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THE YOKONUMA–TEMPERLEY–LIEB ALGEBRA

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Abstract. We define the Yokonuma–Temperley–Lieb algebra as a quotient of the Yokonuma–Hecke algebra over a two-sided ideal generated by an expression analogous to the one of the classical Temperley–Lieb algebra. The main theorem provides necessary and sufficient conditions for the Markov trace defined on the Yokonuma–Hecke algebra to pass through to the quotient algebra, leading to a sequence of knot invariants which coincide with the Jones polynomial.

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Introduction. The Temperley–Lieb algebra appeared originally in Statistical Mechanics and is important in several areas of Mathematics. In his seminal work V. F. R. Jones [12] constructed a Markov trace on the Temperley–Lieb algebra, leading to unexpected applications in knot theory as well as to a fertile interaction between Knot theory and Representation theory. In algebraic terms, the Temperley–Lieb algebra, $\mathrm{TL}_n(u)$, can be defined as a quotient of the Iwahori–Hecke algebra, $\mathrm{H}_n(u)$.

In [14] the Yokonuma–Hecke algebra $\mathrm{Y}_{d,n}(u)$ (defined originally in [25]) has been defined as a quotient of the modular framed braid group $\mathcal{F}_{d,n}$, which comprises framed braids with framings modulo d, over a quadratic relation involving the framing generators t_i by means of certain weighted idempotents e_i (Eqs. (13) and (9)). By setting $d = 1$, the algebra $\mathrm{Y}_{1,n}(u)$ coincides with the Iwahori–Hecke algebra. The Yokonuma–Hecke algebras have been studied in [25, 14, 16, 24, 4]. Further, in [14] the second author found an inductive linear basis for the algebras $\mathrm{Y}_{d,n}(u)$ and constructed a unique Markov trace tr on these algebras depending on parameters $z, x_1, \dots, x_{d-1}$. Aiming to the extraction of framed link invariants from tr, as it turned out in [19], tr does not re-scale directly, according to the framed braid equivalence, leading to conditions that have to be imposed on the trace parameters $x_1, \dots, x_{d-1}$; namely, they had to satisfy a non-linear system of equations, the E-*system* (17). The x_i's being d-th roots of unity is one obvious solution. Gérardin found in [19, Appendix] the full set of solutions of the E-system. Given now any solution of the E-system, 2-variable isotopy invariants for framed, classical and singular links were constructed in [17, 18, 19] respectively, which are studied further in [2, 5].

In this paper we define an analogue for the Temperley–Lieb algebra in the context of framed braids, *the Yokonuma–Temperley–Lieb algebra*, denoted by $\mathrm{YTL}_{d,n}(u)$. It is defined as a quotient of the Yokonuma–Hecke algebra over a two-sided ideal I ((22) and Definition 2), analogous to the classical case. For $d = 1$ the algebra $\mathrm{YTL}_{1,n}(u)$ coincides with the Temperley–Lieb algebra. We first show that I is a principal ideal and we give a presentation for $\mathrm{YTL}_{d,n}(u)$ with non-invertible generators, analogous to the classical case. We then give a spanning set $\Sigma_{d,n}$ for $\mathrm{YTL}_{d,n}(u)$, in which every word contains the highest and lowest index braiding generator exactly once. Moreover, any word in $\Sigma_{d,n}$ inherits the splitting property from $\mathrm{Y}_{d,n}(u)$, that is, it splits into the framing part and the braiding part. We also present the results of Chlouveraki and Pouchin [3] on the dimension and a linear basis for $\mathrm{YTL}_{d,n}(u)$. From the spanning set $\Sigma_{d,n}$, they extracted an explicit basis for $\mathrm{YTL}_{d,n}(u)$ by describing a set of linear dependence relations among the framing parts for each fixed element in the braiding part. Finally, using the dimension results of [3] we find a basis for $\mathrm{YTL}_{2,3}(u)$ different than the basis in [3].

Next, we seek conditions such that the trace tr, defined on the algebras $\mathrm{Y}_{d,n}(u)$, passes to the quotient algebras $\mathrm{YTL}_{d,n}(u)$. More precisely, we compute first the values of the trace parameter z that annihilate the trace of the generator of the defining ideal I. These are the roots of a quadratic equation (47). Then we annihilate the trace values of all elements of $\mathrm{Y}_{d,n}(u)$ that lie in I and so we end up with a system (Σ) of quadratic equations in z (55a)–(55c). If we demand that (Σ) has both roots of (47) as common solutions, we end up with sufficient conditions for the trace tr to pass to the quotient algebras $\mathrm{YTL}_{d,n}(u)$. In particular, Theorem 5 states that if the trace parameters $x_1, \dots, x_{d-1}$

are d-th roots of unity and $z = -\frac{1}{u+1}$ or $z = -1$, then the trace tr passes to the quotient algebras $\mathrm{YTL}_{d,n}(u)$. Note that these two values for z are precisely the ones that Jones computed such that the Ocneanu trace on the algebras $\mathrm{H}_n(u)$ passes to the quotient algebras $\mathrm{TL}_n(u)$. If we let (Σ) to have one common solution for z we obtain the necessary and sufficient conditions for the trace tr to pass through to the quotient algebras $\mathrm{YTL}_{d,n}(u)$. More precisely, Theorem 6 states that the trace tr passes to the quotient algebras $\mathrm{YTL}_{d,n}(u)$ if and only if either the conditions of Theorem 5 are satisfied or the trace parameters $x_1, \ldots, x_{d-1}$ comprise a solution of the E-system (other than d-th roots of unity) and $z = -\frac{1}{2}$. This is our main result.

In [2] it is shown that if the trace parameters $x_1, \ldots, x_{d-1}$ are d-th roots of unity, then the classical link invariants derived from the algebra $\mathrm{Y}_{d,n}(u)$ coincide with the 2-variable Jones or Homflypt polynomial. Using Theorem 6 and the results in [2], we obtain from the invariants for framed and classical links in [18, 19] related to $\mathrm{Y}_{d,n}(u)$, 1-variable framed and classical link invariants through the algebras $\mathrm{YTL}_{d,n}(u)$, which coincide with the Jones polynomial for the case of classical links.

The paper is organized as follows: In Section 1 we recall the definition and basic properties of the classical Temperley–Lieb algebra and the Yokonuma–Hecke algebra. In Section 2 we define the Yokonuma–Temperley–Lieb algebra as a quotient of the Yokonuma–Hecke algebra over a two-sided ideal (Eq. (22) and Definition 2), which we show that is a principal ideal (Lemma 4). Finally, we give a presentation for $\mathrm{YTL}_{d,n}(u)$ with non-invertible generators (Proposition 2). In Section 3 we present the spanning set $\Sigma_{d,n}$ for $\mathrm{YTL}_{d,n}(u)$ and the results of Chlouveraki and Pouchin [3] on the dimension (Proposition 5) and a linear basis for $\mathrm{YTL}_{d,n}(u)$ (Theorem 3). We also give a different basis for $\mathrm{YTL}_{2,3}(u)$. Section 4 focusses on the necessary and sufficient conditions under which the trace tr on $\mathrm{Y}_{d,n}(u)$ passes to the quotient algebra $\mathrm{YTL}_{d,n}(u)$ (Theorems 5 and 6). Finally, in Section 5 we discuss the invariants for classical and framed links that can be constructed through the trace tr and we recover the Jones polynomial.

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1. Preliminaries

1.1. Notation. Throughout the paper we shall fix the following notation. By the term algebra we mean an associative unital (with unity 1) algebra over the field $\mathbb{C}(u)$, where u is an indeterminate. The following two positive integers are also fixed: d and n.

As usual we denote by B_n the braid group on n strands, that is, the group generated by the elementary braids $\sigma_1, \ldots, \sigma_{n-1}$, where σ_i is the positive crossing between the i-th and the $(i+1)$-st strand, satisfying the well-known braid relations: $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ and $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| > 1$.

We denote by S_n the symmetric group on n symbols. Let s_i be the elementary transposition $(i, i+1)$. We denote by l the length function on S_n with respect to the s_i's.

Let $C_d = \langle t \,|\, t^d = 1\rangle$ be the cyclic group of order d. Let $t_i = (1, \ldots, 1, t, 1, \ldots, 1) \in C_d^n$, where t is in the i-th position. Let also

$$C_{d,n} := C_d^n \rtimes S_n,$$

where the action is defined by permutation on the indices of the t_i's, namely $s_i t_j = t_{s_i(j)} s_i$.

Finally, we denote by $\mathrm{H}_n(u)$ the Iwahori–Hecke algebra, that is, the $\mathbb{C}(u)$-algebra defined by generators $h_1, \ldots, h_{n-1}$ which satisfy the relations:

$$h_i h_j h_i = h_j h_i h_j, \quad |i-j| = 1 \tag{1}$$

$$h_i h_j = h_j h_i, \quad |i-j| > 1 \tag{2}$$

$$h_i^2 = (u-1)h_i + u. \tag{3}$$

1.2. The Temperley–Lieb algebra. Originally, the Temperley–Lieb algebra, over $\mathbb{C}$, was defined by generators $f_1, \ldots, f_{n-1}$ subject to the relations:

$$f_i^2 = f_i$$

$$f_i f_j f_i = \delta f_i, \quad |i-j| = 1$$

$$f_i f_j = f_j f_i, \quad |i-j| > 1$$

where δ is an indeterminate (see [8], [11], [12]). The generators f_i are non-invertible. One can define the Temperley–Lieb algebra with the invertible generators (see [11])

$$h_i := (u+1)f_i - 1 \tag{4}$$

where u is defined via the equation $\delta^{-1} = 2 + u + u^{-1}$. The Temperley algebra $\mathrm{TL}_n(u)$, over $\mathbb{C}(u)$, is then defined by generators $h_1, \ldots, h_{n-1}$ (we use the same symbols as for the algebra $\mathrm{H}_n(u)$ by abuse of notation) under the relations (1), (2), (3) and the relations

$$1 + h_i + h_j + h_i h_j + h_j h_i + h_i h_j h_i = 0, \quad |i-j| = 1. \tag{5}$$

