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LOCAL MOVES ON KNOTS AND PRODUCTS OF KNOTS

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Abstract. We show a relation between products of knots, which are generalized from the theory of isolated singularities of complex hypersurfaces, and local moves on knots in all dimensions. We discuss the following problem. Let K be a 1-knot which is obtained from another 1-knot J by a single crossing change (resp. pass-move). For a given knot A, what kind of relation do the products of knots, $K \otimes A$ and $J \otimes A$, have? We characterize these kinds of relation between $K \otimes A$ and $J \otimes A$ by using local moves on high dimensional knots. We also discuss a connection between local moves and knot invariants. We show a new form of identities for knot polynomials associated with a local move.

Contents

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1. Introduction. Let $f : C^n \longrightarrow C$ be a (complex) polynomial mapping with an isolated singularity at the origin of C^n. That is, $f(0) = 0$ and the complex gradient df has an isolated zero at the origin. The *link* of this singularity is defined by the formula $L(f) = V(f) \cap S^{2n-1}$. Here the symbol $V(f)$ denotes the variety of f, and S^{2n-1} is a sufficiently small sphere about the origin of C^n.

Given another polynomial $g : C^m \longrightarrow C$, form $f + g$ with domain $C^{n+m} = C^n \times C^m$ and consider $L(f + g) \subset S^{2n+2m+1}$.

We use a topological construction for $L(f + g) \subset S^{2n+2m+1}$ in terms of $L(f) \subset S^{2n+1}$ and $L(g) \subset S^{2m+1}$. The construction generalizes the algebraic situation. Given nice (to be specified below) codimension-two embeddings $K \subset S^n$ and $L \subset S^m$, we form a product $K \otimes L \subset S^{n+m+1}$. Then $L(f) \otimes L(-g) \cong L(f + g)$.

We will recall and use in this paper a product operation on knots in all dimensions that generalizes this result about singularities [8, 9, 10]. We will also associate geometric equivalence relations, *crossing changes* and *pass equivalence* [9] of classical knots, with local moves on high dimensional knots and links, which were defined and have been researched in [20, 21, 22, 23, 24, 25, 26], and relate this to the knot product construction and to the Arf invariant, the signature, and knot polynomials in higher dimensions. Knot products allow us to consider low dimensional knots and high dimensional knots together (see Section 2.5 and Note 8.13.(2)).

Furthermore we show a new form of identities for knot polynomials associated with a local move: The form is

$$\Delta(K_+) - \Delta(K_-) = \begin{cases} (t-1) \cdot \Delta(K_0) & \text{for some pairs } (n, l) \\ (t+1) \cdot \Delta(K_0) & \text{for the other pairs } (n, l), \end{cases}$$

where $\Delta(\quad)$ is an l-Alexander polynomial of an n-dimensional oriented closed submanifold $\subset S^{n+2}$ (see Theorem 9.2 for detail).

Note 1.1. Local moves on high dimensional knots which we discuss are twist-moves and high dimensional pass-moves. We will show examples of twist-moves after Theorem 7.1

in Section 7, after Theorem 9.1 in Section 9, and after Theorem 9.2 in Section 9. We will show examples of pass-moves on high dimensional knots in Section 3.1.

We will review knot products and local moves on classical knots and on high dimensional knots before we state our main results in Section 4.

2. Products of knots

2.1. Construction of products. In this subsection and the next, we describe the results in references [8, 10, 9]. All manifolds will be smooth. Each ambient sphere S^n comes equipped with an orientation. A *knot* in S^n is any closed oriented codimension-two submanifold K. Given a knot $K \subset S^n$ we may write $S^n = E_K \cup (K \times D^2)$ where E_K is a manifold with boundary equal to $K \times S^1$. If n is larger than 3, we assume that K is connected. Thus, by Alexander duality, $H^1(E_K) \cong \mathbb{Z}$. One may choose $\phi : E_K \longrightarrow S^1$ representing the generator of $H^1(E_K)$ so that ϕ is differentiable and $\phi|\partial E_K$ is a projection on the second factor. If $n = 3$, then K may consist of a collection of disjointly embedded circles. A choice of orientations for these circles determines ϕ so that ϕ^{-1} applied to a regular value is an oriented spanning surface for K which induces the chosen orientations on each component.

A knot K is said to be *spherical* if it is PL homeomorphic to the standard sphere S^n.[1] A knot is said to be *fibered* if there is a choice of ϕ as above so that $\phi : E_K \longrightarrow S^1$ is a locally trivial smooth fibration.

Now suppose that we are given knots $K \subset S^n$ and $L \subset S^m$ and corresponding maps $\phi : E_K \longrightarrow S^1$ and $\psi : E_L \longrightarrow S^1$. If one knot is fibered, then

$$E_K \times_{S^1} E_L = \{(x, y) \in E_K \times E_L \,|\, \phi(x) = \psi(y)\}$$

is a well-defined smooth manifold with boundary. Henceforth, when dealing with a pair of knots, we shall assume that at least one knot is fibered. We now define a manifold $K \otimes L$ and, using its properties, obtain the product knot $K \otimes L \subset S^{n+m+1}$.

DEFINITION. Given knots K and L as above, we define the manifold

$$K \otimes L = (K \times D^{m+1}) \cup (E_K \times_{S^1} E_L) \cup (D^{n+1} \times L).$$

These three pieces are attached according to the following description: Note that

$$\partial(E_K \times_{S^1} E_L) = (K \times E_L) \cup (E_K \times L)$$

and

$$\partial(K \times D^{m+1}) = (K \times D^2 \times L) \cup (K \times E_L),$$

$$\partial(D^{n+1} \times L) = (K \times D^2 \times L) \cup (E_K \times L).$$

Using these boundary identifications, glue the three pieces together to form a closed manifold. The manifold $K \otimes L$ is independent of the choices of maps ϕ and ψ used in its construction.

Now given $\phi : E_L \longrightarrow S^1$, there is an embedding $\hat{\phi} : E_L \longrightarrow D^{m+1} \times S^1$ given essentially by $\hat{\phi}(x) = (x, \phi(x))$. This induces an embedding $K \otimes L \subset K \otimes S^m \cong S^{n+m+1}$.

[1] Let K be homeomorphic to S^n. Then K is diffeomorphic to S^n if $n \leqq 3$ and K is PL homeomorphic to S^n if $n \geqq 5$.

This embedding is well-defined up to ambient isotopy and commutative in K and L. In this way, we obtain a differential topological generalization of the link of the sum of two isolated singularities. In the next section we will make clear how this generalizes the links of singularities.

2.2. The pullback description for knot products. In the previous section we gave a description of the knot product construction in terms of the map $\phi : E_K \longrightarrow S^1$ to the circle associated with the complement of the tubular neighborhood of a knot. In the case of fibered knots this map is a fibration over the circle. For an arbitrary knot we will call ϕ the *classifying map* for the knot $K \subset S^n$. In this section we use the classifying maps to construct maps of balls to the 2-disk that can participate in a pull-back construction for the knot product.

Given any map $f : S^n \longrightarrow D^2$, we can extend it to a map, the *cone* on f,

$$cf : D^{n+1} = CS^n = \{tu \,|\, 0 \leq t \leq 1, u \in S^n\} \longrightarrow D^2$$

defined by the formula $cf(tu) = tf(u)$ where $0 \leq t \leq 1$ and u is a unit vector in R^{n+1}.

Let $\phi : E_K \longrightarrow S^1$ be a classifying map for the knot $K \subset S^n$. Extend ϕ to a map $\phi_1 : S^n = E_K \cup (K \times D^2) \longrightarrow D^2$ by defining it on $K \times D^2$ to be the cartesian projection to D^2. Now extend ϕ_1 to the cone and call this map $\hat{\phi}_K$, *the cone map for* K.

$$\hat{\phi}_K = c\phi_1 : D^{n+1} \longrightarrow D^2.$$

The point about the construction of the cone map for a given knot $K \subset S^n$ is that it produces a differential topological analog of an algebraic singularity whose link is this knot. In particular, we have $\hat{\phi}_K^{-1}(0) = CK \subset D^{n+1} = CS^n$, and this mimics the topology of an isolated singularity. See Fig. 2.2.1.

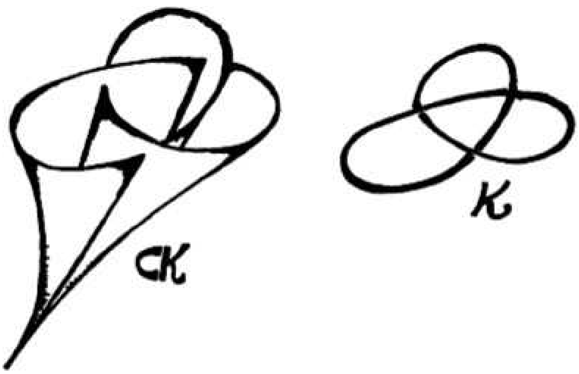


Fig. 2.2.1. Cone on K

Let $\phi : E_K \longrightarrow S^1$ and $\psi : E_L \longrightarrow S^1$ be classifying maps for the knots $K \subset S^n$ and $L \subset S^m$. Assume that ψ is a fibration over the circle, giving a fibered structure for $L \subset S^m$. Let $X[K, L] \subset D^n \times D^m$ be the pull-back as shown below.

$$\begin{array}{ccc} X[K,L] & \xrightarrow{j} & D^{m+1} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \hat{\psi}_L} \\ D^{n+1} & \xrightarrow{\hat{\phi}_K} & D^2 \end{array}$$

The pull-back $X[K,L]$ is the following subset of $D^{n+1} \times D^{m+1}$:

$$X[K,L] = \{(x,y) \in D^{n+1} \times D^{m+1} \,|\, \hat{\phi}_K(x) - \hat{\psi}_L(y) = 0\}.$$

Thus $X[K,L]$ is the differential topological analog of the variety of the sum or difference of two polynomials. Just as with a variety with an isolated singularity, $X[K,L]$ has a singularity at the origin, but the boundary

$$\partial X[K,L] \subset \partial(D^{n+1} \times D^{m+1}) \cong S^{n+m+1}$$

is a smooth submanifold of the $n+m+1$-sphere and this embedding

$$\partial X[K,L] \subset S^{n+m+1}$$

is the same as the knot product defined in the previous section. That is, we have

$$\partial X[K,L] \cong K \otimes L$$

and the embeddings are equivalent. It is by way of this pull-back construction that one can prove that indeed the knot product does generalize the link of the sum of isolated complex hypersurface singularities.

The simplest example of the pull-back construction is given by the following diagram.

$$\begin{tikzcd} X[a,b] \arrow[r, "j"] \arrow[d, "i"'] & D^2 \arrow[d, "{[b]}"'] \\ D^2 \arrow[r, "{[a]}"] & D^2 \end{tikzcd}$$

In this diagram we have indicated the knot product construction in its lowest dimensional case. The maps on the disks are of the form $[n] : D^2 \longrightarrow D^2$ where $[n](z) = z^n$, n is a natural number and z is a complex variable. We take D^2 as the unit disk in the complex plane. Then the maps on spheres in this case are maps of degree a and degree b from circles to themselves. The individual knots are empty and the spanning manifolds consist in a and b points respectively. We refer to $[a]$ and $[b]$ (regarding the restriction to the circles as defining the maps) as the *empty knots of degree a and degree b*. We see that

$$\partial(X[a,b]) = [a] \otimes [b] \subset S^3$$

is the corresponding knot product and it is easy to see that $[a] \otimes [b]$ *is a torus link of type* (a,b). Continuing in this vein one discovers that the Brieskorn manifolds [9] the links of singularities

$$\Sigma(a_1, \ldots, a_n) = L(z_1^{a_1} + \ldots + z_n^{a_n}) \subset S^{2n-1},$$

are given by the formula

$$\Sigma(a_1, \ldots, a_n) = [a_1] \otimes \ldots \otimes [a_n] \subset S^{2n-1}.$$

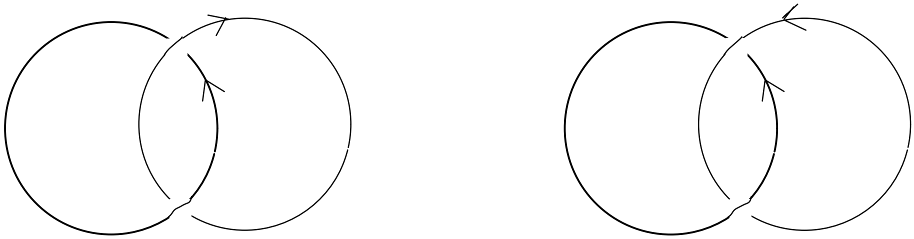
in other words the Brieskorn manifolds and their embeddings in spheres are constructed as products of empty knots of chosen degrees. This completes our description of the elements of the knot product construction.

