega \left[\int_0^s \left| \int_0^t [H(\tau,x;u_1) - H(\tau,x;u_2)]\, d\tau \right|^2 dt \right] dx \\
&\le \text{const} \int_\Omega \Big[\int_0^s |H(\tau,x;u_1) - H(\tau,x;u_2)|^2\, d\tau \Big] dx \le \tilde{M}(K) \int_0^s \|w(\tau\|_H^2\, d\tau \qquad (64)
\end{aligned}$$

if $\|u_j\|_{L^\infty(0,T;V)} \le K$ where $\tilde{M}(K)$ is a constant depending on K.

Choosing sufficiently small $\varepsilon > 0$, we obtain from (54), (56)–(59), (62)–(64)

$$\begin{aligned}
\|w(s)\|_H^2 + \|w_1(s)\|_V^2 &\le \tilde{\chi}(F_j, u_0^j, u_1^j) \int_0^s [\|w(t)\|_H^2 + \|w_1(t)\|_V^2]\, dt \\
&\qquad + \text{const} \left[\|f_1 - f_2\|_{L^2(Q_s)}^2 + \|w(0)\|_V^2 + \|w'(0)\|_H^2 \right].
\end{aligned}$$

Hence by Gronwall's lemma we obtain (48). ∎

Remark 4.2. By using Examples in Section 2 it is not difficult to formulate examples satisfying the assumptions of Theorem 4.1.

Remark 4.3. By a usual argument (Cantor's trick) one obtains: if the solution is unique (by the above theorem) then not only a subsequence but also the original sequence (u_m) obtained by Galerkin's method converges to the solution u weakly in $L^\infty(0,T;V)$, strongly in $L^2(Q_T)$ and $(u_m') \to u'$ weakly in $L^\infty(0,T;H)$.

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