Note that, for $n \geq 3$, relations (5) are symmetric with respect to the indices i, j, so relations (1) follow from relations (5). Relations (1)–(3) are the well-known defining relations of the Iwahori–Hecke algebra $\mathrm{H}_n(u)$. Therefore, $\mathrm{TL}_n(u)$ can be considered as a quotient of $\mathrm{H}_n(u)$ via the morphism: $h_i \mapsto h_i$. It turns out that the set

$$\left\{(h_{j_1} h_{j_1-1} \ldots h_{j_1-k_1})(h_{j_2} h_{j_2-1} \ldots h_{j_2-k_2}) \ldots (h_{j_p} h_{j_p-1} \ldots h_{j_p-k_p})\right\}$$

where $1 \leq j_1 < j_2 < \ldots < j_p \leq n-1$ and $1 \leq j_1 - k_1 < j_2 - k_2 < \ldots < j_p - k_p$, furnishes a linear basis for $\mathrm{TL}_n(u)$ and the dimension of $\mathrm{TL}_n(u)$ is equal to the n-th Catalan number $c_n = \frac{1}{n+1}\binom{2n}{n}$ [11, 12]. Recall finally that, Ocneanu constructed in [7] a unique Markov trace on the algebras $\mathrm{H}_n(u)$:

Theorem 1 (Ocneanu). *For any $\zeta \in \mathbb{C}^\times$ there exists a linear trace τ on $\bigcup_{n=1}^{\infty} \mathrm{H}_n(u)$ defined uniquely by the inductive rules*:

1. $\tau(ab) = \tau(ba), \quad a, b \in \mathrm{H}_n(u),$
2. $\tau(1) = 1,$
3. $\tau(ah_n) = \zeta\tau(a), \quad a \in \mathrm{H}_n(u).$

Jones' technique for redefining his Markov trace on the Temperley–Lieb algebra as factoring of the Ocneanu trace on the Iwahori–Hecke algebra [11] tells us that the least requirement is that the Ocneanu trace respects the defining relations (5). This requirement implies:

$$\zeta = -\frac{1}{u+1} \quad \text{or} \quad \zeta = -1. \tag{6}$$

The Ocneanu trace is used in [11] for constructing the Homflypt polynomial invariant for classical knots and links. Then, by specializing ζ to $-\frac{1}{u+1}$ the Jones polynomial was recovered.

1.3. The Yokonuma–Hecke algebra. The group $\mathbb{Z}^n$ is generated by the "framing generators" $t_1, \ldots, t_n$, the standard multiplicative generators of $\mathbb{Z}^n$. In this notation an element $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ in the additive notation can be expressed as $t_1^{a_1} \ldots t_n^{a_n}$. The *framed braid group* on n strands is then defined as

$$\mathcal{F}_n = \mathbb{Z}^n \rtimes B_n,$$

where the action of B_n on $\mathbb{Z}^n$ is given by the permutation induced by a braid on the indices:

$$\sigma_i t_j = t_{s_i(j)} \sigma_i. \tag{7}$$

In particular, $\sigma_i t_i = t_{i+1}\sigma_i$ and $t_{i+1}\sigma_i = \sigma_i t_i$. A word w in $\mathcal{F}_n$ has thus the "splitting property", that is, it splits into the "framing" part and the "braiding" part:

$$w = t_1^{a_1} \ldots t_n^{a_n} \, \sigma,$$

where $\sigma \in B_n$ and $a_i \in \mathbb{Z}$. So w is a classical braid with an integer attached to each strand. Topologically, an element of $\mathbb{Z}^n$ is identified with a framed identity braid on n strands, while a classical braid in B_n is viewed as a framed braid with all framings 0. The multiplication in $\mathcal{F}_n$ is defined by placing one braid on top of the other and collecting the total framing of each strand to the top.

For a fixed positive integer d, the *d-modular framed braid group* on n strands, $\mathcal{F}_{d,n}$, is defined as the quotient of $\mathcal{F}_n$ over the *modular relations*:

$$t_i^d = 1 \quad (i = 1, \ldots, n). \tag{8}$$

Thus, $\mathcal{F}_{d,n} = C_d^n \rtimes B_n$, where C_d^n is isomorphic to $(\mathbb{Z}/d\mathbb{Z})^n$ but with multiplicative notation. Framed braids in $\mathcal{F}_{d,n}$ have framings modulo d.

Passing now to the group algebra $\mathbb{C}\mathcal{F}_{d,n}$, we have the following elements $e_i \in \mathbb{C}C_d^n$ (see [16] for diagrammatic interpretations), which are idempotents (cf. [16, Lemma 4]):

$$e_i := \frac{1}{d} \sum_{s=0}^{d-1} t_i^s t_{i+1}^{-s}, \qquad i = 1, \ldots, n-1. \tag{9}$$

The definition of the idempotent e_i can be generalized in the following ways. For any indices i, j and any $m \in \mathbb{Z}/d\mathbb{Z}$, we define the following elements in $\mathbb{C}C_d^n$:

$$e_{i,j} := \frac{1}{d} \sum_{s=0}^{d-1} t_i^s t_j^{-s}, \tag{10}$$

and

$$e_i^{(m)} := \frac{1}{d} \sum_{s=0}^{d-1} t_i^{m+s} t_{i+1}^{-s} \tag{11}$$

(notice that $e_i = e_{i,i+1} = e_i^{(0)}$). The following lemma collects some of the relations among the e_i's, the t_i's and the σ_i's. These relations will be used in the paper.

Remark 1. Later on we are going to use the elements defined above inside the group algebras $\mathbb{C}(u)G$, where G could be $C_{d,n}$ or $\mathcal{F}_{d,n}$. We will use for these elements the same symbols along the paper, as $C_{d,n}$ injects in all the algebras we will be considering.

Lemma 1. *For the idempotents* e_i *and for* $1 \leq i, j \leq n-1$ *the following relations hold*:

$$\begin{aligned} t_j e_i &= e_i t_j \\ e_{i+1}\sigma_i &= \sigma_i e_{i,i+2} \\ e_i \sigma_j &= \sigma_j e_i \qquad \text{for } j \neq i-1, i+1 \\ e_j \sigma_i \sigma_j &= \sigma_i \sigma_j e_i \qquad \text{for } |i-j| = 1 \\ e_i e_{i+1} &= e_i e_{i,i+2} \\ e_i e_{i+1} &= e_{i,i+2} e_{i+1}. \end{aligned}$$

Proof. All relations are immediate consequences of the definitions. The proofs for the first four relations can be found, for example, in [17, Lemma 2.1]. For the sixth relation we have:

$$e_i e_{i+1} = \frac{1}{d} \sum_{s=0}^{d-1} t_i^s t_{i+1}^{-s} \frac{1}{d} \sum_{m=0}^{d-1} t_{i+1}^m t_{i+2}^{-m} = \frac{1}{d^2} \sum_{s=0}^{d-1} \sum_{m=0}^{d-1} t_i^s t_{i+1}^{m-s} t_{i+2}^{-m}. \tag{12}$$

Setting now $k = m - s$ we obtain

$$\begin{aligned} e_i e_{i+1} &= \frac{1}{d^2} \sum_{s=0}^{d-1} \sum_{k=0}^{d-1} t_i^s t_{i+1}^k t_{i+2}^{-k-s} \\ &= \frac{1}{d} \sum_{s=0}^{d-1} t_i^s t_{i+2}^{-s} \frac{1}{d} \sum_{k=0}^{d-1} t_{i+1}^k t_{i+2}^{-k} = e_{i,i+2} e_{i+1}. \end{aligned}$$

The fifth relation is proved in an analogous way. ∎

Remark 2. Concerning the proof of the fifth and sixth relation, the following alternative proof was suggested by the Referee, which is the underlying explanation for the relations, and so adds to their understanding: These relations, in the group algebra $\mathbb{C}C_d^n$, express the fact that in the group algebra of any abelian group the product of the idempotents associated to two subgroups H, K is the idempotent associated to the product of the groups. Here in additive terms the fifth and sixth relations express the fact that the subgroup generated by $(1,-1,0)$ and $(0,1,-1)$ in C_d^3 is also generated by $(1,-1,0)$ and $(1,0,-1)$ or by $(0,1,-1)$ and $(1,0,-1)$.

The *Yokonuma–Hecke algebra* $\mathrm{Y}_{d,n}(u)$ is defined [14, 16] as the quotient of the group algebra $\mathbb{C}(u)\mathcal{F}_{d,n}$ over the two-sided ideal generated by the elements:

$$\sigma_i^2 - 1 - (u-1)e_i - (u-1)e_i\sigma_i, \quad \text{for all } i. \tag{13}$$

Let g_i be the image of σ_i in the quotient of $\mathbb{C}(u)\mathcal{F}_{d,n}$ by the two-sided ideal defined above. The ideal relations imply the following quadratic relations in $\mathrm{Y}_{d,n}(u)$:

$$g_i^2 = 1 + (u-1)\, e_i + (u-1)\, e_i\, g_i \tag{14}$$

(see [16] for diagrammatic interpretations). Since the quadratic relations do not change the framing we have $\mathbb{C}C_d^n \subset \mathbb{C}(u)C_d^n \subset \mathrm{Y}_{d,n}(u)$ and we keep the same notation for the elements of $\mathbb{C}C_d^n$ and for the elements e_i in $\mathrm{Y}_{d,n}(u)$. The elements g_i are invertible:

$$g_i^{-1} = g_i + (u^{-1} - 1)\, e_i + (u^{-1} - 1)\, e_i g_i.$$

For $d = 1$ we have $t_j = 1$ and $e_i = 1$, and in this case the quadratic relations (14) become $g_i^2 = (u-1)g_i + u$, which are the quadratic relations of the Iwahori–Hecke algebra $\mathrm{H}_n(u)$. So, $\mathrm{Y}_{1,n}(u)$ coincides with the algebra $\mathrm{H}_n(u)$. Further, there is an obvious epimorphism of the Yokonuma–Hecke algebra $\mathrm{Y}_{d,n}(u)$ onto the algebra $\mathrm{H}_n(u)$ via the map:

$$\begin{aligned} g_i &\mapsto h_i \\ t_j &\mapsto 1. \end{aligned} \tag{15}$$

We can alternatively define the algebra $\mathrm{Y}_{d,n}(u)$ as a u-deformation of the algebra $\mathbb{C}C_{d,n}$. More precisely, let $w \in S_n$ and let $w = s_{i_1} \dots s_{i_k}$ be a reduced expression for w. The generators g_i of $\mathrm{Y}_{d,n}(u)$ satisfy the same braiding relations as the generators of S_n, hence together with the well-known theorem of Matsumoto [20], it follows that $g_w := g_{i_1} \dots g_{i_k}$ is well-defined. Notice that the defining generators g_i correspond to g_{s_i}. We have the following multiplication rule in $\mathrm{Y}_{d,n}(u)$ (see [13, Proposition 2.4]):

$$g_{s_i} g_w = \begin{cases} g_{s_i w} & \text{for } l(s_i w) > l(w) \\ g_{s_i w} + (u-1) e_i g_{s_i w} + (u-1) e_i g_w & \text{for } l(s_i w) < l(w) \end{cases} \tag{16}$$

We also write g_{t_i} for t_i and we define: $g_{t_i w} = g_{t_i} g_w = t_i g_w$. Using the above multiplication formulas the second author proved in [14] that $\mathrm{Y}_{d,n}(u)$ has the following standard basis:

$$\{t_1^{a_1} \dots t_n^{a_n} g_w \mid a_i \in \mathbb{Z}/d\mathbb{Z},\ w \in S_n\}.$$

Further, we have an inductive basis of the Yokonuma–Hecke algebra, which is used in the proof of the main theorem.