2.3. The empty knots and the Hopf link. Let $A \otimes^{\mu} B$ mean $A \otimes B \otimes \ldots \otimes B$, which is composed of one copy of A and μ copies of B, where $\mu \in \mathbb{N} \cup \{0\}$. Let $\otimes^{\mu} B$ mean $B \otimes \ldots \otimes B$, which is composed of μ copies of B, where $\mu \in \mathbb{N} \cup \{0\}$.

DEFINITION 2.3.1 ([8, 10]). Let $n \in \mathbb{N}$. The *empty knot* $[n]$ is the smooth map $S^1 \to S^1$ such that $\theta \mapsto n\theta$, where $S^1 = \{e^{2\pi i\theta} \,|\, \theta \in \mathbb{R}\}$.

We regard a Seifert hypersurface of the empty knot $[n]$ as a set of n points $\subset S^1$. We can regard the empty knot $[n]$ as a fibred knot. In [8, 10] is defined a knot product of the empty knot and an n-dimensional closed oriented submanifold $\subset S^{n+2}$.

The *positive Hopf link* or the *linking number* $+1$ *Hopf link* is as shown below on the left. The *negative Hopf link* or the *linking number* -1 *Hopf link* is as shown below on the right. In this paper the Hopf link means the negative Hopf link.



THEOREM 2.3.2 ([8, 10]). *Let $[n]$ denote the empty knot of degree n. Then we have*

$$[2] \otimes [2] = \textit{the negative Hopf link.}$$

For $\mu \in \mathbb{N}$, we have

$$\otimes^{2\mu}[2] = \otimes^{\mu} \quad (\textit{the negative Hopf link}).$$

For any n-dimensional closed oriented submanifold $K \subset S^{n+2}$,

$$K \otimes^{2\mu} [2] = K \otimes^{\mu} \quad (\textit{the negative Hopf link}).$$

NOTE. See line -12 of page 389 and line 18 of page 391 of [10].

2.4. Passing bands in low and high dimensions. In three dimensions a *bandpass* is a replacement of one band crossing over another band by that band crossing underneath the other band. See the following figure for an illustration.

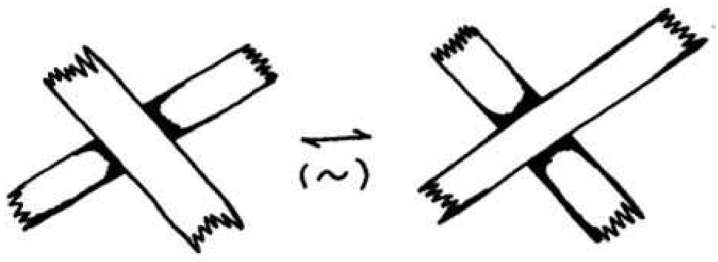


Fig. 2.4.1. Band pass

We usually assume that the bands are part of oriented surfaces spanning a link. This means that the local orientation on the two edges of each band are in opposite directions. We say that two oriented knots or links are *pass equivalent* if one can be obtained from another by a sequence of ambient isotopies and band passes. It is not necessary to construct spanning surfaces for the links in order to perform the band passes, since this is a local operation on the diagrams. The surface interpretation is useful for proving facts about pass equivalence. One can show that any classical knot is pass equivalent to either the unknot or the trefoil knot. One can also show that two classical knots are pass equivalent if and only if they have the same Arf invariant. See Theorem 2.4.1 and [9] for more information on this topic.

In this paper we will relate crossing changes and pass equivalence to local moves on higher dimensional knots and links and interrelate them with the knot product. Furthermore we show a connection between the local moves and invariants and polynomials of high dimensional knots.

We review the following theorem.

THEOREM 2.4.1 ([9, 19]). *Let* $L_1 = (K_{1,1}, \ldots, K_{1,l})$ *and* $L_2 = (K_{2,1}, \ldots, K_{2,l})$ *be* l*-component* 1*-links* ($l \in \mathbb{N}$)*. Then* L_1 *and* L_2 *are pass-move-equivalent if and only if* L_1 *and* L_2 *satisfy one of the following conditions* (1) *and* (2)*.*

(1) *Both* L_1 *and* L_2 *are proper links, and*

$$\mathrm{Arf}(L_1) = \mathrm{Arf}(L_2).$$

(2) *Neither* L_1 *nor* L_2 *is a proper link, and*

$$\mathrm{lk}(K_{1j}, L_1 - K_{1j}) \equiv \mathrm{lk}(K_{2j}, L_2 - K_{2j}) \bmod 2 \quad \text{for all } j.$$

In [20] a result is shown on a relation between high dimensional pass-moves and knot invariants.

We end this subsection with an example in Fig. 2.4.2. This example is given in more detail in [9] but here we can point to our results in this paper that make the low dimensional band-passing that we are about to discuss, actual high dimensional band-passing that accomplishes these results in high dimensional manifolds. The result is an 8-fold periodicity in the list of Brieskorn manifolds $\Sigma(k, 2, 2, 2, \ldots, 2)$ where there are an odd number of 2's. Let Σ_k^{4n+1} denote such a Brieskorn manifold with $2n+1$ symbols that are 2's. Then Σ_k^{4n+1} bounds a handle-body whose structure is analogous to the spanning surface for a $(2, k)$ torus link, and the operation of band-exchange results in a diffeomorphism of this handle-body, hence a diffeomorphism of its boundary. See Fig. 2.4.2 for an illustration of the $(2, k)$ torus links, here called K_k and the banded surfaces that bound these links. In [9] we exploit this relationship with the low dimensional topology to prove by band-passing that K_{k+8} is pass-equivalent to K_k, and so prove, up to diffeomorphism, that the list of manifolds Σ_k^{4n+1} is periodic of period 8 in k. By applying the results of this paper, we can make this conclusion directly by using the higher dimensional versions of pass-moves. (Outline of the proof: A $(4n+1)$-submanifold $\mathcal{K}_k = K_k \otimes^n$ (the Hopf link) in S^{4n+3} is diffeomorphic to Σ_k^{4n+1}. $\mathcal{K}_k$ is high dimensional pass-move-equivalent to $\mathcal{J}$ with the following properties: A Seifert matrix associated with a Seifert hypersurface $V_{\mathcal{J}}$ for $\mathcal{J}$ is the same as a Seifert matrix associated with a Seifert hypersurface $V_{\mathcal{K}_{k+8}}$ for $\mathcal{K}_{k+8}$. $V_{\mathcal{J}}$ and $V_{\mathcal{K}_{k+8}}$ consist of a $(4n+2)$-dimensional 0-handle and $(4n+2)$-dimensional $(2n+1)$-handles. Of course $V_{\mathcal{J}}$ and $V_{\mathcal{K}_{k+8}}$ are compact oriented parallelizable and have the same intersection matrix on the $(2n+1)$-th homology groups. Therefore, by surgery theory, Σ_k^{4n+1} is diffeomorphic to Σ_{k+8}^{4n+1}.)

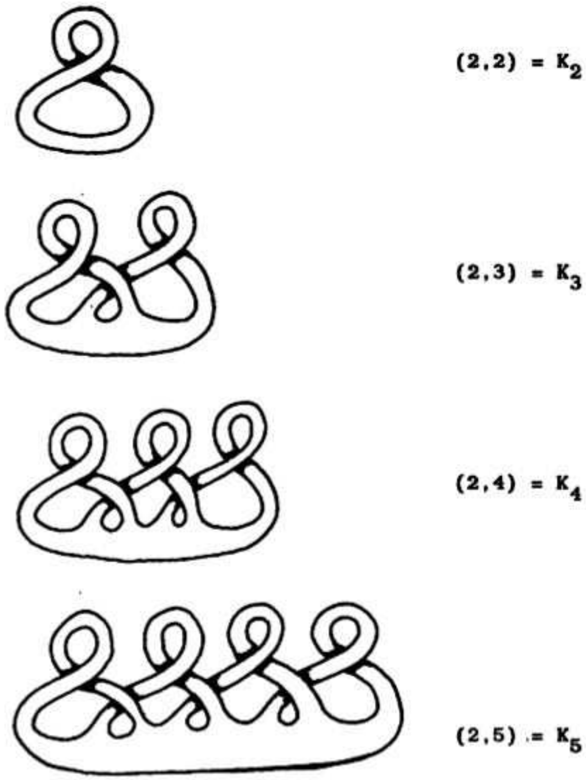


Fig. 2.4.2. The $(2,k)$ torus knots in band representation

The details of this band exchange, illustrated in three-dimensions are interesting, and we refer the reader to [9] for more about this aspect of the example. We could investigate $\Sigma(a,b,2,2,2,\ldots,2)$, where there are an even number of 2's, by using high dimensional pass-moves because the (a,b) torus knots are classified by pass-equivalence. Recall Note 1.1.

2.5. The main problem. It is natural to consider the following problem: Let K be a 1-knot which is obtained from another 1-knot J by a single crossing change (resp. pass-move). For a given knot A, what kind of relation do $K\otimes A$ and $J\otimes A$ have? In this paper we characterize these kinds of relation between $K\otimes A$ and $J\otimes A$ by using local moves on high dimensional knots.

By considering this problem of the effect on higher dimensional knots of changes from lower dimensions, via knot products, we raise many questions that deserve further investigations.

3. Local moves on high dimensional knots

3.1. Examples. Local moves on high dimensional knots were defined and have been researched in [20, 21, 22, 23, 24, 25, 26]. We review the definition of local moves on high dimensional knots after showing an example. Recall Note 1.1.

LEMMA. *Letting B^p denote a p-dimensional ball, we can write*

$$S^p = B^p_u \cup B^p_d$$
$$S^p \times S^q = (B^p_u \cup B^p_d) \times (B^q_u \cup B^q_d).$$

Thus

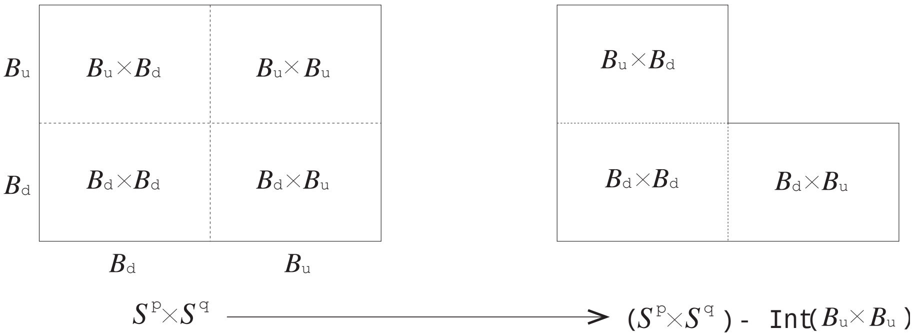
$$S^p \times S^q = (B^p_u \times B^q_u) \cup (B^p_u \times B^q_d) \cup (B^p_d \times B^q_u) \cup (B^p_d \times B^q_d).$$

Proof. Use the fact $(X \cup Y) \times Z = (X \times Z) \cup (Y \times Z)$. ∎

Now let

$$F = (S^p \times S^q) - \mathrm{Int}(B^p_u \times B^q_u).$$

We indicate F in the figure below and abbreviate $B^\sharp_\star$ to $B_\star$.



F is drawn in another way as below. Note that we can bend the corner of $B^p_u \times B^q_u$ and change it into the $(p+q)$-dimensional ball. Let $p + q = n + 1$. Hence the boundary of F is S^n.

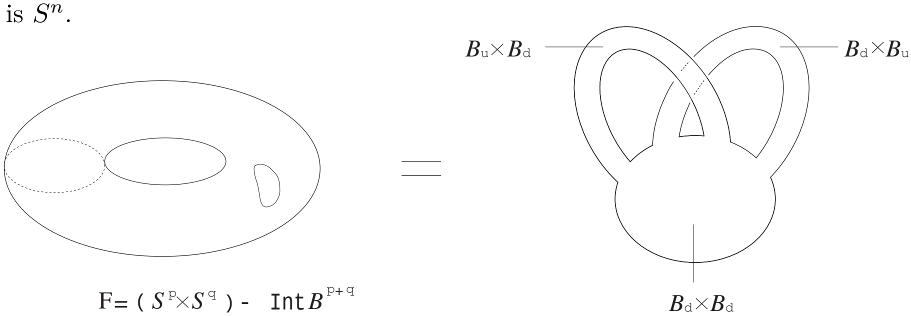


Fig. 3.1.1. $(S^p \times S^q) - \mathrm{Int}\, B^{p+q}$

We can regard $B^p_d \times B^q_d$ as a $(p+q)$-dimensional 0-handle, $B^p_u \times B^q_d$ as a $(p+q)$-dimensional p-handle, and $B^p_d \times B^q_u$ as a $(p+q)$-dimensional q-handle.

Let $F \subset S^p \times S^q \subset S^{n+2}$. This is indicated in Fig. 3.1.2 below. The boundary of F in S^{n+2} is an n-knot. Furthermore it is the trivial n-knot.

Carry out a 'local move' on this n-knot in an $(n+2)$-ball, which is denoted by a dotted circle in Fig. 3.1.3.

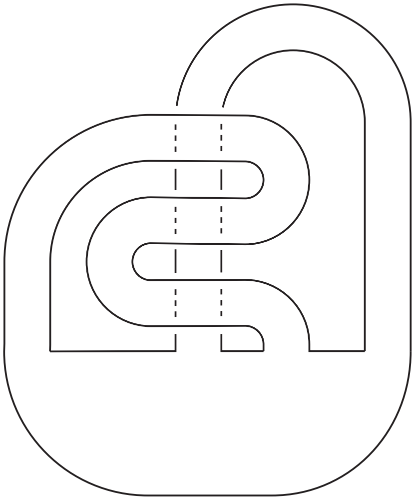


Fig. 3.1.2. A trivial n-knot

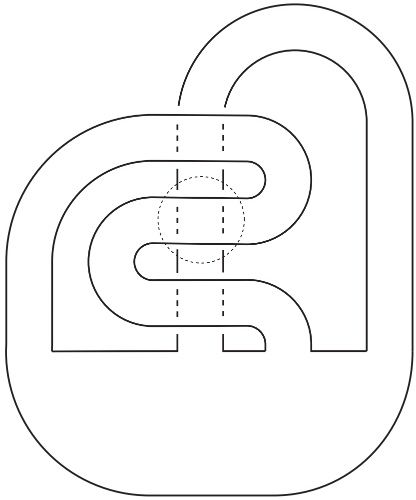


Fig. 3.1.3. A local move will be carried out in the dotted $(n+2)$-ball. The resulting n-knot is a nontrivial n-knot

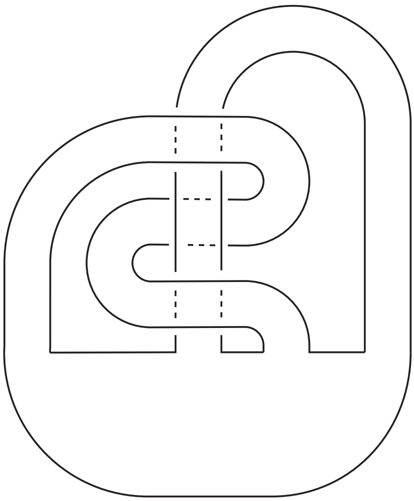


Fig. 3.1.4. A nontrivial n-knot

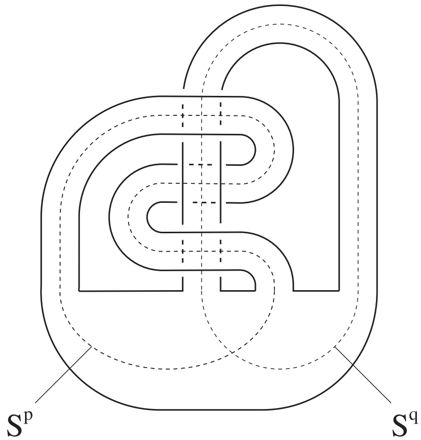


Fig. 3.1.5. S^p and S^q in F whose boundary is the n-knot

We can prove that the knot in Fig. 3.1.4 is nontrivial by using Seifert matrices and the Alexander polynomial. We use the fact that S^p and S^q can be 'linked' in S^{p+q+1}. Recall that $p+q+1=n+2$. Note that S^q and S^p are included in F as shown in Fig. 3.1.5.

Note that the above operation is done only in an $(n+2)$-ball. This operation is an example of (p,q)-pass-moves.

Local moves on high dimensional submanifolds are exciting ways of explicit construction of high dimensional figures. They are also generalization of local moves on 1-links. They are useful to research link cobordism, knot cobordism, and the intersection of submanifolds (see [20]) etc. There remain many exciting problems. Some of them are proper in high dimension and others are analogous to one-dimensional case. For example, we do not know a local move on high dimensional knots which is an unknotting operation.

3.2. (p,q)-pass-moves. We review (p,q)-pass-moves on n-knots ($p,q\in\mathbb{N}$, $p+q=n+1$) on high dimensional knots. [20, 22, 24] defined them. See also [23, 25, 26]. Confirm that, if $(p,q)=(1,1)$, (p,q)-pass-moves are pass-moves on 1-links.

We first define (p,q)-pass-moves on n-knots ($p,q\in\mathbb{N}$, $p+q=n+1$). Let K_+, K_-, K_0 be n-dimensional closed oriented submanifolds $\subset S^{n+2}$ ($n\in\mathbb{N}$). Let B be an $(n+2)$-ball trivially embedded in S^{n+2}. Suppose that K_+ coincides with K_-, K_0 in $\overline{S^{n+2}-B}$.

Take an $(n+1)$-dimensional p-handle h^p_* ($*=+,-$) and an $(n+1)$-dimensional $(n+1-p)$-handle h^{n+1-p} in B with the following properties.

(1) $h^p_*\cap\partial B$ is the attaching part of h^p_*, $h^{n+1-p}\cap\partial B$ is the attaching part of h^{n+1-p}.

(2) h^p_* (resp. h^{n+1-p}) is embedded trivially in B.

(3) $h^p_*\cap h^{n+1-p}=\emptyset$.

(4) The attaching part of h^p_+ coincides with that of h^p_-. The linking number (in B) of

$$[h^p_+\cup(-h^p_-)]\quad\text{and}\quad[h^{n+1-p}\text{ whose attaching part is fixed in }\partial B]$$

is one if an orientation is given.

Let K_* ($*=+,-$) satisfy $K_*\cap\operatorname{Int}B=(\partial h^p_*-\partial B)\cup(\partial h^{n+1-p}-\partial B)$. Note the following. When we define K_+, h_+ exists in B and h_- does not exist in B. When we define K_-, h_- exists in B and h_+ does not exist in B.

Let

$$P=K_+\cap(S^{n+2}-\operatorname{Int}B),\qquad Q=h^p_+\cap\partial B,\qquad R=h^{n+1-p}\cap\partial B,$$
$$T=P\cup Q\cup R.$$

Then T is an n-dimensional oriented closed submanifold $\subset(S^{n+2}-\operatorname{Int}B)\subset S^{n+2}$. Let K_0 be $T\subset S^{n+2}$. Then we say that $(K_+,\ K_-,\ K_0)$ is related by a single $(p,n+1-p)$-*pass-move* in B. We also say that $(K_+,\ K_-,\ K_0)$ is a $(p,n+1-p)$-*pass-move-triple*. We say that K_+ and K_- differ by a single $(p,n+1-p)$-*pass-move* in B. We showed examples of pass-moves on high dimensional knots in Section 3.1.

If (K_+,K_-,K_0) is a $(p,n+1-p)$-pass-move-triple, then we also say that (K_-,K_+,K_0) is a $(p,n+1-p)$-*pass-move-triple*. If K_+ and K_- differ by a single $(p,n+1-p)$-pass-move in B, then we also say that K_- and K_+ differ by a single $(p,n+1-p)$-*pass-move* in B.

Let (K_+,K_-,K_0) be related by a single $(p,n+1-p)$-pass-move in B. Then there is a Seifert hypersurface V_* for K_* ($*=+,-,0$) with the following properties.

$$V_\sharp=V_0\cup h^p_\sharp\cup h^{n+1-p}(\sharp=+,-).\tag{1}$$
$$V_\sharp\cap B=h^p_\sharp\cup h^{n+1-p}.$$
$$V_0\cap\operatorname{Int}B=\emptyset.\tag{2}$$
$$V_0\cap\partial B\text{ is the attaching part of }h^p_\sharp\cup h^{n+1-p}.$$

(The idea of the proof is the Thom–Pontrjagin construction.)

Then the ordered set (V_+, V_-, V_0) is called a $(p, n+1-p)$*-pass-move-triple of Seifert hypersurfaces* for (K_+, K_-, K_0). We say that an ordered set (V_+, V_-, V_0) is related by a single $(p, n+1-p)$*-pass-move* in B. We say that V_- (resp. V_+) is obtained from V_+ (resp. V_-) by a single $(p, n+1-p)$*-pass-move* in B.

Fig. 3.2.1 is a diagram of a (p, q)-pass-move.

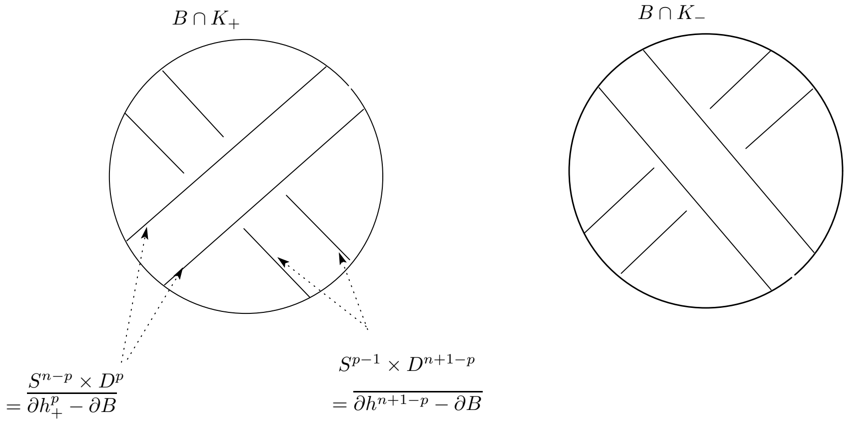
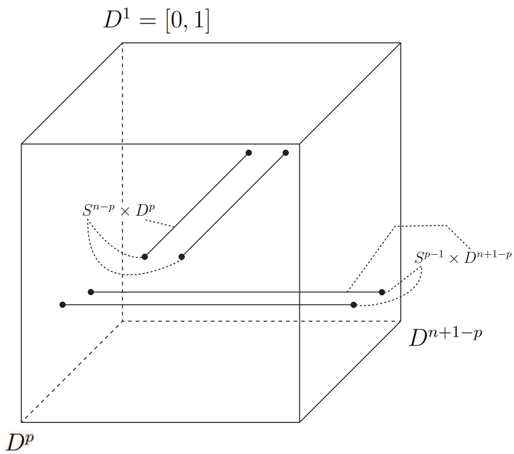


Fig. 3.2.1. A $(p, n+1-p)$-pass-move on an n-dimensional submanifold $\subset S^{n+2}$

Note $B = B^{n+2} = D^{n+2} \subset S^{n+2}$. The left (resp. right) figure includes h_+^p (resp. h_-^p) and h^{n+1-p}.