Proposition 1 ([14], Proposition 8). *Every element in $\mathrm{Y}_{d,n+1}(u)$ is a unique linear combination of words, each of one of the following types*:

$$\mathfrak{m}_n g_n g_{n-1} \dots g_i t_i^k \quad \textit{or} \quad \mathfrak{m}_n t_{n+1}^k,$$

where $k \in \mathbb{Z}/d\mathbb{Z}$ and $\mathfrak{m}_n$ is a word in the inductive basis of $\mathrm{Y}_{d,n}(u)$.

1.4. A Markov trace on $\mathbf{Y}_{d,n}(u)$. We will denote the elements of the additive group $\mathbb{Z}/d\mathbb{Z}$ by $\{0, 1, \dots, d-1\}$.

Using the above basis, the second author constructed in [14] a linear Markov trace on the algebra $\mathrm{Y}_{d,n}(u)$. Namely:

Theorem 2 ([14] Theorem 12). *For indeterminates z, x_s, where $s \in \mathbb{Z}/d\mathbb{Z}$, $s \neq 0$, there exists a unique linear Markov trace* tr:

$$\mathrm{tr} : \bigcup_{n=1}^{\infty} \mathrm{Y}_{d,n}(u) \longrightarrow \mathbb{C}(u)[z, x_1, \dots, x_{d-1}]$$

defined inductively on n by the following rules:

$$\begin{aligned} \mathrm{tr}(ab) &= \mathrm{tr}(ba) \\ \mathrm{tr}(1) &= 1 \\ \mathrm{tr}(ag_n) &= z\,\mathrm{tr}(a) && (\textit{Markov property}) \\ \mathrm{tr}(at_{n+1}^s) &= x_s\,\mathrm{tr}(a) && (s = 1, \ldots, d-1) \end{aligned}$$

where $a, b \in \mathrm{Y}_{d,n}(u)$.

Note that the first rule of tr is the standard rule for a trace function, the second rule is the basis of the inductive computation of tr, the third rule is the so-called *Markov property* that takes care of the highest index braiding generator in the word, whilst the fourth rule takes care of the highest index framing generator in the word.

Remark 3. We will define $x_0 := 1$, so x_s is defined for all $s \in \mathbb{Z}/d\mathbb{Z}$.

By direct computation, $\mathrm{tr}(e_i)$ takes the same value for all i. We denote this value by E, that is:

$$E := \mathrm{tr}(e_i) = \frac{1}{d}\sum_{s=0}^{d-1} x_s x_{d-s}.$$

For all $m \in \mathbb{Z}/d\mathbb{Z}$, we also define:

$$E^{(m)} := \mathrm{tr}(e_i^{(m)}) = \frac{1}{d}\sum_{s=0}^{d-1} x_{m+s} x_{d-s},$$

where $e_i^{(m)}$ is defined in (11). Notice that $E = E^{(0)}$.

1.5. The E-system. In order for an invariant for framed knots and links to be constructed through the trace on $\mathrm{Y}_{d,n}(u)$, tr should be normalized and rescaled properly. In [19] it is proved that such a rescaling is possible if the trace parameters x_i are solutions of a non-linear system of equations, the so-called E-system.

Definition 1. We say that the set of complex numbers $\{x_0, x_1, \ldots, x_{d-1}\}$ (where x_0 is always equal to 1) satisfies the E-*condition* if $x_1, \ldots, x_{d-1}$ satisfy the following E-*system* of non-linear equations in $\mathbb{C}$:

$$E^{(m)} = x_m E \qquad (1 \leq m \leq d-1),$$

or equivalently,

$$\sum_{s=0}^{d-1} x_{m+s} x_{d-s} = x_m \sum_{s=0}^{d-1} x_s x_{d-s} \qquad (1 \leq m \leq d-1). \tag{17}$$

In [19, Appendix] it is proved that the solutions of the E-system are the functions x_S from $\mathbb{Z}/d\mathbb{Z}$ to $\mathbb{C}$, parametrized by the non-empty subsets S of the cyclic group $\mathbb{Z}/d\mathbb{Z}$ as follows:

$$x_S = \frac{1}{|S|}\sum_{s \in S} \exp_s, \tag{18}$$

where $\exp_s(k) = \exp(2i\pi sk/d)$ and exp denotes the usual complex exponential function.

Remark 4. It is worth noting that the solution of the E-system can be interpreted as a generalization of the Ramanujan's sum. Indeed, by taking the subset P of $\mathbb{Z}/d\mathbb{Z}$ consisting of the numbers coprime to d, then the solution parametrized by P is, up to the factor $|P|$, the Ramanujan's sum $c_d(k)$ (see [21]).

Equivalently, x_S can be seen as an element in $\mathbb{C}C_d$, namely

$$x_S = \sum_{k=0}^{d-1} x_k t^k, \tag{19}$$

where $x_k = \frac{1}{|S|}\sum_{s\in S}\chi_s(t^k)$, $k = 0,\dots,d-1$, and χ_s is the character of C_d defined as $\chi_s : t^m \mapsto \exp_s(m)$. So, the coefficient x_k of t^k in (19) corresponds to $x_S(k)$ in (18).

Recall now that on the group algebra $\mathbb{C}G$ of the finite group G, we have two products, one of them is the multiplication coordinate-wise, also called the multiplications of the values, which is defined as:

$$\Big(\sum_{g\in G} a_g g\Big)\cdot\Big(\sum_{g\in G} b_g g\Big) = \sum_{g\in G} a_g b_g g.$$

and the other product is the convolution product:

$$\Big(\sum_{g\in G} a_g g\Big) * \Big(\sum_{h\in G} b_h h\Big) = \sum_{g\in G}\sum_{h\in G} a_g b_h gh = \sum_{g\in G}\Big(\sum_{h\in G} a_h b_{gh^{-1}}\Big) g. \tag{20}$$

Lemma 2. *In $\mathbb{C}C_d$ consider the element $x = \sum_{0\le k\le d-1} x_k t^k$. We have:*

$$x * x = d \sum_{0\le \ell\le d-1} E^{(\ell)} t^\ell$$

and

$$x * x * x = d^2 \sum_{0\le \ell\le d-1} \mathrm{tr}(e_1^\ell e_2) t^\ell.$$

Proof. The expression for $x*x$ follows immediately by direct computation. For the second expression we have

$$\begin{aligned} x*x*x &= d\sum_{0\le \ell\le d-1} E^{(\ell)} t^\ell * x = d \sum_{0\le \ell\le d-1} E^{(\ell)} t^\ell * \sum_{0\le k\le d-1} x_k t^k \\ &= d \sum_{0\le \ell,k\le d-1} E^{(\ell)} x_k t^{\ell+k} = d \sum_{0\le \ell,k,s\le d-1} x_s x_{\ell-s} x_k t^{\ell+k} \\ &= d \sum_{0\le \ell,k,s\le d-1} x_s x_{\ell-s-k} x_k t^\ell = d^2\, \mathrm{tr}(e_1^{(\ell)} e_2). \quad \blacksquare \end{aligned}$$

For each $a\in\mathbb{Z}/d\mathbb{Z}$ the character χ_a defines, with respect to the convolution product, an element $\mathbf{i}_a$ of $\mathbb{C}C_d$,

$$\mathbf{i}_a := \sum_{0\le s\le d-1} \chi_a(s) t^s.$$

One can verify that

$$\mathbf{i}_a * \mathbf{i}_b = \begin{cases} d\,\mathbf{i}_a & \text{if } a = b \\ 0 & \text{if } a\ne b \end{cases}$$

that is, $\mathbf{i}_a/d$ is an idempotent element. On the other hand, if we regard $\delta_a := t^a$ as element in $\mathbb{C}C_d$, it is clear that

$$\delta_a \cdot \delta_b = \begin{cases} \delta_a & \text{if } a = b \\ 0 & \text{if } a \neq b. \end{cases}$$

The connection between the two products on $\mathbb{C}C_d$ is given by the *Fourier transform.* More precisely, the Fourier transform is the linear automorphism on $\mathbb{C}C_d$, defined as:

$$x := \sum_{0\le s\le d-1} a_s t^s \mapsto \widehat{x} := (x * \mathbf{i}_s)(0) = \sum_{0\le \ell\le d-1} a_\ell\, \chi_s(d-\ell). \tag{21}$$

With the above notation we have

LEMMA 3. *The following hold in* $\mathbb{C}C_d$:

$$\widehat{x * y} = \widehat{x} \cdot \widehat{y}, \qquad \widehat{x \cdot y} = d^{-1}\widehat{x} * \widehat{y},$$
$$\widehat{\delta_a} = \mathbf{i}_{-a}, \qquad \widehat{\mathbf{i}_a} = d\,\delta_a, \qquad \widehat{\widehat{x}}(u) = d\,x(-u).$$

Proof. The proof is just a straightforward computation (see [23]). ∎

2. The Yokonuma–Temperley–Lieb algebra. In this section we define a framed analogue of the Temperley–Lieb algebra, as quotient of $\mathrm{Y}_{d,n}(u)$ over an appropriate two-sided ideal.

2.1. The Hecke algebra, $\mathrm{H}_n(u)$, can be considered as a u-deformation of the $\mathbb{C}S_n$, while $\mathrm{TL}_n(u)$ is the quotient of $\mathrm{H}_n(u)$ over the two-sided ideal:

$$J = \langle h_{i,j} \,|\, \text{for all } i,j \text{ such that } |i-j| = 1\rangle,$$

where $h_{i,j}$'s are the Steinberg elements

$$h_{i,j} := 1 + h_i + h_j + h_ih_j + h_jh_i + h_ih_jh_i.$$

It is well-known that J is a principal ideal. Indeed,

$$J = \langle h_{1,2}\rangle.$$

Notice now that $h_{i,j}$ can be rewritten as

$$h_{i,j} = \sum_{\alpha\in W_{i,j}} h_\alpha,$$

where $W_{i,j}$ is the subgroup of S_n generated by s_i and s_j (clearly, $W_{i,j}$ is isomorphic to S_3). On the other hand, $\mathrm{Y}_{d,n}(u)$ can be regarded as a u-deformation of $\mathbb{C}[C_d^n \rtimes S_n]$. The symmetric group S_n can be considered as a subgroup of $C_d^n \rtimes S_n$, therefore the subgroups $W_{i,j}$ of S_n can be also regarded as subgroups of $C_d^n \rtimes S_n$. Thus, in analogy to the ideal J of $\mathrm{H}_n(u)$, it is natural to consider the following ideal I of $\mathrm{Y}_{d,n}(u)$:

$$I := \langle g_{i,j} \,|\, \text{for all } i,j \text{ such that } |i-j| = 1\rangle, \tag{22}$$

where

$$g_{i,j} := \sum_{\alpha\in W_{i,j}} g_\alpha = 1 + g_i + g_j + g_ig_j + g_jg_i + g_ig_jg_i. \tag{23}$$

We then introduce the definition:

DEFINITION 2. For $n \geq 3$, the *Yokonuma–Temperley–Lieb* algebra, $\mathrm{YTL}_{d,n}(u)$, is defined as the quotient:

$$\mathrm{YTL}_{d,n}(u) = \frac{\mathrm{Y}_{d,n}(u)}{I}.$$

In other words, the algebra $\mathrm{YTL}_{d,n}(u)$ can be presented by the generators $g_1, \ldots, g_{n-1}$, $t_1, \ldots, t_n$ (by some abuse of notation), subject to the following relations:

$$g_i g_j = g_j g_i, \quad |i-j| > 1 \tag{24}$$

$$g_{i+1} g_i g_{i+1} = g_i g_{i+1} g_i \tag{25}$$

$$g_i^2 = 1 + (u-1)e_i + (u-1)e_i g_i \tag{26}$$

$$t_i t_j = t_j t_i \quad \text{for all } i, j \tag{27}$$

$$t_i^d = 1 \quad \text{for all } i \tag{28}$$

$$g_i t_i = t_{i+1} g_i \tag{29}$$

$$g_i t_{i+1} = t_i g_i \tag{30}$$

$$g_i t_j = t_j g_i \quad \text{for } j \neq i \text{ and } j \neq i+1 \tag{31}$$

$$1 + g_i + g_{i+1} + g_i g_{i+1} + g_{i+1} g_i + g_i g_{i+1} g_i = 0. \tag{32}$$

We shall refer to relations (32) as *the Steinberg relations.*

Notice that relations (24)–(31) are the defining relations of the algebra $\mathrm{Y}_{d,n}(u)$. Note also that relations (32) are symmetric with respect to the indices i, $i+1$, that is:

$$g_i g_{i+1} g_i = -1 - g_i - g_{i+1} - g_i g_{i+1} - g_{i+1} g_i = g_{i+1} g_i g_{i+1},$$

so for $n \geq 3$ relations (25) follow from relations (32).

REMARK 5. In analogy to the Yokonuma–Hecke algebra, $\mathrm{YTL}_{1,n}(u)$ coincides with the algebra $\mathrm{TL}_n(u)$. Further, the epimorphism (15) induces an epimorphism of the Yokonuma–Temperley–Lieb algebra $\mathrm{YTL}_{d,n}(u)$ onto the algebra $\mathrm{TL}_n(u)$. Also, by relations (29) and (30), any monomial in $\mathrm{YTL}_{d,n}(u)$ inherits the *splitting property* of $\mathrm{Y}_{d,n}(u)$, that is, it can be written in the form

$$w = t_1^{a_1} \ldots t_n^{a_n} g_{i_1} \ldots g_{i_k}, \tag{33}$$

where $a_1, \ldots, a_n \in \mathbb{Z}/d\mathbb{Z}$.

We now have:

LEMMA 4. *The ideal I is principal.*

Proof. Observe first that (σ_1, σ_2) is conjugate to (σ_i, σ_{i+1}) in the braid group, hence also in $\mathcal{F}_{d,n}$. This proves that the pairs (g_1, g_2) and (g_i, g_{i+1}) are conjugate in $\mathrm{Y}_{d,n}$. This conjugation maps the elements $g_{1,2}$ to $g_{i,i+1}$, and the ideal I is principal. ∎

Corollary 1. *$\mathrm{YTL}_{d,n}(u)$ is the $\mathbb{C}(u)$-algebra generated by the set*

$$\{t_1, \ldots, t_n, g_1, \ldots, g_{n-1}\}$$

whose elements are subject to the defining relations of $\mathrm{Y}_{d,n}(u)$ and the relation

$$g_{1,2} = 0.$$

Proof. The result follows by the multiplication rule defined on $\mathrm{Y}_{d,n}(u)$ and Lemma 4. ∎

2.2. A presentation with non-invertible generators. In analogy with (4) one can obtain a presentation for the Yokonuma–Temperley–Lieb algebra $\mathrm{YTL}_{d,n}(u)$ with the non-invertible generators:

$$l_i := \frac{1}{u+1}(g_i + 1). \tag{34}$$

In particular we have:

Proposition 2. *$\mathrm{YTL}_{d,n}(u)$ can be viewed as the algebra generated by the elements $l_1, \ldots, l_{n-1}$, $t_1, \ldots, t_n$, which satisfy the following defining relations*:

$$t_i^d = 1, \qquad \text{for all } i \tag{35}$$

$$t_i t_j = t_j t_i, \qquad \text{for all } i, j \tag{36}$$

$$l_i t_j = t_j l_i, \qquad \text{for } j \neq i \text{ and } j \neq i+1 \tag{37}$$

$$l_i t_i = t_{i+1} l_i + \frac{1}{u+1}(t_i - t_{i+1}) \tag{38}$$

$$l_i t_{i+1} = t_i l_i + \frac{1}{u+1}(t_{i+1} - t_i) \tag{39}$$

$$l_i^2 = \frac{(u-1)e_i + 2}{u+1}\, l_i \tag{40}$$

$$l_i l_j = l_j l_i, \qquad |i-j| > 1 \tag{41}$$

$$l_i l_{i+1} l_i = \frac{(u-1)e_i + 1}{(u+1)^2}\, l_i. \tag{42}$$

Proof. Obviously, $\mathrm{YTL}_{d,n}(u)$ is generated by the l_i's and the t_i's. It is a straightforward computation to see that relations (24)–(32) are transformed into the relations (35)–(42). However, we shall show here how it works for the quadratic relations (35) and the Steinberg relations (42). From (34) we obtain

$$g_i = (u+1)l_i - 1. \tag{43}$$

We then have

$$g_i^2 = ((u+1)l_i - 1)^2,$$

which is equivalent to

$$1 + (u-1)e_i + (u-1)e_i g_i = (u+1)^2 l_i^2 - 2(u+1)l_i + 1$$

or, equivalently,

$$(u-1)(u+1)e_i l_i = (u+1)^2 l_i^2 - 2(u+1)l_i,$$

which leads to Eq. (40).

For the Steinberg elements $g_{i,i+1}$ using (43) we have

$g_{i,i+1} = 1+g_i+g_{i+1}+g_ig_{i+1}+g_{i+1}g_i+g_ig_{i+1}g_i = (u+1)^3l_il_{i+1}l_i-(u+1)^2l_i^2+(u+1)l_i.$

From the Steinberg relation (32) and (40) we have

$$(u+1)^2l_il_{i+1}l_i = ((u-1)e_i+1)l_i$$

or, equivalently,

$$l_il_{i+1}l_i = \frac{(u-1)e_i+1}{(u+1)^2}\,l_i,$$

which is (42). ∎

Remark 6. Setting $d=1$ in the presentation of $\mathrm{YTL}_{d,n}(u)$ in Proposition 2, one obtains the classical presentation of $\mathrm{TL}_n(u)$, as discussed in Subsection 1.2. Note also that, substituting in the braid relations (25) the g_i's using (43), we obtain the equation:

$$l_il_{i+1}l_i - \frac{(u-1)e_i+1}{(u+1)^2}\,l_i = l_{i+1}l_il_{i+1} - \frac{(u-1)e_{i+1}+1}{(u+1)^2}\,l_{i+1}$$

which becomes superfluous, since it can be deduced from (42). This was to be expected, since the braid relations (25) were also superfluous.

3. A spanning set for the Yokonuma–Temperley–Lieb algebra. In this section we discuss various properties of a word in $\mathrm{YTL}_{d,n}(u)$ and we present a spanning set for $\mathrm{YTL}_{d,n}(u)$ (Proposition 4). Furthermore, using the work of Chlouveraki and Pouchin in [3] we give their formula for the dimension of $\mathrm{YTL}_{d,n}(u)$ (Proposition 5) and we also discuss their results on the linear basis of $\mathrm{YTL}_{d,n}(u)$ (Theorem 3). We finally compute a basis for $\mathrm{YTL}_{2,3}(u)$ different than the one of Theorem 3.

3.1. We have the following definition:

Definition 3. In $\mathrm{YTL}_{d,n}(u)$ we define a *length function* l as follows:

$$l(t^ag_{i_1}\dots g_{i_k}) := l'(s_{i_1}\dots s_{i_k}),$$

where l' is the usual length function of S_n and $t^a := t_1^{a_1}\dots t_n^{a_n}\in C_d^n$. A word in $\mathrm{YTL}_{d,n}(u)$ of the form (33) will be called *reduced* if it is of minimal length with respect to relations (24)–(26), (32).