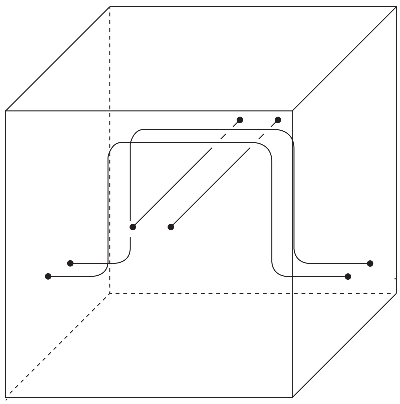
Note. When we construct K_- and K_0 from K_+, we make a change only in B and we do not impose any requirement on diffeomorphism type or homeomorphism type of K_-, K_0 other than the change only in B. In this sense, we use the word 'local' in the above definition.



This cube is $B = D^{n+2} = D^1 \times D^p \times D^{n+1-p}$

$B \cap K_+$

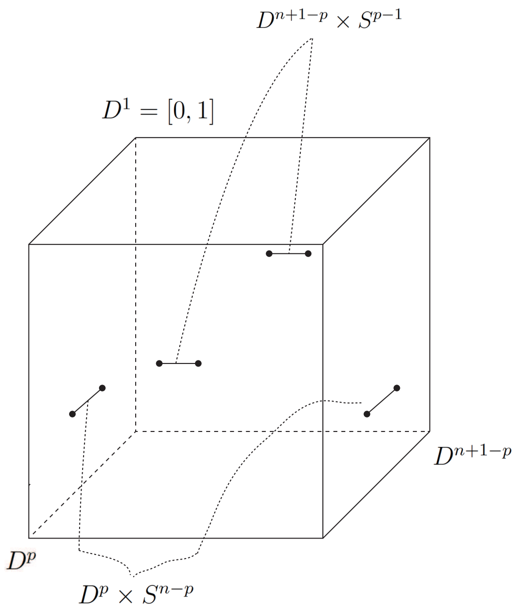
Fig. 3.2.2. (1) A $(p, n+1-p)$-pass-move-triple



$B \cap K_-$

Fig. 3.2.2.

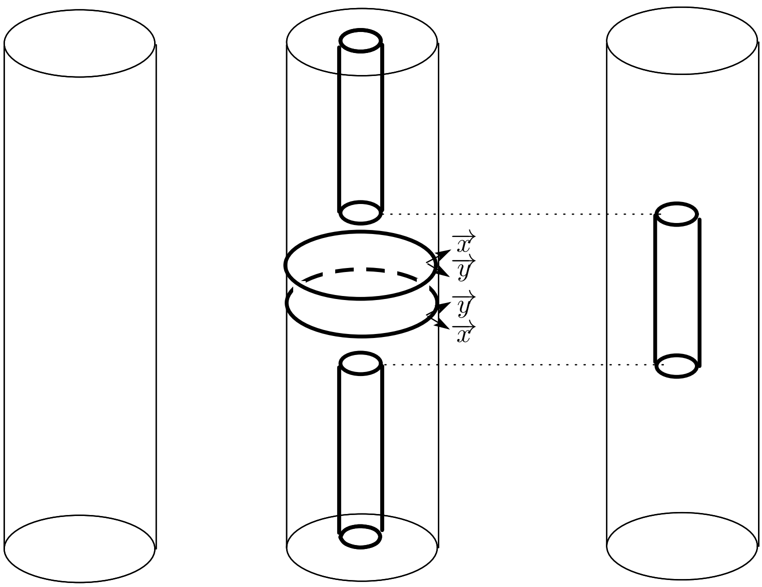
(2) A $(p, n+1-p)$-pass-move-triple



$B \cap K_0$

Fig. 3.2.2. (3) A $(p, n+1-p)$-pass-move-triple

Fig. 3.2.2, which consists of the three figures (1), (2) and (3), is a diagram of a (p, q)-pass-move-triple.



$t = -0.5$ $\quad$ $t = 0$ $\quad$ $t = 0.5$

K_+

Fig. 3.2.3. (1) A (1,2)-pass-move-triple

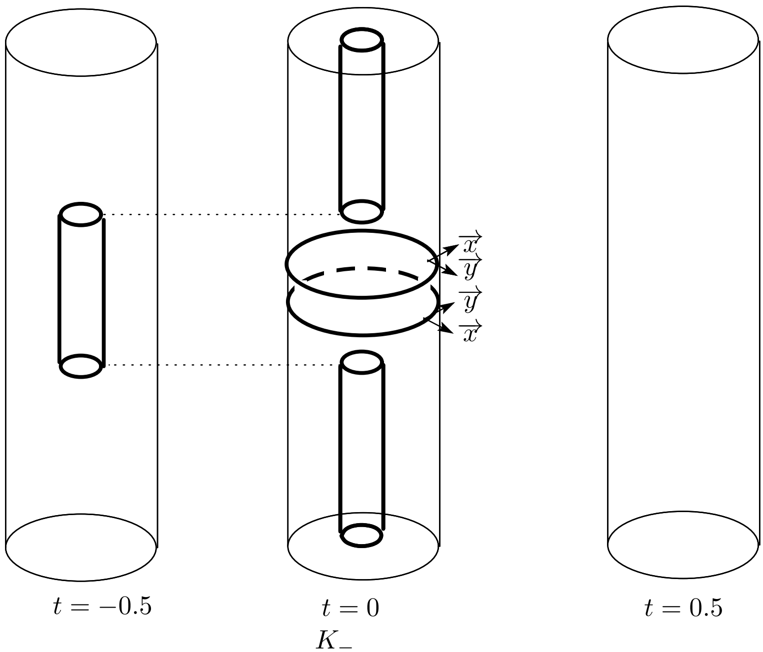


Fig. 3.2.3. (2) A (1,2)-pass-move-triple

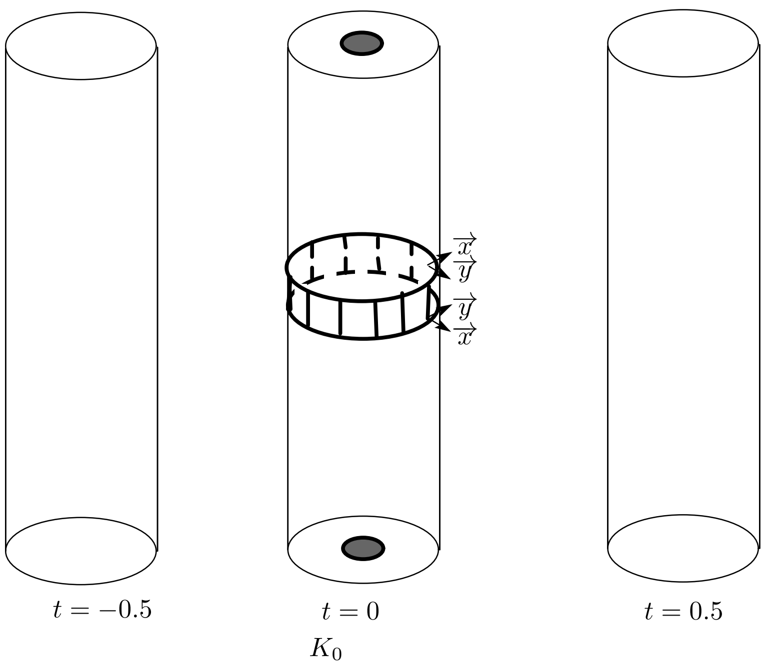


Fig. 3.2.3. (3) A (1,2)-pass-move-triple

In Fig. 3.2.3, which consists of the three figures (1), (2) and (3), we draw a $(1,2)$-pass-move-triple (the $p = 1$ and $n = 2$ case). Since (K_+, K_-, K_0) is related by a single $(1,2)$-pass-move in B, B has the following properties. We regard B as (2-disc)$\times[0,1] \times \{t \,|\, -1 \leqq t \leqq 1\}$.

(i) $K_+ - B$, $K_- - B$, and $K_0 - B$ coincide each other.
(ii) $B \cap K_+$, $B \cap K_-$, $B \cap K_0$ are shown as above.

In the above figures we draw $B_{-0.5} \cap K_*$, $B_0 \cap K_*$, $B_{0.5} \cap K_*$, where $B_{t_0} =$ (2-disc) $\times [0,1] \times \{t \,|\, t = t_0\}$. We suppose that each vector $\overrightarrow{x}$, $\overrightarrow{y}$ in the above figures is a tangent vector of each disc at a point. (Note that we use $\overrightarrow{x}$ (resp. $\overrightarrow{y}$) for different vectors.) The orientation of each disc in the above figures is determined by the each set $\{\overrightarrow{x}, \overrightarrow{y}\}$. In [22], near Figures 4.1 and 4.2, more explanation of the structure of $B \cap K_+$ and that of $B \cap K_-$ are given.

In [22] one more local move was discussed, which is called the 'ribbon-move', and the following results were proved. Let K and K' be two-dimensional closed oriented submanifolds $\subset S^4$. The following conditions (1) and (2) are equivalent.
(1) K is (1,2)-pass-move-equivalent to K'.
(2) K is ribbon-move-equivalent to K'.

Furthermore, if K is obtained from K' by a single ribbon-move, then K is obtained from K' by a single (1,2)-pass-move.

3.3. Twist-moves. We next review twist-moves on high dimensional knots, which are defined in [24]. Let K_+, K_-, K_0 be $(2p+1)$-dimensional closed oriented submanifold $\subset S^{2p+3}$ $(p \in \mathbb{N} \cup \{0\})$. Let B be a $(2p+3)$-ball trivially embedded in S^{2p+3}. Suppose that K_+ coincides with K_-, K_0 in $\overline{S^{2p+3} - B}$. Take a single $(2p+2)$-dimensional $(p+1)$-handle h_+ (resp. h_-) embedded in B such that

$$[\text{the handle}] \cap \partial B \text{ is the attaching part of the handle.}$$

Note. [4, 5, 32, 33] etc. imply that the core of h_+ (resp. h_-) is trivially embedded in B under the above condition.

Suppose that ($h_+ -$ its attaching part) $\cap$ ($h_- -$ its attaching part) $= \emptyset$. Suppose that their attaching parts coincide. Thus we can suppose that we regard $h_+ \cup h_-$ as an oriented $(2p+2)$-submanifold $\subset S^{2p+1}$ if we give the opposite orientation to h_-. Then we can define a $(p+1)$-Seifert matrix for the $(2p+2)$-submanifold $h_+ \cup h_-$. We can suppose that the Seifert matrix is the matrix (1).

Let K_* $(* = +, -)$ satisfy $K_* \cap \operatorname{Int} B = (\partial h_* - \partial B)$. Note the following. When we define K_+, h_+ exists in B and h_- does not exist in B. When we define K_-, h_- exists in B and h_+ does not exist in B. Let $P = K_+ \cap (S^{2p+3} - \operatorname{Int} B)$. Let $Q = h_+ \cap \partial B$. Let $T = P \cup Q$. Then T is an $(2p+1)$-dimensional oriented closed submanifold in $S^{2p+3} - \operatorname{Int} B$. Let K_0 be T in S^{2p+3}. Then we say that an ordered set (K_+, K_-, K_0) is related by a single *twist-move.* (K_+, K_-, K_0) is called a *twist-move-triple.* We say that K_+ and K_- differ by a single *twist-move* in B. If (K_+, K_-, K_0) is a twist-move-triple, then we also say that (K_-, K_+, K_0) is a *twist-move-triple.* If K_+ and K_- differ by a single twist-move in B, we also say that K_- and K_+ differ by a single *twist-move* in B. Recall Note 1.1.

Note. The $XXII$-move in [24] is the twist-move in the '$p =$ even' case.

Note. Suppose that p is an odd natural number, put $p = 2k+1$. The twist-move for $(4k+3)$-submanifolds $\subset S^{4k+5}$ ($4k+3 \in \mathbb{N}$, $k \in \mathbb{N} \cup \{0\}$) has the following property: Suppose that K_+ is made into K_- by the twist-move. Then K_- is a nonspherical knot in general even if K_+ is a spherical knot. Furthermore the $H_*(K_-;\mathbb{Z})$ is not congruent to $H_*(K_+;\mathbb{Z})$ in general. Example: A Seifert hypersurface V_* for a 3-knot K_* $(* = +, -)$. Framed link representation of V_+ is the Hopf link such that the framing of one component is zero and that of the other is two. Framed link representation of V_- is the Hopf link such that the framing of each component is two.

Let (K_+, K_-, K_0) be related by a single twist-move in B. Then there is a Seifert hypersurface V_* for K_* $(* = +, -, 0)$ with the following properties.