Proposition 3. *Each word in* $\mathrm{YTL}_{d,n}(u)$ *can be written as a sum of monomials, where the highest and lowest index of the generators* g_i *appear at most once.*

Proof. An analogous statement holds for the Yokonuma–Hecke algebra $\mathrm{Y}_{d,n}(u)$ where only the highest index generators appear at most once [14, Proposition 8]. Since $\mathrm{YTL}_{d,n}(u)$ is a quotient of the algebra $\mathrm{Y}_{d,n}(u)$ the highest index property passes through to the algebra $\mathrm{YTL}_{d,n}(u)$. The idea is analogous to [12, Lemma 4.1.2] and is based on induction on the length of reduced words, use of the braid relations and reduction of length using the quadratic relations (26). For the case of the lowest index generator g_i we use induction on the length of reduced words and the Steinberg relations (32). Indeed, clearly, the statement is true for all words of length ≤ 2, namely for words of the form t^a, t^ag_i, $t^ag_ig_j$.

For words of length 3: Let $w = t^a g_i g_j g_i$. Applying relation (25) will violate the highest index property of the word, so we must use the Steinberg relation (32) and we have

$$t^a g_i g_j g_i = -t^a - t^a g_i - t^a g_j - t^a g_i g_j - t^a g_j g_i.$$

We assume that the lowest index generator appears at most once in all reduced words of length $\leq r$, and we will show the lowest index property for words of length $r+1$. Let $w = t^a g_{i_1} g_{i_2} \dots g_{i_{r+1}}$ be a reduced word in $\mathrm{YTL}_{d,n}(u)$ of length $r+1$, and $l = \min\{i_1, \dots, i_{r+1}\}$.

Let first $w = t^a w_1 g_l w_2 g_l w_3$, and suppose that w_2 does not contain g_l. We then have two possibilities:

If w_2 does not contain g_{l+1}, then g_l commutes with all the g_i's in w_2 and since there cannot be a g_l^2 term in a reduced word, we have, using the induction hypothesis:

$$\begin{aligned} w = t^a w_1 g_l w_2 g_l w_3 &= t^a w_1 w_2 g_l^2 w_3 \\ &= t^a w_1 w_2 (1 + (u-1)e_l + (u-1)e_l g_l)) w_3 \\ &= t^a w_1 w_2 w_3 + (u-1) t^a w_1 w_2 e_l w_3 + (u-1) t^a w_1 w_2 e_l g_l w_3. \end{aligned}$$

If w_2 does contain g_{l+1}, then, by the induction hypothesis, w_2 has the form $w_2 = v_1 g_{l+1} v_2$, where in v_1, v_2 the lowest index generator is at least g_{l+2}, hence

$$w = t^a w_1 g_l v_1 g_{l+1} v_2 g_l w_3 = t^a w_1 v_1 g_l g_{l+1} g_l v_2 w_3.$$

Applying now the Steinberg relation (32) we obtain a linear combination of words each of which has at least one less occurrence of g_l than w. Note also that in the case where $l+1 = m = \max\{i_1, \dots, i_{r+1}\}$, no contradiction is caused with respect to the highest index generator. Continuing in the same manner for all possible pairs of g_l in the word we reduce to having g_l at most once. ■

The following proposition gives us a precise spanning set for $\mathrm{YTL}_{d,n}(u)$.

PROPOSITION 4. *The set of reduced words*

$$\Sigma_{d,n} = \left\{ t^a (g_{i_1} g_{i_1 - 1} \dots g_{i_1 - k_1})(g_{i_2} g_{i_2 - 1} \dots g_{i_2 - k_2}) \dots (g_{i_p} g_{i_p - 1} \dots g_{i_p - k_p}) \right\}, \tag{44}$$

where

$$t^a = t_1^{a_1} \dots t_n^{a_n} \in C_d^n, \quad 1 \leq i_1 < i_2 < \dots < i_p \leq n-1,$$
$$\text{and} \quad 1 \leq i_1 - k_1 < i_2 - k_2 < \dots < i_p - k_p,$$

spans the Yokonuma–Temperley–Lieb algebra $\mathrm{YTL}_{d,n}(u)$. *The highest index generator is* g_{i_p} *of the rightmost cycle and the lowest index generator is* $g_{i_1 - k_1}$ *of the leftmost cycle of a word in* $\Sigma_{d,n}$.

Proof. We present here an elegant proof suggested by the referee. An element w in a group is called fully commutative if any reduced expression for w can be obtained from any other by means of braid relations that only involve commuting generators. Through relations (24)–(32) any word is a linear combination of words of the form $t^a g_{i_1} \dots g_{i_k}$, where $g_{i_1} \dots g_{i_k}$ is the image of a fully commutative word of the braid monoid and it is well-known that a fully commutative word can be written under the form given in the statement of Proposition 4. For facts about fully commutative elements the reader is referred to [22], [10], [6], [1].

The interested reader could also find our original direct proof (by induction on the length of a word) in [9]. ■

M. Chlouveraki and G. Pouchin in [3] have computed the dimension for $\mathrm{YTL}_{d,n}(u)$ by using the representation theory of the Yokonuma–Hecke algebra [4]. More precisely, they proved the following result.

PROPOSITION 5. *The dimension of the Yokonuma–Temperley–Lieb algebra is*

$$\dim(\mathrm{YTL}_{d,n}(u)) = dc_n + \frac{d(d-1)}{2}\sum_{k=1}^{n-1}\binom{n}{k}^2,$$

where c_n is the n-th Catalan number.

3.2. To find an explicit basis for $\mathrm{YTL}_{d,n}(u)$ Chlouveraki and Pouchin in [3] worked as follows: As mentioned in Remark 5 each word in $\mathrm{YTL}_{d,n}(u)$ inherits the splitting property. For each fixed element in the braiding part, they described a set of linear dependence relations among the framing parts (see [3, Proposition 5]). Using these relations they extracted from $\Sigma_{d,n}$ (recall (44)) a smaller spanning set for $\mathrm{YTL}_{d,n}(u)$ and showed that the cardinality of this smaller spanning set is equal to the dimension of the algebra. Thus, it is a basis for $\mathrm{YTL}_{d,n}(u)$. Before describing this basis, we will need the following notation:

Let $\underline{i}$ and $\underline{k}$ be the p-tuples:

$$\underline{i} = (i_1, \ldots, i_p) \quad \text{and} \quad \underline{k} = (k_1, \ldots, k_p)$$

and let $\mathcal{I}$ be the set of pairs $(\underline{i}, \underline{k})$ such that

$$1 \leq i_1 < \ldots < i_p \leq n-1 \quad \text{and} \quad 1 \leq i_1 - k_1 < \ldots < i_p - k_p \leq n-1.$$

We also denote by $g_{\underline{i},\underline{k}}$ the element

$$g_{\underline{i},\underline{k}} := (g_{i_1}g_{i_1-1}\ldots g_{i_1-k_1})(g_{i_2}g_{i_2-1}\ldots g_{i_2-k_2})\ldots(g_{i_p}g_{i_p-1}\ldots g_{i_p-k_p}).$$

Then the set $\Sigma_{d,n}$ can be written as

$$\Sigma_{d,n} = \{t_1^{r_1}\ldots t_n^{r_n}\, g_{\underline{i},\underline{k}} \,|\, r_1, \ldots, r_n \in \mathbb{Z}/d\mathbb{Z}, (\underline{i}, \underline{k}) \in \mathcal{I}\}.$$

The *degree of a word* $w = t_1^{r_1}\ldots t_n^{r_n} g_{i_1}\ldots g_{i_m}$ in $\mathrm{Y}_{d,n}(u)$, denoted by $\deg(w)$, is defined to be the integer m. Set:

$$\Sigma_{d,n}^{<w} := \{s \in \Sigma_{d,n} \mid \deg(s) < \deg(w)\}.$$

The group algebra $\mathbb{C}(u)(\mathbb{Z}/d\mathbb{Z})^n$ is isomorphic to the subalgebra of $\mathrm{Y}_{d,n}(u)$ that is generated by the t_i's but not to the subalgebra of $\mathrm{YTL}_{d,n}(u)$ that is generated by the t_i's. Further, the group algebra $\mathbb{C}(u)(\mathbb{Z}/d\mathbb{Z})^n$ has a natural basis, $B_{d,n}$, given by monomials in $t_1, \ldots, t_n$:

$$B_{d,n} = \{t_1^{r_1}\ldots t_n^{r_n} \,|\, r_1, \ldots, r_n \in \mathbb{Z}/d\mathbb{Z}\}.$$

Thus, any element of $\mathbb{C}(u)(\mathbb{Z}/d\mathbb{Z})^n$ can be written as a linear combination of words in $B_{d,n}$. There is a surjective algebra morphism from $\mathbb{C}(u)(\mathbb{Z}/d\mathbb{Z})^n$ to the subalgebra of $\mathrm{YTL}_{d,n}(u)$ that is generated by the t_i's. We will denote by $\overline{b}$ the image of an element $b \in B_{d,n}$ into the subalgebra of $\mathrm{YTL}_{d,n}(u)$ that is generated by the t_i's. We then have the following theorem:

THEOREM 3 (Chlouveraki–Pouchin). *The following set is a linear basis for* $\mathrm{YTL}_{d,n}(u)$:

$$S_{d,n} = \{\bar{b}_{\underline{i},\underline{k}}\, g_{\underline{i},\underline{k}} \,|\, (\underline{i},\underline{k}) \in \mathcal{I},\ b_{\underline{i},\underline{k}} \in \mathcal{B}_{d,n}(g_{\underline{i},\underline{k}})\},$$

where $\mathcal{B}_{d,n}(g_{\underline{i},\underline{k}})$ *is a proper subset of* $B_{d,n}$ *such that*

$$\{b_{\underline{i},\underline{k}} + R(g_{\underline{i},\underline{k}}) \,|\, b_{\underline{i},\underline{k}} \in \mathcal{B}_{d,n}(g_{\underline{i},\underline{k}})\}$$

is a basis of the quotient space $\mathbb{C}(u)(\mathbb{Z}/d\mathbb{Z})^n/R(g_{\underline{i},\underline{k}})$, *and where* $R(w)$ *is the following ideal of* $\mathbb{C}(u)(\mathbb{Z}/d\mathbb{Z})^n$*:*

$$R(w) = \left\{\mathfrak{m} \in \mathbb{C}(u)(\mathbb{Z}/d\mathbb{Z})^n \,|\, \bar{\mathfrak{m}}\, w \in \mathrm{Span}_{\mathbb{C}(u)}(\Sigma_{d,n}^{<w})\right\}.$$

3.3. For $d = 2$, $n = 3$ it is relatively easy to find a basis for $\mathrm{YTL}_{2,3}(u)$. We will give here a basis different than the one in Theorem 3. Before continuing, we need the following technical lemma that will be also used in the proof of Theorem 5.

LEMMA 5 (cf. [15, Lemma 7.5]). *For the element* $g_{1,2}$ *we have in* $\mathrm{Y}_{d,n}(u)$ (*recall* (10) *for* $e_{1,3}$):

(1) $$g_1 g_{1,2} = [1 + (u-1)e_1]g_{1,2}$$

(2) $$g_2 g_{1,2} = [1 + (u-1)e_2]g_{1,2}$$

(3) $$g_1 g_2 g_{1,2} = [1 + (u-1)e_1 + (u-1)e_{1,3} + (u-1)^2 e_1 e_2]g_{1,2}$$

(4) $$g_2 g_1 g_{1,2} = [1 + (u-1)\, e_2 + (u-1)e_{1,3} + (u-1)^2 e_1 e_2]g_{1,2}$$

(5) $$g_1 g_2 g_1 g_{1,2} = [1 + (u-1)(e_1 + e_2 + e_{1,3}) + (u-1)^2(u+2)\, e_1 e_2]g_{1,2}.$$

Analogous relations hold for multiplications with $g_{1,2}$ *from the right.*

Proof. The idea is to expand the left-hand side of each equation and then use (26) and Lemma 1. We will demonstrate the proof for the indicative cases (1) and (4). The other cases are proved similarly.

For case (1) we have:

$$\begin{aligned} g_1 g_{1,2} &= g_1 + g_1^2 + g_1 g_2 + g_1^2 g_2 + g_1 g_2 g_1 + g_1^2 g_2 g_1 \\ &= g_1 + [1 + (u-1)e_1 + (u-1)e_1 g_1] \\ &\quad + g_1 g_2 + [g_2 + (u-1)e_1 g_2 + (u-1)e_1 g_1 g_2] \\ &\quad + g_1 g_2 g_1 + [g_2 g_1 + (u-1)e_1 g_2 g_1 + (u-1)e_1 g_1 g_2 g_1] \\ &= g_{1,2} + (u-1)e_1 g_{1,2}. \end{aligned}$$

Case (2) is completely analogous. In order to prove case (4) we will use cases (1) and (2):

$$\begin{aligned} g_2 g_1 g_{1,2} &= g_2(g_{1,2} + (u-1)e_1 g_{1,2}) \\ &= g_2 g_{1,2} + (u-1)e_{1,3} g_2 g_{1,2} \quad \text{(Lemma 1)} \\ &= [1 + (u-1)e_2]g_{1,2} + (u-1)e_{1,3}(1 + (u-1)e_2)g_{1,2} \\ &= [1 + (u-1)e_2]g_{1,2} + (u-1)e_{1,3}g_{1,2} + (u-1)^2 e_{1,3} e_2 g_{1,2} \quad \text{(Lemma 1)} \\ &= [1 + (u-1)e_2 + (u-1)e_{1,3} + (u-1)^2 e_1 e_2]g_{1,2}. \ \blacksquare \end{aligned}$$

To find a basis for $\mathrm{YTL}_{2,3}(u)$: From Proposition 5 we have $\dim(\mathrm{YTL}_{2,3}(u)) = 28$. On the other hand, the spanning set $\Sigma_{2,3}$ of $\mathrm{YTL}_{2,3}(u)$ of Proposition 4, contains 40 elements.

Thus, any relation $w_1 g_{1,2} w_2 = 0$ with $w_1,\ w_2 \in \mathrm{Y}_{2,3}(u)$ reduces to having $w_1, w_2 \in \Sigma_{2,3}$. Further, if any of w_1, w_2 contains braiding generators, then by Lemma 5 (after pushing framing generators in w_2 to the right) these get absorbed by $g_{1,2}$. Thus, and since $e_{i,j} = \frac{1}{2}(1 + t_i t_j)$ for $d = 2$, it suffices to consider the system of equations

$$w_1 g_{1,2} w_2 = 0 \qquad w_1, w_2 \in \mathcal{T}, \tag{45}$$

where $\mathcal{T} := \{1, t_1, t_2, t_3, t_1t_2, t_1t_3, t_2t_3, t_1t_2t_3\}$. For finding all possible linear dependencies in $\Sigma_{2,3}$, after substituting $g_1g_2g_1$ by $-1 - g_1 - g_2 - g_1g_2 - g_2g_1$ in (45), note that some of these 64 equations reduce trivially to $g_{1,2} = 0$; for example if $w_2 = 1$ or $w_2 = t_1t_2t_3$ (since it commutes with $g_{1,2}$). From the rest one can extract 12 linearly independent equations which, applied on the spanning set $\Sigma_{2,3}$, lead to the following basis for $\mathrm{YTL}_{2,3}(u)$:

$$\begin{aligned}\mathcal{S}_{2,3} = \big\{&1, t_1, t_2, t_1t_2, g_1, t_2g_1, t_3g_1, t_2t_3g_1, g_2, t_1g_2, t_3g_2, t_1t_3g_2,\\ &g_1g_2, t_1g_1g_2, t_2g_1g_2, t_3g_1g_2, t_1t_2g_1g_2, t_1t_3g_1g_2, t_2t_3g_1g_2, t_1t_2t_3g_1g_2,\\ &g_2g_1, t_1g_2g_1, t_2g_2g_1, t_3g_2g_1, t_1t_2g_2g_1, t_1t_3g_2g_1, t_2t_3g_2g_1, t_1t_2t_3g_2g_1\big\}.\end{aligned}$$

4. A Markov trace on $\mathbf{YTL}_{d,n}(u)$. The following section is dedicated to finding the necessary and sufficient conditions for the trace tr on $\mathrm{Y}_{d,n}(u)$ to pass to the quotient algebra $\mathrm{YTL}_{d,n}(u)$, in analogy to the classical case, where the Ocneanu trace on $\mathrm{H}_n(u)$ passes to the quotient algebra $\mathrm{TL}_n(u)$ if the trace parameter ζ satisfies some appropriate condition.

4.1. It is clear by now that if the trace passes to $\mathrm{YTL}_{d,n}(u)$ then it has to kill the generator $g_{1,2}$ of the principal ideal through which the quotient is defined, that is, if $\mathrm{tr}(g_{1,2}) = 0$. We have the following lemma:

LEMMA 6. *For the element $g_{1,2}$ we have*:

$$\mathrm{tr}(g_{1,2}) = (u+1)z^2 + ((u-1)E + 3)z + 1. \tag{46}$$

Proof. The proof is a straightforward computation:

$$\begin{aligned}\mathrm{tr}(g_{1,2}) &= \mathrm{tr}(1) + \mathrm{tr}(g_1) + \mathrm{tr}(g_2) + \mathrm{tr}(g_1g_2) + \mathrm{tr}(g_2g_1) + \mathrm{tr}(g_1g_2g_1)\\ &= 1 + 2z + 2z^2 + z + (u-1)Ez + (u-1)z^2\\ &= (u+1)z^2 + ((u-1)E + 3)z + 1. \ \blacksquare\end{aligned}$$

Lemma 6, together with the equation

$$\mathrm{tr}(g_{1,2}) = (u+1)z^2 + ((u-1)E + 3)z + 1 = 0 \tag{47}$$

gives us the following values for z:

$$z_{\pm} = \frac{-\left((u-1)E + 3\right) \pm \sqrt{\left((u-1)E + 3\right)^2 - 4(u+1)}}{2(u+1)}. \tag{48}$$

We shall do now the analysis for all conditions that must be imposed on the trace parameters in order that tr passes to $\mathrm{YTL}_{d,n}(u)$. If we have in mind Corollary 1 and the linearity of tr, it follows that tr passes to $\mathrm{YTL}_{d,n}(u)$ if and only if the following equations are satisfied for all monomials $\mathfrak{m}$ in the inductive basis of $\mathrm{Y}_{d,n}(u)$:

$$\mathrm{tr}(\mathfrak{m}\, g_{1,2}) = 0. \tag{49}$$

Let us first consider the case $n = 3$. By Proposition 1 the elements in the inductive basis of $\mathrm{Y}_{d,3}(u)$ are of the following forms:

$$t_1^a t_2^b t_3^c, \quad t_1^a g_1 t_1^b t_3^c, \quad t_1^a t_2^b g_2 g_1 t_1^c, \quad t_1^a t_2^b g_2 t_2^c, \quad t_1^a g_1 t_1^b g_2 t_2^c, \quad t_1^a g_1 t_1^b g_2 g_1 t_1^c. \tag{50}$$

Using Lemma 5 and the following notation:

$$Z_{a,b,c} := (u+1)z^2 x_{a+b+c} + \big((u-1)E^{(a+b+c)} + x_a x_{b+c} + x_b x_{a+c} + x_c x_{a+b}\big)z + x_a x_b x_c$$

$$V_{a,b+c} := (u+1)z^2 x_{a+b+c} + (u+1)zE^{(a+b+c)} + z\, x_a x_{b+c} + x_a E^{(b+c)}$$

$$W_{a,b,c} := (u+1)z^2 x_{a+b+c} + (u+2)zE^{(a+b+c)} + \mathrm{tr}\big(e_1^{(a+b+c)} e_2\big)$$

we obtain by (49) and (50) the following equations, for any $a, b, c \in \mathbb{Z}/d\mathbb{Z}$:

$$Z_{a,b,c} = 0 \tag{51}$$

$$Z_{a,b,c} + (u-1)V_{a,b+c} = 0 \tag{52}$$

$$Z_{a,b,c} + (u-1)[V_{a,b+c} + V_{b,a+c} + W_{a,b,c}] = 0 \tag{53}$$

$$Z_{a,b,c} + (u-1)[V_{a,b+c} + V_{b,a+c} + V_{c,a+b} + W_{a,b,c}] = 0. \tag{54}$$

Equations (51)–(54) reduce to the following system of equations of $z, x_1, \ldots, x_{d-1}$ for any $a, b, c \in \mathbb{Z}/d\mathbb{Z}$:

$$(\Sigma) \qquad \begin{cases} Z_{a,b,c} = 0 & \text{(55a)} \\ V_{a,b+c} = 0 & \text{(55b)} \\ W_{a,b,c} = 0 & \text{(55c)} \end{cases}$$

Notice that for $a = b = c = 0$ Eq. (51) becomes Eq. (47). If, now, we require both solutions in (48) to participate in the solutions of (Σ), then we are led to sufficient conditions for tr to pass to $\mathrm{YTL}_{2,3}(u)$ (Section 4.2). If not, then we are led to necessary and sufficient conditions for tr to pass to $\mathrm{YTL}_{2,3}(u)$ (Section 4.3).