(1) $$V_\sharp = V_0 \cup h_\sharp \ (\sharp = +, -), \quad V_\sharp \cap B = h_\sharp.$$

(2) $$V_0 \cap \operatorname{Int} B = \emptyset.$$

$$V_0 \cap \partial B \text{ is the attaching part of } h^p_\sharp.$$

(The idea of the proof is the Thom–Pontrjagin construction.)

The ordered set (V_+, V_-, V_0) is called a *twist-move-triple of Seifert hypersurfaces* for (K_+, K_-, K_0). We say that V_- (resp. V_+) is obtained from V_+ (resp. V_-) by a single *twist-move* in B.

Fig. 3.3.1, which consists of the three figures (1), (2) and (3), is a diagram of a twist-move-triple. The upper half of Fig. 3.3.2 is another diagram of a twist-move triple. Compare the upper half of Fig. 3.3.2 and the lower half. If $p = 0$ (hence $n = 2p+1 = 1$), the left figure in the upper half and that in the lower half are the same. That is, if $p = 0$ (hence $n = 2p+1 = 1$), a twist-move-triple is a crossing-change-triple of 1-links. Note that we move $B \cap K_0$ by isotopy in the right B in the upper half of Fig. 3.3.2. Recall Note 1.1.

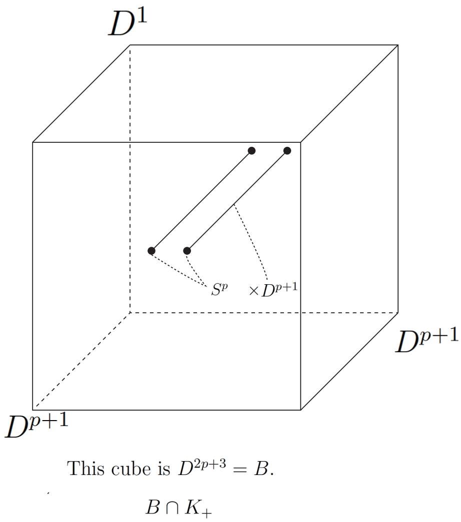


Fig. 3.3.1. (1) A twist-move-triple

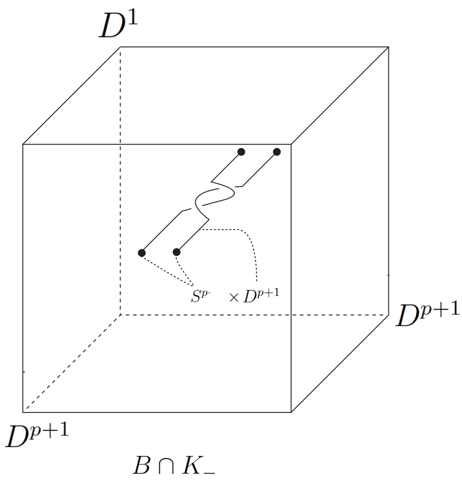
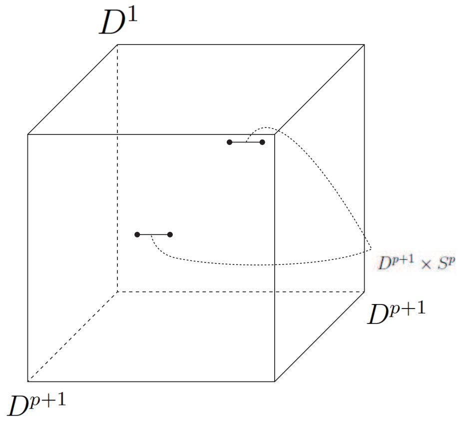
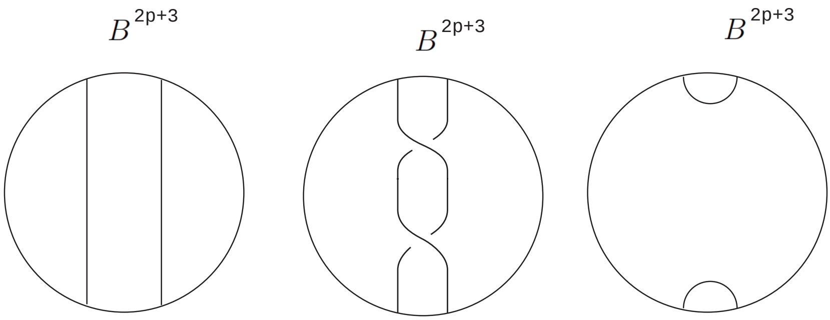


Fig. 3.3.1. (2) A twist-move-triple



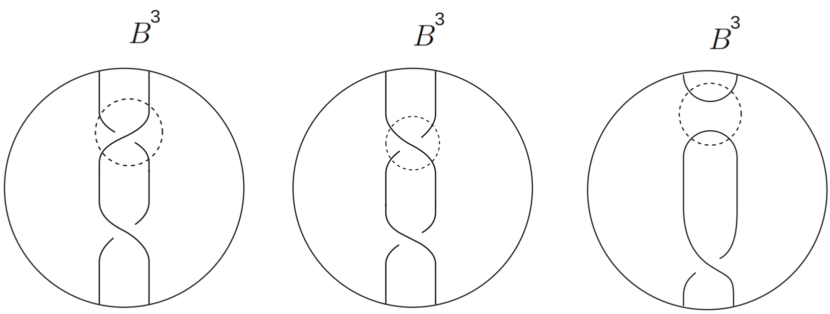
$B \cap K_0$

Fig. 3.3.1. (3) A twist-move-triple



$B \cap K_+^{2p+1}$ $\quad$ $B \cap K_-^{2p+1}$ $\quad$ $B \cap K_0^{2p+1}$

If $p = 0$



The triple of three makes a crossing-change-triple of a 1-dimensional link.

Fig. 3.3.2. A twist-move-triple of 1-links is a crossing-change-triple of 1-links

3.4. An overview of the main results. One of our main theorems is the following. If a 1-link K is obtained from a 1-link K' by a single crossing-change, then the knot product, $K \otimes$ (the Hopf link), is obtained from the knot product, $K' \otimes$ (the Hopf link), by a single twist-move (see Theorems 4.1 and 7.1). Other results in this paper are as follows: If a 1-knot K is obtained from a 1-knot K' by a single pass-move, then the knot product, $K \otimes$ (the Hopf link), is obtained from the knot product, $K' \otimes$ (the Hopf link), by a single $(3,3)$-pass-move (see Theorem 8.1). Let K and K' be 1-knots. The 1-knot K is pass-move-equivalent to the 1-knot K' if and only if the knot product, $K \otimes$(the Hopf link), is $(3,3)$-pass-move-equivalent to the knot product, $K' \otimes$ (the Hopf link) (see Theorems 8.1 and 8.10). Of course we show more results in other high dimensional cases.

4. Main results — technical statements. We work in the smooth category. Let $L = (K_1, \ldots, K_m)$ be an m-component n-(dimensional) oriented ordered submanifold $\subset S^{n+2}$. If $m = 1$ and if L is PL homeomorphic to the standard sphere, then L is called an n-dimensional spherical knot. (Note the footnote in Section 2.1.) If each K_i is a spherical knot, then L is called an n-dimensional spherical link. Let $\mathrm{id} : S^{n+2} \to S^{n+2}$ be the identity map. We say that n-submanifolds L and L' are *identical* if $\mathrm{id}(L) = L'$ and $\mathrm{id}|_L : L \to L'$ is an orientation and order preserving identity map. We say that n-submanifolds L and L' are *equivalent* if there exists an orientation preserving diffeomorphism $f : S^{n+2} \to S^{n+2}$ such that $f(L) = L'$ and $f|_L : L \to L'$ is an orientation and order preserving diffeomorphism. An m-component n-submanifold $L = (L_1, \ldots, L_m)$ is called a *trivial* $(n\text{-})$*link* if each L_i bounds an $(n+1)$-ball B_i trivially embedded in S^{n+2} and if $B_i \cap B_j = \emptyset$ $(i \neq j)$. If $m = 1$, then L is called a *trivial* $(n\text{-})$*knot*.

The following theorems are special cases of our results. We first prove relations between the crossing-change on 1-links and the twist-move on high dimensional knots.

THEOREM 4.1. *Suppose that two* 1*-links J and K differ by a single crossing-change. Then the knot products, $J \otimes^{\mu}$ (the Hopf link) and $K \otimes^{\mu}$ (the Hopf link), differ by a single twist-move, where $\mu \in \mathbb{N} \cup \{0\}$.*

NOTE.

(1) The fact that the two knots differ means that the two knots are not identical. There are two cases that the two knots are not equivalent and that the two knots are equivalent.

(2) The above Theorem 4.1 follows from the following Theorem 7.1 by Theorem 2.3.2.

(3) In Section 7 we will show an example of the phenomenon which Theorems 4.1 and 7.1 assert.

THEOREM 7.1. *Let $m \in \mathbb{N} \cup \{0\}$. Suppose that two* (*not necessarily connected*) $(2m+1)$-*dimensional closed oriented submanifolds $\subset S^{2m+3}$, J and K, differ by a single twist-move. Then the $(2m+2\nu+1)$-submanifolds $\subset S^{2m+2\nu+3}$, $J \otimes^{\nu} [2]$ and $K \otimes^{\nu} [2]$, differ by a single twist-move.*

THEOREM 7.3. *Let $k \in \mathbb{N}$. Let K* (*resp. J*) *be $(4k+5)$-dimensional smooth submanifold $\subset S^{4k+7}$. Suppose that K and J differ by a single twist-move and are nonequivalent. Suppose that K is equivalent to $A \otimes^{k+1}$* (*the Hopf link*) *for a* 1*-knot A. Then there is* *a unique equivalence class of simple $(4k+1)$-knots for K (resp. J) with the following properties.*

(i) *There is a representative element K' of the above equivalence class for K such that K is equivalent to $K' \otimes$ (the Hopf link).*
(ii) *There is a representative element J' of the above equivalence class for J such that J is equivalent to $J' \otimes$ (the Hopf link).*
(iii) *K' and J' differ by a single twist-move and are nonequivalent.*

Note. Let K be an n-dimensional spherical knot $\subset S^{n+2}$. If $\pi_1(S^{n+2} - K) = \mathbb{Z}$ and if $\pi_i(S^{n+2} - K) = 0$ $(2 \leqq i < \frac{n}{2},\ i \in \mathbb{N})$, then we call K a *simple knot*. See [16].

Note. Let $p \in \mathbb{N} \cup \{0\}$. There are countably infinitely many nonequivalent $(2p+5)$-dimensional spherical knots which are not the product of any $(2p+1)$-dimensional closed oriented submanifold $\subset S^{2p+3}$ and the Hopf link by [8, 10].

Note. If $k = 0$, we have a different situation: By Theorem 7.2 there are nonequivalent 1-knots K' and J' with the following properties.
(1) K' and J' differ by two crossing-changes not by a single crossing-change.
(2) Let $\mu \in \mathbb{N}$. $K' \otimes^{\mu}$ (the Hopf link) and $J' \otimes^{\mu}$ (the Hopf link) differ by a single twist-move and are nonequivalent. (Recall that the twist-move on 1-links is the crossing-change on 1-links. See Fig. 3.3.2.)

We also prove relations between the pass-move on 1-knots and the (p,q)-pass-move on high dimensional knots.