4.2. Suppose that both solutions for z from (48) participate in the solution set of (Σ). Then we have the following proposition:

PROPOSITION 6. *If the trace parameters x_i are d-th roots of unity, $x_i = x_1^i$, $1 \le i \le d-1$, and $z = -\frac{1}{u+1}$ or $z = -1$, then the trace* tr *defined on $\mathrm{Y}_{d,3}(u)$ passes to the quotient $\mathrm{YTL}_{d,3}(u)$.*

Proof. Suppose that (Σ) has both solutions for z from (48). This implies that there exist λ in $\mathbb{C}(u)(x_1, \ldots, x_{d-1})$ such that

$$Z_{a,b,c} = \lambda Z_{0,0,0}.$$

From this we deduce that

$$\lambda = x_{a+b+c}$$

$$x_a x_{b+c} + x_b x_{a+c} + x_c x_{a+b} = 3x_{a+b+c}$$

$$E^{(a+b+c)} = x_{a+b+c} E \tag{56}$$

$$x_{a+b+c} = x_a x_b x_c. \tag{57}$$

Since this holds for any $a, b, c \in \mathbb{Z}/d\mathbb{Z}$, by taking $b = c = 0$ in (56) we have

$$E^{(a)} = x_a E, \tag{58}$$

which is exactly the E-system. Moreover, by taking $c = 0$ in (57) we obtain

$$x_a x_b = x_{a+b}. \tag{59}$$

This implies that the x_i's are d-th roots of unity, $x_i = x_1^i$, $1 \leq i \leq d-1$, which is equivalent to $E = 1$ [19, Appendix]. In order to conclude the proof it is enough to verify that these conditions for the x_i's satisfy also (55b)–(55c) of (Σ). Since the x_i's are solutions of the E-system, Eq. (55b) is immediately satisfied. We will finally check (55c). One has $\mathrm{tr}(e_1^{(m)} e_2) = x_m E^2$ as soon as the x_m satisfy the E-system. Once this has been noticed, Eq. (55c) becomes the same as (51) by using (57) and $E = 1$. ■

Using induction on n one can prove the general case of the sufficient conditions for tr to pass to $\mathrm{YTL}_{d,n}(u)$. Indeed we have:

THEOREM 4. *If the trace passes to the quotient for $n = 3$ then it passes for all $n > 3$.*

Proof. By induction on n. In Proposition 6 we proved the case where $n = 3$. Assume that the statement holds for all $\mathrm{YTL}_{d,k}(u)$, where $k \leq n$, that is,

$$\mathrm{tr}(a_k\, g_{1,2}) = 0$$

for all $a_k \in \mathrm{Y}_{d,k}(u)$, $k \leq n$. We will show the statement for $k = n+1$. It suffices to prove that the trace vanishes on any element in the form $a_{n+1} g_{1,2}$, where a_{n+1} belongs to the inductive basis of $\mathrm{Y}_{d,n+1}(u)$ (recall Proposition 1), given the conditions of the theorem. Namely:

$$\mathrm{tr}(a_{n+1}\, g_{1,2}) = 0.$$

Since a_{n+1} is in the inductive basis of $\mathrm{Y}_{d,n+1}(u)$, it is of one of the following forms:

$$a_{n+1} = a_n g_n \ldots g_i t_i^k \quad \text{or} \quad a_{n+1} = a_n t_{n+1}^k,$$

where a_n is in the inductive basis of $\mathrm{Y}_{d,n}(u)$. For the first case we have

$$\mathrm{tr}(a_{n+1} g_{1,2}) = \mathrm{tr}(a_n g_n \ldots g_i t_i^k g_{1,2}) = z\, \mathrm{tr}(a_n g_{n-1} \ldots g_i t_i^k\, g_{1,2})$$

and the result follows by induction. Therefore the statement is proved. The second case is proved similarly. Hence, the proof is concluded. ■

The above theorem allows us to state the following:

THEOREM 5. *For $n \geq 3$, if the trace parameters x_i are d-th roots of unity, $x_i = x_1^i$, $1 \leq i \leq d-1$, and $z = -\frac{1}{u+1}$ or $z = -1$, then the trace* tr *defined on $\mathrm{Y}_{d,n}(u)$ passes to the quotient $\mathrm{YTL}_{d,n}(u)$.*

4.3. Moving on, we investigate the possibility of the x_i's being solutions of the E-system, other than d-th roots of unity. We have the following:

THEOREM 6. *The trace* tr *passes to the quotient $\mathrm{YTL}_{d,n}(u)$ if and only if the x_i's are solutions of the* E*-system and one of the two cases holds*:

(i) *For some $0 \leq m_1 \leq d-1$ the x_ℓ's are expressed as*

$$x_\ell = \exp_{m_1}(\ell) \quad (0 \leq \ell \leq d-1).$$

In this case the x_ℓ's are d-th roots of unity and $z = -\frac{1}{u+1}$ or $z = -1$.

(ii) *For some* $0 \leq m_1, m_2 \leq d-1$*, where* $m_1 \neq m_2$*, the* x_ℓ*'s are expressed as*

$$x_\ell = \frac{1}{2}\big(\exp_{m_1}(\ell) + \exp_{m_2}(\ell)\big) \quad (0 \leq \ell \leq d-1).$$

In this case we have $z = -\frac{1}{2}$.

Note that case (i) captures Theorem 5.

Proof. Observe that the x_ℓ's expressed by (i) are indeed solutions of the system (Σ). We will now assume that our solutions are not of this form. This implies that $x_a \neq E^{(a)}$ for some $0 \leq a \leq d-1$, and this will allow us to have this quantity in denominators later.

We will use induction on n. We will first prove the case $n = 3$. Suppose that trace tr passes to the quotient algebra $\mathrm{YTL}_{d,3}(u)$. This means that (Σ) has solutions for z any one of those in (48), for any $a, b, c \in \mathbb{Z}/d\mathbb{Z}$. Subtracting (55a) from (55b) we obtain:

$$\big(x_a x_{b+c} + x_b x_{a+c} - 2E^{(a+b+c)}\big)z = -\left(x_a x_b x_c - x_c E^{(a+b)}\right). \tag{60}$$

For $b = c = 0$ in (60) and since we assumed that there is an a such that $x_a \neq E^{(a)}$ we obtain: $z = -\frac{1}{2}$. On the other hand, subtracting Eq. (55a) from (55c) we have

$$\big(3E^{(a+b+c)} - x_a x_{b+c} - x_b x_{a+c} - x_c x_{a+b}\big)z = x_a x_b x_c - \mathrm{tr}(e_1^{(a+b+c)} e_2). \tag{61}$$

For the value a such that $x_a - E^{(a)} \neq 0$ and for $b = c = 0$ in (61) we obtain

$$z = -\frac{x_a - \mathrm{tr}(e_1^{(a)} e_2)}{3(x_a - E^{(a)})}. \tag{62}$$

By combining (60) and (62) we have

$$\frac{1}{2} = \frac{x_a - \mathrm{tr}(e_1^{(a)} e_2)}{3(x_a - E^{(a)})}$$

or, equivalently,

$$3(x_a - E^{(a)}) = 2(x_a - \mathrm{tr}(e_1^{(a)} e_2)).$$

By Lemma 2, this is equivalent to

$$3x - \frac{3}{d}\, x * x = 2x - \frac{2}{d^2}\, x * x * x.$$

Taking the Fourier transform (see Lemma 3) we arrive at

$$\frac{2}{d^2}\,\widehat{x}^3 - \frac{3}{d}\,\widehat{x}^2 + \widehat{x} = 0.$$

Assuming that $\widehat{x} = \sum_{0 \leq \ell \leq d-1} y_\ell t^\ell$ we have the following expression for the coefficients y_ℓ in the expansion of $\widehat{x}$:

$$y_\ell\Big(\frac{2}{d^2} y_\ell^2 - \frac{3}{d} y_\ell + 1\Big) = 0.$$

So either $y_\ell = 0$ or $y_\ell = d$ or $y_\ell = \frac{1}{2}d$. So, if we take a partition of the set $\{\ell : 0 \leq \ell \leq d-1\}$ into sets S_0, S_1, $S_{1/2}$ such that y_ℓ takes the value $i \cdot d$ on S_i $(i = 0, 1, 1/2)$, we then have from Lemma 3:

$$x = \sum_{m \in S_1} \mathbf{i}_{-m} + \frac{1}{2} \sum_{m \in S_{1/2}} \mathbf{i}_{-m}.$$

From $x_0 = 1$ we obtain the conditions

$$1 = x(0) = |S_1| + \frac{1}{2}|S_{1/2}|.$$

This means that either S_1 has only one element and $S_{1/2} = \emptyset$ or $S_1 = \emptyset$ and $S_{1/2}$ has two elements. The first case corresponds to the case (i) where the x_ℓ's are d-th roots of unity. In the second case, if $S_{1/2} = \{m_1, m_2\}$ we obtain the following solution of the E-system:

$$x_\ell = \frac{1}{2}\big(\exp_{m_1}(\ell) + \exp_{m_2}(\ell)\big), \qquad (0 \le \ell \le d-1), \tag{63}$$

which corresponds to $z = -\frac{1}{2}$.