Theorem 8.1. *Suppose that two 1-knots J and K differ by a single pass-move. Let $\mu \in \mathbb{N} \cup \{0\}$. Then the $(4\mu+1)$-submanifolds $\subset S^{4\mu+3}$, $J \otimes^{\mu}$ (the Hopf link) and $K \otimes^{\mu}$ (the Hopf link), differ by a single $(2\mu+1, 2\mu+1)$-pass-move.*

Theorem 8.5. *Let $l \in \mathbb{N}$. Let J, K be simple $(2l+1)$-knots. Suppose that J and K differ by a single $(l+1, l+1)$-pass-move. Let $\mu \in \mathbb{N} \cup \{0\}$. Then the $(2l+4\mu+1)$-submanifolds $\subset S^{2l+4\mu+3}$, $J \otimes^{\mu}$ (the Hopf link) and $K \otimes^{\mu}$ (the Hopf link), differ by a single $(l+2\mu+1, l+2\mu+1)$-pass-move.*

Theorem 8.10. *Let $\mu \in \mathbb{N}$. Let K (resp. J) be a $(4\mu+1)$-submanifold $\subset S^{4\mu+3}$. Let K and J be $(2\mu+1, 2\mu+1)$-pass-move-equivalent. Suppose that K is equivalent to $K' \otimes^{\mu}$ (the Hopf link) for a 1-knot K'. Then there is a 1-knot J' with the following properties.*

(i) *J is equivalent to $J' \otimes^{\mu}$ (the Hopf link).*
(ii) *K' and J' are pass-move-equivalent.*

Note. In the above theorem, if such K' exists, there are countably infinitely many different pairs $([P],[Q])$ of different equivalence-classes of 1-knots such that $P \otimes^{\mu}$ (the Hopf link) (resp. $Q \otimes^{\mu}$ (the Hopf link)) is equivalent to K (resp. J) and that P and Q are pass-move-equivalent, where $[A]$ is the equivalence class of a 1-knot A. We prove this in Section 12.

Theorem 8.11. *Let $p \in \mathbb{N}$. Let K and J be $(2p+5)$-dimensional smooth submanifolds $\subset S^{2p+7}$. Suppose that K and J differ by a single $(p+3,p+3)$-pass-move and are nonequivalent. Suppose that K is equivalent to*

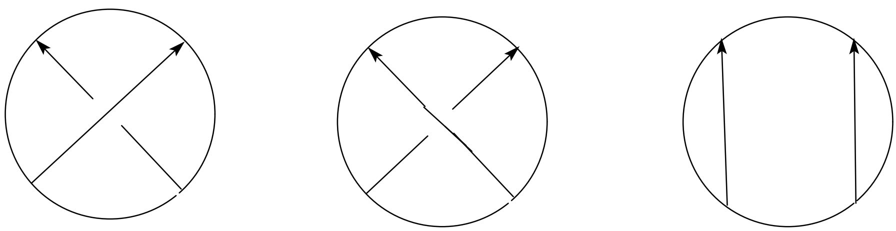
$$\begin{cases} A \otimes^{p/2+1} (\textit{the Hopf link}) \textit{ for a 1-knot } A & \textit{if } p \textit{ is even} \\ A \otimes (\textit{the Hopf link}) \textit{ for a simple 3-knot } A & \textit{if } p=1 \ (\textit{and hence } 2p+5=7) \\ A \otimes^{(p-1)/2} (\textit{the Hopf link}) \textit{ for a simple 7-knot } A & \textit{if } p \textit{ is odd and } p \neq 1. \end{cases}$$

Then there is a unique equivalence class of simple $(2p+1)$-knots for K (resp. J) with the following properties.

(i) *There is a representative element K' of the above equivalence class for K such that K is equivalent to $K' \otimes$ (the Hopf link).*
(ii) *There is a representative element J' of the above equivalence class for J such that J is equivalent to $J' \otimes$ (the Hopf link).*
(iii) *K' and J' differ by a single $(p+1,p+1)$-pass-move and are nonequivalent.*

Next we discuss a relation between polynomial invariants of 1-links and those of high dimensional knots related by knot products.

Suppose that 1-links K_+, K_-, K_0 differ only in a 3-ball B as shown below.



K_+ K_- K_0

Then the ordered set (K_+, K_-, K_0) is called a *crossing-change-triple*. We also say that the ordered set (K_+, K_-, K_0) is related by a single *crossing-change* in B.

Let $A(K)$ be the Alexander–Conway polynomial of 1-links K. It is well-known that

$$A(K_+) - A(K_-) = (t-1) \cdot A(K_0).$$

Note that there is another kind of setting of the variable. Here, we have the following.

Theorem 9.1. *Let K_+, K_-, K_0 be as above.*

(1) *Let $\mu \in \mathbb{N} \cup \{0\}$. There is a polynomial $\Delta_{2\mu+1}(K_* \otimes^{\mu}$ (the Hopf link)$) \in \mathbb{Q}[t,t^{-1}]$ whose $\mathbb{Q}[t,t^{-1}]$-balanced class is the $(2\mu+1)$-$\mathbb{Q}[t,t^{-1}]$-Alexander polynomial $A_{2\mu+1}(K_* \otimes^{\mu}$ (the Hopf link)$)$ $(*=+,-,0)$ such that*

$$\Delta_{2\mu+1}(K_+ \otimes^{\mu} (\textit{the Hopf link})) - \Delta_{2\mu+1}(K_- \otimes^{\mu} (\textit{the Hopf link})) \\ = (t-1) \cdot \Delta_{2\mu+1}(K_0 \otimes^{\mu} (\textit{the Hopf link})).$$

(2) *Let* $\nu \in \mathbb{N} \cup \{0\}$. *There is a polynomial* $\Delta_{\nu+1}(K_* \otimes^\nu [2]) \in \mathbb{Q}[t,t^{-1}]$ *whose* $\mathbb{Q}[t,t^{-1}]$-*balanced class is the* $(\nu+1)$-$\mathbb{Q}[t,t^{-1}]$-*Alexander polynomial* $A_{\nu+1}(K_* \otimes^\nu [2])$ $(* = +,-,0)$ *such that*

$$\Delta_{\nu+1}(K_+ \otimes^\nu [2]) - \Delta_{\nu+1}(K_- \otimes^\nu [2]) = (t + (-1)^{\nu+1}) \cdot \Delta_{\nu+1}(K_0 \otimes^\nu [2]),$$

where $[2]$ *denotes the empty knot of degree two.*

Part (1) of Theorem 9.1 follows from part (2) by Theorem 2.3.2.

NOTE.
(1) We review the p-$\mathbb{Q}[t,t^{-1}]$-Alexander polynomial A_p for n-dimensional closed oriented submanifolds $\subset S^{n+2}$ in Section 5.
(2) We will show an example of Theorem 9.1.(2) in Section 9.

The above Theorem 9.1.(2) is related to the following Theorem 9.2. Compare the example of Theorem 9.1.(2) and that of Theorem 9.2. (Both examples are in Section 9.) The '$l =$ even' case of Theorem 9.2 is proved in [24]. In this paper we prove the '$l =$ odd' case of Theorem 9.2.

THEOREM 9.2. *Let* $l \in \mathbb{N} \cup \{0\}$. *Let* K_+ *be a* $(2l+1)$-*dimensional spherical knot* $\subset S^{2l+3}$. *Let* K_-, K_0 *be* $(2l+1)$-*dimensional submanifolds* $\subset S^{2l+3}$. *Let* (K_+, K_-, K_0) *be a twist-move-triple.*

Then there is a polynomial $\Delta_{l+1}(K_*) \in \mathbb{Q}[t,t^{-1}]$ *whose* $\mathbb{Q}[t,t^{-1}]$-*balanced class is the* $(l+1)$-$\mathbb{Q}[t,t^{-1}]$-*Alexander polynomial* $A_{l+1}(K_*)$ $(* = +,-,0)$ *and*

$$\Delta_{l+1}(K_+) - \Delta_{l+1}(K_-) = (t + (-1)^{l+1}) \cdot \Delta_{l+1}(K_0).$$

NOTE.
(1) We defined the twist-move-triple in Section 3.
(2) We will show an example of Theorem 9.2 in Section 9.
(3) The identity in Theorem 9.2 (resp. Theorem 9.1.(2)) has a periodicity in dimensions. The identity in the '$l =$ odd' case of Theorem 9.2 (resp. Theorem 9.1.(2)) has a different form from the identities in the '$l =$ even' case of Theorem 9.2 (resp. Theorem 9.1.(2)), in Theorem 6.1, in Theorem 6.2 and in the well-known case of classical links that is quoted above.

5. Review of the $\mathbb{Q}[t,t^{-1}]$-Alexander polynomials for n-knots and n-dimensional closed oriented submanifolds. We review the $\mathbb{Q}[t,t^{-1}]$-Alexander polynomials for n-knots and n-links and n-dimensional closed oriented submanifolds, Seifert matrices, Alexander matrices, etc. See [1, 14, 15, 16].

Let $K = (K_1, \ldots, K_\xi)$ be an n-dimensional closed oriented submanifold of S^{n+2} $(n \in \mathbb{N})$. Let each K_i be connected. It is known that any tubular neighborhood of K is diffeomorphic to $K \times D^2$ (see pages 49, 50 of [13]). Let $X = \overline{S^{n+2} - K \times D^2}$. By using the orientation of S^{n+2} and that of K, we can determine an orientation of ∂D^2. Take a homomorphism $\alpha : H_1(X;\mathbb{Z}) \to \mathbb{Z}$ to carry all $[\partial D^2]$ with the orientations to $+1$. Take the infinite cyclic covering $\pi : \widetilde{X} \to X$ associated with α. $\widetilde{X}$ is called the *canonical cyclic covering space* of K. We can regard $H_p(\widetilde{X};\mathbb{Z})$ as a $\mathbb{Z}[t,t^{-1}]$-module by using the covering translation $\widetilde{X} \to \widetilde{X}$ defined by α. It is called the $\mathbb{Z}[t,t^{-1}]$-*p-Alexander module*. We can also regard $H_p(\widetilde{X};\mathbb{Q})$ as a $\mathbb{Q}[t,t^{-1}]$-module. It is called the $\mathbb{Q}[t,t^{-1}]$-*p-Alexander module*.

According to module theory, it holds that any $\mathbb{Q}[t,t^{-1}]$-module is congruent to

$$(\mathbb{Q}[t,t^{-1}]/\lambda_1) \oplus \ldots \oplus (\mathbb{Q}[t,t^{-1}]/\lambda_l) \oplus (\oplus^k \mathbb{Q}[t,t^{-1}]),$$

where
(1) $\lambda_* \in \mathbb{Q}[t,t^{-1}]$ is not zero,
(2) λ_* is not the $\mathbb{Q}[t,t^{-1}]$-balanced class of 1,
(3) k is the rank of the free part.

Two polynomials $f(t), g(t) \in \mathbb{Q}[t,t^{-1}]$ are said to be $\mathbb{Q}[t,t^{-1}]$-*balanced* if there is an integer n and a nonzero rational number r such that $f(t) = r \cdot t^n \cdot g(t)$.

Let $H_p(\widetilde{X};\mathbb{Q})$ be as above. Then the $\mathbb{Q}[t,t^{-1}]$-*p-Alexander polynomial* is

$$\begin{cases} \text{the } \mathbb{Q}[t,t^{-1}]\text{-balanced class} & \\ \qquad\qquad \text{of the product } \lambda_1 \cdots \lambda_l & \text{if } k=0 \text{ and } H_p(\widetilde{X};\mathbb{Q}) \text{ is nontrivial} \\ 0 & \text{if } k \neq 0 \\ 1 & \text{if } H_p(\widetilde{X};\mathbb{Q}) \cong 0. \end{cases}$$

A *Seifert hypersurface* for an n-dimensional oriented closed submanifold K in S^{n+2} is an $(n+1)$-dimensional oriented connected compact submanifold in S^{n+2} whose boundary is K ($n \in \mathbb{N}$). Note that there are two cases that K is not connected and that K is connected.

Let V be a Seifert hypersurface for the above n-submanifold K. Note that the orientation of V is compatible with that of K. Recall that Seifert hypersurfaces are connected by the definition (see Section 2.3). Let $x_1, \ldots, x_\mu$ be p-cycles in V which are basis of $H_p(V;\mathbb{Z})/\mathrm{Tor}$. Let $y_1, \ldots, y_\nu$ be $(n+1-p)$-cycles in V which are basis of $H_{n+1-p}(V;\mathbb{Z})/\mathrm{Tor}$. Push y_i to the positive direction of the normal bundle of V. Call it y_i^+. Push y_i to the negative direction of the normal bundle of V. Call it y_i^-. A $(p, n+1-p)$-*positive Seifert matrix* for the above submanifold K associated with V represented by an ordered basis $\{x_1, \ldots, x_\mu\}$ and an ordered basis $\{y_1, \ldots, y_\nu\}$, is a $(\mu \times \nu)$-matrix

$$S = (s_{ij}) = (\mathrm{lk}(x_i, y_j^+)).$$

We sometimes abbreviate $(p, n+1-p)$-positive Seifert matrix to p-Seifert matrix if it is clear from the context. We sometimes let $S_p(K)$ denote a positive p-Seifert matrix for a closed oriented submanifold K and V and $\{x_i\}$ and $\{y_j\}$ if we know what V and $\{x_i\}$ and $\{y_j\}$ are.