We can now check that these solutions satisfy the system (Σ). Since $z = -\frac{1}{2}$ and $E = \frac{1}{2}$, we have $E^{(\ell)} = x_\ell/2$, $V_{c,a+b} = W_{a,b,c} = 0$. Thus $Z_{a,b,c} = 0$ reduces to

$$x_a x_{b+c} + x_b x_{a+c} + x_c x_{a+b} = x_{a+b+c} + 2x_a x_b x_c,$$

which can be checked to be satisfied by the values x_ℓ given in (63).

The rest of the proof (the induction on n) follows by Theorem 4. ∎

Remark 7. The values for the trace parameter z in Theorems 5 and 6, $z = -\frac{1}{u+1}$ and $z = -1$, in order that tr on $\mathrm{Y}_{d,n}(u)$ passes to the quotient $\mathrm{YTL}_{d,n}(u)$ are the same as the values in (6) for ζ of the Ocneanu trace τ on $\mathrm{H}_n(u)$, so that τ passes to the quotient $\mathrm{TL}_n(u)$ (recall Section 1.2).

5. The Jones polynomial from $\mathbf{YTL}_{d,n}(u)$. The 2-variable Jones or Homflypt polynomial, $P(\lambda, u)$, can be defined through the Ocneanu trace on $\mathrm{H}_n(u)$ [11]. Indeed, for any braid $\alpha \in \bigcup_\infty B_n$ we have

$$P(\lambda, u)(\widehat{\alpha}) = \Big(-\frac{1-\lambda u}{\sqrt{\lambda}(1-u)}\Big)^{n-1} \big(\sqrt{\lambda}\big)^{\varepsilon(\alpha)} \tau(\pi(\alpha)),$$

where $\lambda = \frac{1-u+\zeta}{u\zeta}$, π is the natural epimorphism of $\mathbb{C}(u)B_n$ onto $\mathrm{H}_n(u)$ that sends the braid generator σ_i to h_i, and $\varepsilon(\alpha)$ is the algebraic sum of the exponents of the σ_i's in α. Further, the Jones polynomial, $V(u)$, related to the algebras $\mathrm{TL}_n(u)$, can be redefined through the Homflypt polynomial, by specializing ζ to $-\frac{1}{u+1}$, see [11]. This is the non-trivial value for ζ, for which the Ocneanu trace τ passes to the quotient $\mathrm{TL}_n(u)$. Namely:

$$V(u)(\widehat{\alpha}) = \Big(-\frac{1+u}{\sqrt{u}}\Big)^{n-1} \big(\sqrt{u}\big)^{\varepsilon(\alpha)} \tau(\pi(\alpha)) = P(u, u)(\widehat{\alpha}).$$

As mentioned in Section 1.5, given a solution of the E-system parametrized by a subset S of $\mathbb{Z}/d\mathbb{Z}$, one can obtain an invariant for framed knots and links [19]:

$$\Gamma_{d,S}(w, u)(\widehat{\alpha}) = \Big(-\frac{1-wu}{\sqrt{w}(1-u)E}\Big)^{n-1} \big(\sqrt{w}\big)^{\varepsilon(\alpha)} \operatorname{tr}(\gamma(\alpha)), \tag{64}$$

where $w = \frac{z+(1-u)E}{uz}$, γ the natural epimorphism of the framed braid group algebra $\mathbb{C}(u)\mathcal{F}_n$ onto the algebra $\mathrm{Y}_{d,n}(u)$, and $\alpha \in \bigcup_\infty \mathcal{F}_n$. Note that if the input braids α have all framings zero, then $\Gamma_{d,s}(w, u)$ restrict to invariants of classical knots and links, denoted by $\Delta_{d,s}(w, u)$. In [2] it is shown that for generic values of the parameters u, z the invariants $\Delta_{d,S}(w, u)$ do not coincide with the Homflypt polynomial except in the trivial cases $u = 1$ or $E = 1$. More precisely, for $E = 1$ an algebra homomorphism can be defined, $h : \mathrm{Y}_{d,n}(u) \longrightarrow \mathrm{H}_n(u)$, and the composition $\tau \circ h$ is a Markov trace on $\mathrm{Y}_{d,n}(u)$ which takes the same values as the specialized trace tr, whereby the x_i's are specialized to the d-th roots of unity. For details see [2, §3]. Yet, as computational data [5] indicate, they may still be topologically equivalent to the Homflypt polynomial.

Recalling now the conditions of Theorem 6 for the trace tr to pass to the quotient $\mathrm{YTL}_{d,n}(u)$, we note that in both cases the x_i's are solutions of the E-system, as required by [19], in order to proceed with defining link invariants. We do not take into consideration case (i) for $z = -1$ and case (ii), where $z = -\frac{1}{2}$, since crucial braiding information is lost and therefore they are of no topological interest. For example, the trace tr for these two values of z gives the same result for all even (resp. odd) powers of the g_i's, as it becomes clear from the following formulas from [19], for $m \in \mathbb{Z}^{>0}$:

$$\mathrm{tr}(g_i^m) = \Big(\frac{u^m - 1}{u+1}\Big) z + \Big(\frac{u^m - 1}{u+1}\Big) E + 1 \qquad \text{if } m \text{ is even}$$

and

$$\mathrm{tr}(g_i^m) = \left(\frac{u^m - 1}{u+1}\right) z + \left(\frac{u^m - 1}{u+1}\right) E - E \qquad \text{if } m \text{ is odd,}$$

since, for $z = -1$ and $z = -\frac{1}{2}$ we find from (47) $E = 1$ and $E = \frac{1}{2}$ respectively. The only remaining case of interest is case (i) of Theorem 6, where the x_ℓ's are the d-th roots of unity and $z = -\frac{1}{u+1}$. This implies that $E = 1$ and $w = u$ in (64). So, by [2] and [11], the invariant $\Delta_{d,s}(u, u)$ coincides with the Jones polynomial.

References

[1] R. Biagioli, F. Jouhet, P. Nadeau, *Fully commutative elements in finite and affine Coxeter groups*, arXiv:1402/2166 [math.CO].

[2] M. Chlouveraki, S. Lambropoulou, *The Yokonuma–Hecke algebras and the Homflypt polynomial*, J. Knot Theory Ramifications 22 (2013), 1350080, 35 pp.

[3] M. Chlouveraki, G. Pouchin, *Determination of the representations and a basis for the Yokonuma–Temperley–Lieb algebra*, Algebr. Represent. Theory, published online; see arXiv:1311.5626 [math.RT].

[4] M. Chlouveraki, L. Poulain d'Adency, *Representation theory of the Yokonuma–Hecke algebra*, Adv. Math. 259 (2014), 134–172.

[5] S. Chmutov, S. Jablan, J. Juyumaya, K. Karvounis, S. Lambropoulou, *On the knot invariants from the Yokonuma–Hecke algebras*, in preparation; see http://www.math.ntua.gr/˜sofia/yokonuma/index.html.

[6] C. K. Fan, *A Hecke algebra quotient and properties of commutative elements of a Weyl group*, Ph.D. thesis, M.I.T., 1995.

[7] P. Freyd, D. Yetter, J. Hoste, W. B. R. Lickorish, K. Millett, A. Ocneanu, *A new polynomial invariant of knots and links*, Bull. Amer. Math. Soc. 12 (1985), 239–246.

[8] F. M. Goodman, P. de la Harpe, V. F. R. Jones, *Coxeter Graphs and Towers of Algebras*, Springer, New York 1989.

[9] D. Goundaroulis, J. Juyumaya, A. Kontogeorgis, S. Lambropoulou, *The Yokonuma–Temperley–Lieb Algebra*, arXiv: 1012.1557v4.

[10] J. Graham, *Modular Representations of Hecke Algebras and Related Algebras*, Ph.D. thesis, University of Sydney, 1995.

[11] V. F. R. Jones, *Hecke algebra representations of braid groups and link polynomials*, Ann. of Math. (2) 126 (1987), 335–388.

[12] V. F. R. Jones, *Index for subfactors*, Invent. Math. 72 (1983), 1–25.

[13] J. Juyumaya, *Sur les nouveaux générateurs de l'algèbre de Hecke* $\mathcal{H}(G, U, 1)$, J. Algebra 204 (1998), 49–68.

[14] J. Juyumaya, *Markov trace on the Yokonuma–Hecke algebra*, J. Knot Theory Ramifications 13 (2004), 25–39.

[15] J. Juyumaya, *A partition Temperley–Lieb algebra*, arXiv: 1304.5158.

[16] J. Juyumaya, S. Lambropoulou, *p-adic framed braids*, Topology Appl. 154 (2007), 1804–1826.

[17] J. Juyumaya, S. Lambropoulou, *An invariant for singular knots*, J. Knot Theory Ramifications 18 (2009), 825–840.

[18] J. Juyumaya, S. Lambropoulou, *An adelic extension of the Jones polynomial*, in: The Mathematics of Knots, Theory and Application, Contrib. Math. Comput. Sci. 1, Springer, Heidelberg 2010, 125–142.

[19] J. Juyumaya, S. Lambropoulou, *p-adic framed braids* II, Adv. Math. 234 (2013), 149–191.

[20] H. Matsumoto, *Générateurs et relations des groupes de Weyl généralisés*, C. R. Acad. Sci. Paris 258 (1964), 3419–3422.

[21] S. Ramanujan, *On certain trigonometric sums and their applications in the theory of numbers*, Transactions of the Cambridge Philosophical Society 22, No. 13 (1918), 259–276.

[22] J. R. Stembridge, *On the fully commutative elements of Coxeter groups*, J. Algebraic Combin. 5 (1996), 353–385.

[23] A. Terras, *Fourier Analysis of Finite Groups and Applications*, London Math. Soc. Stud. Texts 43, Cambridge Univ. Press, Cambridge 1999.

[24] N. Thiem, *Unipotent Hecke algebras of* $\mathbf{GL}_n(\mathbb{F}_q)$, J. Algebra 284 (2005), 559–577.

[25] T. Yokonuma, *Sur la structure des anneux de Hecke d'un groupe de Chevalley fini*, C. R. Acad. Sc. Paris Sér. A-B 264 (1967), A344–A347.