A $(p, n+1-p)$-*negative Seifert matrix* for the above submanifold K associated with V represented by an ordered basis $\{x_1, \ldots, x_\mu\}$ and an ordered basis $\{y_1, \ldots, y_\nu\}$, is a matrix

$$N = (n_{ij}) = (\mathrm{lk}(x_i, y_j^-)).$$

We sometimes let $N_p(V)$ denote a negative p-Seifert matrix for a closed oriented submanifold K and V and $\{x_i\}$ and $\{y_j\}$ if we know what K and $\{x_i\}$ and $\{y_j\}$ are. Let S_p and N_p be as above. Then we have the following. $S_p - N_p$ represents the map $\{H_p(V;\mathbb{Z})/\mathrm{Tor}\} \times \{H_{n+1-p}(V;\mathbb{Z})/\mathrm{Tor}\} \to \mathbb{Z}$, which is defined by the intersection product. We call $t \cdot S_p - N_p$ the *p-Alexander matrix* for K associated with V represented by an ordered basis $\{x_1, \ldots, x_\mu\}$ and an ordered basis $\{y_1, \ldots, y_\nu\}$.

Note that we sometimes define it to be $S_p - t \cdot N_p$. The difference of both is only setting the variables because we mainly discuss $\mathbb{Q}[t, t^{-1}]$-balanced-classes as follows. All we have to do is to change t with t^{-1}.

PROPOSITION 5.1. *Let K be an n-dimensional oriented closed submanifold $\subset S^{n+2}$. Let S_p be a $(p, n+1-p)$-positive Seifert matrix for K associated with V represented by an ordered basis $\{x_1, \ldots, x_\mu\}$ and an ordered basis $\{y_1, \ldots, y_\nu\}$. Let N_p be a $(p, n+1-p)$-negative Seifert matrix for K associated with V represented by an ordered basis $\{x_1, \ldots, x_\mu\}$ and an ordered basis $\{y_1, \ldots, y_\nu\}$. Suppose $\mu = \nu$. Suppose that the linear map defined by a $(p-1)$-Alexander matrix is injective.*

Then the p-$\mathbb{Q}[t, t^{-1}]$-Alexander polynomial is the $\mathbb{Q}[t, t^{-1}]$-balanced class of 'the determinant of p-Alexander matrix'

$$\det(t \cdot S_p - N_p).$$

NOTE. Of course $\mu \neq \nu$ in general.

Proof. Take the above $X = \overline{S^{n+2} - K \times D^2}$, $\widetilde{X}$, V. Let $V \times [-1, 1]$ be the tubular neighborhood of V in X. Let $Y = X - V$. Consider the Meyer–Vietoris exact sequence:

$$H_\natural(\amalg_{-\infty}^{\infty} V \times [-1,1]; \mathbb{Q}) \to H_\natural(\amalg_{-\infty}^{\infty} Y; \mathbb{Q}) \to H_\natural(\widetilde{X}; \mathbb{Q}),$$

where $\amalg_{-\infty}^{\infty} V \times [-1,1]$ is the lift of $V \times [-1,1]$, and $\amalg_{-\infty}^{\infty} Y$ is the lift of Y. This completes the proof. ■

Let N_p be a $(p, n+1-p)$-negative Seifert matrix for K associated with V represented by an ordered basis $\{x_1, \ldots, x_\mu\}$ and an ordered basis $\{y_1, \ldots, y_\nu\}$. Let S_{n+1-p} be a $(n+1-p, p)$-positive Seifert matrix for K associated with V represented by an ordered basis $\{y_1, \ldots, y_\nu\}$ and an ordered basis $\{x_1, \ldots, x_\mu\}$. By the definition of x_i^+ and y_i^-, $\mathrm{lk}(y_i, x_j^+) = \mathrm{lk}(y_i^-, x_j)$. By page 541 of [14], $\mathrm{lk}(y_i^-, x_j) = (-1)^{p(n+1-p)+1} \mathrm{lk}(x_j, y_i^-)$. Hence

$$N_p = (-1)^{p \cdot n+1} S_{n+1-p}$$

(note that $p(1-p)$ is an even number).

PROPOSITION 5.2. *Let K be a $(2m+1)$-dimensional closed oriented submanifold $\subset S^{2m+3}$. Let S be an $(m+1, m+1)$-Seifert matrix. We have*

$$S = (-1)^m \cdot {}^tS.$$

The *signature* $\sigma(K)$ for a $(2p+1)$-dimensional closed oriented submanifold $K \subset S^{2p+3}$ ($p \in \mathbb{N} \cup \{0\}$) is the signature of the matrix $S_{p+1}(K) + {}^tS_{p+1}(K)$. Therefore, we have the following.

CLAIM 5.3. *Let K be a $(4k+3)$-dimensional closed oriented submanifold $\subset S^{4k+5}$ ($k \in \mathbb{N} \cup \{0\}$). Let V be a Seifert hypersurface for K. Then the signature of K coincides with the signature of V.*

Let K be a $(4k+1)$-dimensional spherical knot ($4k+1 \geq 1$, $k \in \mathbb{N} \cup \{0\}$). We regard naturally $(H_{2k+1}(V; \mathbb{Z})/\mathrm{Tor}) \otimes \mathbb{Z}_2$ as a subgroup of $H_{2k+1}(V; \mathbb{Z}_2)$. Then we can take basis $x_1, \dots, x_\nu, y_1, \dots, y_\nu$ of $(H_{2k+1}(V;\mathbb{Z})/\mathrm{Tor}) \otimes \mathbb{Z}_2$ such that $x_i \cdot x_j = 0$, $y_i \cdot y_j = 0$, $x_i \cdot y_j = \delta_{ij}$ for any pair (i,j), where $\cdot$ denotes the $\mathbb{Z}_2$-intersection product. The *Arf invariant* of K is

$$\Big(\sum_{i=1}^{\nu} \mathrm{lk}(x_i, x_i^+) \cdot \mathrm{lk}(y_i, y_i^+)\Big) \bmod 2.$$

Let $L = (L_1, \dots, L_\mu)$ be a $(4k+1)$-link ($4k+1 \geq 1$, $k \in \mathbb{N} \cup \{0\}$, $\mu \in \mathbb{N} - \{1\}$). We define the Arf invariant of L. There are two cases.

(1) Let $4k+1 \geq 5$. The Arf invariant of L is defined in the same manner as the knot case.

(2) Let $4k+1 = 1$. See Appendix of [13] and Note right above Note 1.2.1 of [21].

We use the following proposition.

PROPOSITION 5.4. *Let K be an n-dimensional closed oriented submanifold $\subset S^{n+2}$. Then*

(1) $$S_{p+1}(K \otimes [2]) = (-1)^{(n-p)} S_p(K) \otimes S_0([2]),$$

(2) $$N_{p+1}(K \otimes [2]) = (-1)^{(n-p)} N_p(K) \otimes N_0([2]),$$

(3) $$S_0([2]) = (1),$$

(4) $$N_0([2]) = (-1),$$

(5) $$S_{\nu-1}(\otimes^\nu [2]) = (-1)^{(\nu-1)\nu/2}$$

(note that $\otimes^\nu[2]$ is a $(2\nu-3)$-submanifold $\subset S^{2\nu-1}$),

(6) $$S_{p+\nu}(K \otimes^\nu [2]) = (-1)^{(n-p)\nu + \nu(\nu-1)/2}\, S_p(K),$$

(7) $$S_{p+2\mu}(K \otimes^\mu (\text{the Hopf link})) = (-1)^\mu S_p(K)$$

(Let $\nu = 2\mu$ in (6). We obtain (7).),

(8) *if α, β are disjoint cycles of dimension p and q in S^{p+q+1}, then*
$$\mathrm{lk}(\alpha, \beta) = (-1)^{pq+1} \mathrm{lk}(\beta, \alpha).$$

Proof. Section 6 of [10] implies (1)–(7). Page 541 of [14] implies (8). ∎

6. Some results on invariants of n-knots and local moves on n-knots

THEOREM 6.1 ([24]). *Let K_+, K_- be spherical n-knots $\subset S^{n+2}$ ($n \in \mathbb{N}$). Let K_0 be an n-submanifold $\subset S^{n+2}$. Suppose that (K_+, K_-, K_0) is related by a $(p, n+1-p)$-pass-move. Let $p \neq n+1-p$. Then there is a polynomial $\Delta_p(K_*) \in \mathbb{Q}[t, t^{-1}]$ whose $\mathbb{Q}[t, t^{-1}]$-balanced class is the p-$\mathbb{Q}[t, t^{-1}]$-Alexander polynomial $A_p(K_*)$ for the submanifold K_* ($* = +, -, 0$) such that*

$$\Delta_p(K_+) - \Delta_p(K_-) = (t-1) \cdot \Delta_p(K_0).$$

THEOREM 6.2 ([24]). *Let K_+, K_- be spherical $(4k+1)$-knots ($4k+1 \in \mathbb{N}$, $k \in \mathbb{N} \cup \{0\}$). Let K_0 be a $(4k+1)$-submanifold $\subset S^{4k+3}$. Suppose that (K_+, K_-, K_0) is related by a single twist-move. Then there is a polynomial $\Delta_{2k+1}(K_*) \in \mathbb{Q}[t, t^{-1}]$ whose $\mathbb{Q}[t, t^{-1}]$-balanced class is the $(2k+1)$-$\mathbb{Q}[t, t^{-1}]$-Alexander polynomial $A_{2k+1}(K_*)$ ($* = +, -, 0$) such that*

$$\Delta_{2k+1}(K_+) - \Delta_{2k+1}(K_-) = (t-1) \cdot \Delta_{2k+1}(K_0).$$

Recall the two sentences right before Theorem 9.2 in Section 4. The 'l = even' case of Theorem 9.2 is the above Theorem 6.2.

THEOREM 6.3 ([24]). *Let K_+, K_-, K_0 be as in Theorem* 6.2. *Let $k \in \mathbb{N} \cup \{0\}$. Let bP_{4k+2} be the bP-subgroup $\subset \Theta^{4k+1}$. Suppose that bP_{4k+2} is not congruent to the trivial group. Then*

$$\operatorname{Arf} K_+ - \operatorname{Arf} K_- = \{|bP_{4k+2} \cap I(K_0)| + 1\} \bmod 2,$$

where $I(K_0)$ is the inertia group of an oriented smooth manifold which is orientation preserving diffeomorphic to K_0 and the symbol $|\;|$ denotes the order of a group.

NOTE. See Section 5 and [12, 13, 14] for the Arf invariant. See [12] for the homotopy sphere group $\Theta^\star$ and the bP-subgroup $\subset \Theta^\star$. See [2, 11] for the inertia group.

We state a problem.

PROBLEM 6.4. Let K_+ be an n-dimensional spherical knot. Suppose that (K_+, K_-, K_0) is a twist-move-triple (resp. $(p, n+1-p)$-pass-move-triple, where $p \neq n+1-p$). Let $\alpha(K)$ be an invariant of K as a submanifold and be a $\mathbb{Q}[t, t^{-1}]$-balanced class. Suppose that there are $f_+, f_-, f_0 \in \mathbb{Q}[t, t^{-1}]$ such that the $\mathbb{Q}[t, t^{-1}]$-balanced class of f_* is $\alpha(K_*)$ $(* = +, -, 0)$ and that

$$\begin{cases} f_+ - f_- = (t-1) \cdot f_0 & \text{in the other cases than the following} \\ f_+ - f_- = (t+1) \cdot f_0 & \text{if } n = 4k+3,\ k \in \mathbb{N} \cup \{0\} \text{ and if we consider the twist-move.} \end{cases}$$

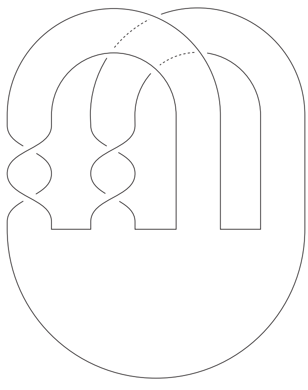
Then is $\alpha(K)$ the $\mathbb{Q}[t, t^{-1}]$-Alexander polynomial or what is determined by the $\mathbb{Q}[t, t^{-1}]$-Alexander polynomial?

NOTE. It is well-known that the Alexander–Conway polynomial (resp. the Jones polynomial, the HOMFLY polynomial) of 1-links is essentially characterized by the well-known local move identity and the fact that it is trivial for the trivial knot.

7. Theorems on relations between crossing-changes and knot products

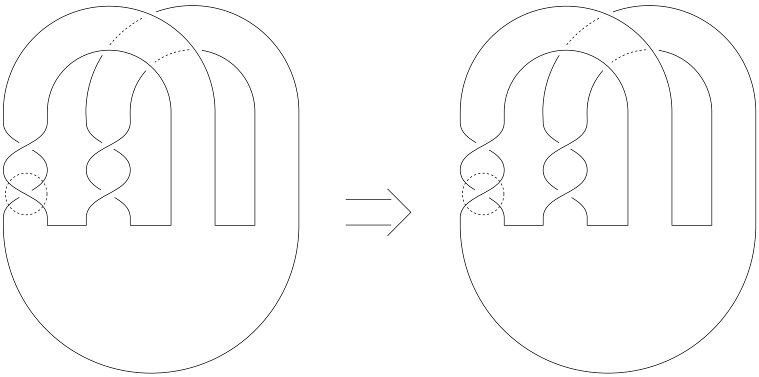
THEOREM 7.1. *Let $m \in \mathbb{N} \cup \{0\}$. Suppose that two* (*not necessarily connected*) *$(2m+1)$-dimensional closed oriented submanifolds $\subset S^{2m+3}$, J and K, differ by a single twist-move. Then the $(2m+2\nu+1)$-submanifolds $\subset S^{2m+2\nu+3}$, $J \otimes^\nu [2]$ and $K \otimes^\nu [2]$, differ by a single twist-move, where* [2] *denotes the empty knot of degree two.*

We show an example of the phenomenon in the '$m = 0$ and $\nu = 1$ case' of Theorem 7.1. The following knot T is the 1-dimensional trivial knot.



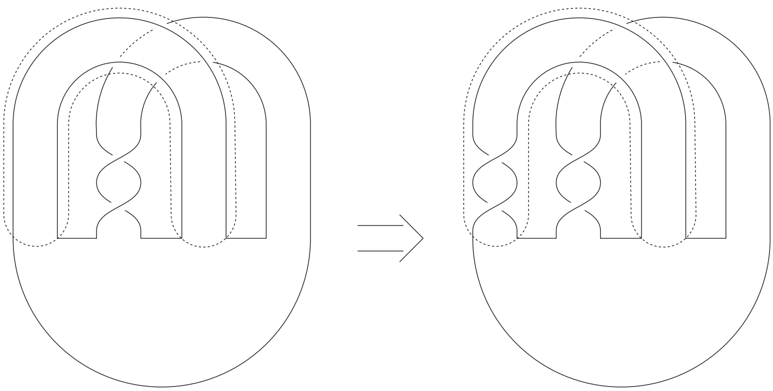
T

Carry out a crossing-change in the 3-ball which is represented by the dotted circle in the following figure. We obtain a new knot K. Note that K is the trefoil knot.



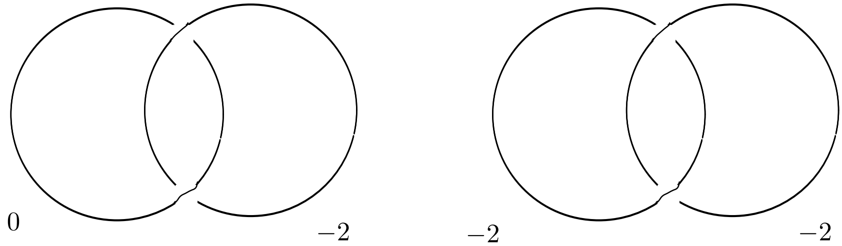
T K

Note that this crossing-change is the same as the twist-move in the 3-ball which is represented by the dotted circle in the following figure.

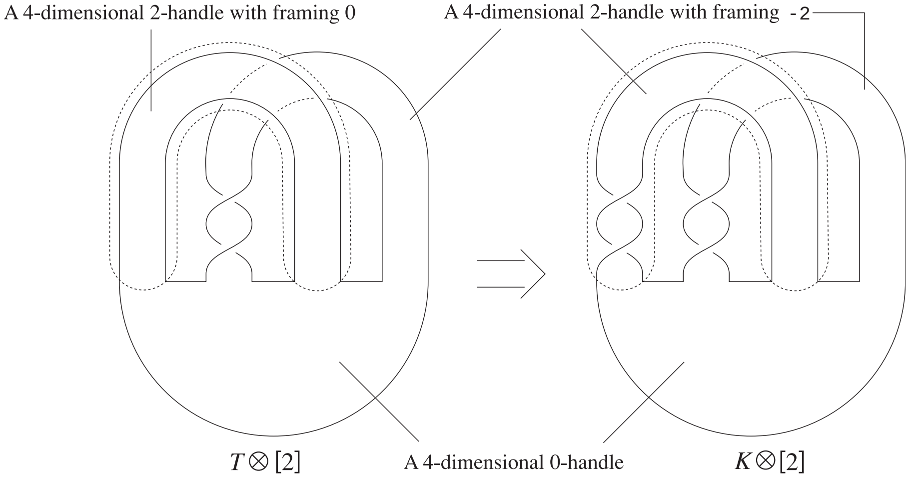


T K

Take $T \otimes [2]$ and $K \otimes [2]$ in S^5. By [8, 10] we can determine the embedding type of $T \otimes [2]$ (resp. $K \otimes [2]$). A Seifert hypersurface for $T \otimes [2]$ is diffeomorphic to the following 4-manifold (on the left) and its associated Seifert matrix is $\left(\begin{smallmatrix} 0 & -1 \\ 0 & -1 \end{smallmatrix}\right)$ by Proposition 5.4. A Seifert hypersurface for $K \otimes [2]$ is diffeomorphic to the following 4-manifold (on the right) and its associated Seifert matrix is $\left(\begin{smallmatrix} -1 & -1 \\ 0 & -1 \end{smallmatrix}\right)$ by Proposition 5.4.

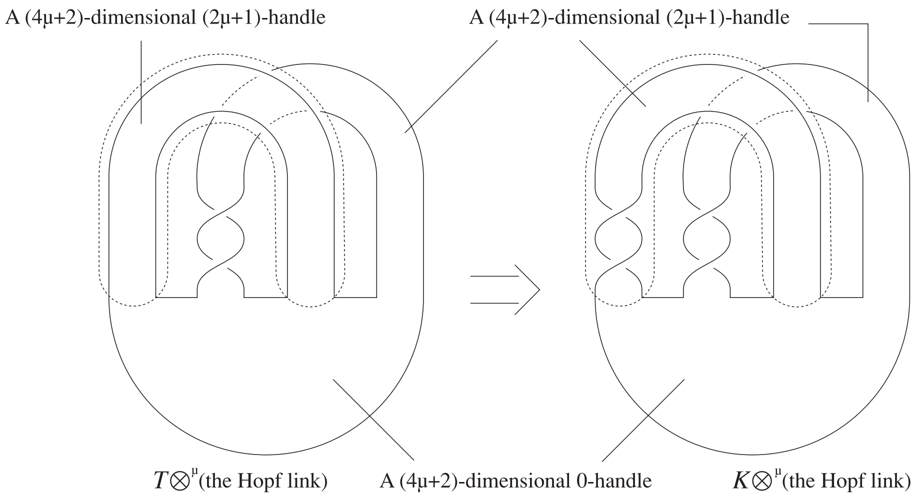


$K \otimes [2]$ is obtained from $T \otimes [2]$ by a single twist-move in the 5-ball which is represented by the dotted circle in the following figure. (The readers should not mind the twisting or the framing there so much.)



Note that $K \otimes [2]$ is not homeomorphic to $T \otimes [2]$. Twist moves of $(4k+3)$-knots $(k \in \mathbb{N} \cup \{0\})$ change the homeomorphism types of submanifolds in general but we can determine the new embedding types which we obtain by twist-moves.

We show an example of '$m = 0$ and $\nu = 2\mu$' case of Theorem 7.1. Let T and K be as above. $K \otimes^{2\mu} [2] = K \otimes^{\mu}$ (the Hopf link) in $S^{4\mu+3}$ is obtained from $T \otimes^{2\mu} [2] = T \otimes^{\mu}$ (the Hopf link) in $S^{4\mu+3}$ by a single twist-move in the $(4\mu+3)$-ball which is represented by the dotted circle in the following figure. (The readers should not mind the twisting or the framing there so much.)



Note that a Seifert hypersurface for $T \otimes^{\mu}$ (the Hopf link) is diffeomorphic to the plumbing of the trivial $D^{2\mu+1}$-bundle over $S^{2\mu+1}$ and the $D^{2\mu+1}$-bundle over $S^{2\mu+1}$ associated with the tangent bundle.

Note that a Seifert hypersurface for $K \otimes^{\mu}$ (the Hopf link) is diffeomorphic to the plumbing of two copies of the $D^{2\mu+1}$-bundle over $S^{2\mu+1}$ associated with the tangent bundle.

For some μ, $K \otimes^\mu$ (the Hopf link) is not diffeomorphic but homeomorphic to $T \otimes^\mu$ (the Hopf link). Then $K \otimes^\mu$ (the Hopf link) is an exotic sphere. For other μ, $K \otimes^\mu$ (the Hopf link) is diffeomorphic to $T \otimes^\mu$ (the Hopf link). Recall the discussion associated with the bP-subgroup in [12].

THEOREM 7.2.

(1) *There is a nontrivial* 1*-knot* K *which is obtained from the trivial knot by a single crossing-change with the following property. Let* $\nu \in \mathbb{N}$, $\nu \geqq 2$. *The* $(2\nu+1)$*-submanifold* $\subset S^{2\nu+3}$, $K \otimes^\nu [2]$, *is equivalent to the trivial* $(2\nu+1)$*-knot.*

(2) *There is a nontrivial* 1*-knot* P *which is obtained from the trivial knot by two crossing-changes not by a single crossing-change with the following property. Let* $\nu \in \mathbb{N}$, $\nu \geqq 2$. *The* $(2\nu+1)$*-submanifold* $\subset S^{2\nu+3}$, $P \otimes^\nu [2]$, *is equivalent to the trivial* $(2\nu+1)$*-knot.*

(3) *There are nontrivial* 1*-knots* P *and* Q *with the following properties.*
  (i) P *and* Q *differ by a single crossing-change and are nonequivalent.*
  (ii) *Let* $\nu \in \mathbb{N}$ *and* $\nu \geqq 2$. *The* $(2\nu+1)$*-submanifolds* $\subset S^{2\nu+3}$, $P \otimes^\nu [2]$ *and* $Q \otimes^\nu [2]$, *are equivalent spherical knots.*

(4) *There is a nontrivial* 1*-knot* P *with the following properties.*
  (i) P *is obtained from the trivial* 1*-knot by two crossing-changes not by a single crossing-change.*
  (ii) *Let* $\mu \in \mathbb{N}$. *The* $(4\mu+1)$*-submanifold* $\subset S^{4\mu+3}$, $P \otimes^\mu$ (*the Hopf link*) *is obtained from the trivial knot by a single twist-move and is a nontrivial knot.*

(5) *There are nontrivial* 1*-knots* P *and* Q *with the following properties.*
  (i) P *and* Q *differ by a single crossing-change and are nonequivalent.*
  (ii) *Let* $\mu \in \mathbb{N}$. *The* $(4\mu+1)$*-submanifolds* $\subset S^{4\mu+3}$, $P \otimes^\mu$ (*the Hopf link*) *and* $Q \otimes^\mu$ (*the Hopf link*) *are equivalent spherical knots and nontrivial knots.*

NOTE. Recall that, if a 1-knot K is obtained from a nonequivalent 1-knot J by a single crossing-change, then the 1-knot K is obtained from the nonequivalent knot J by a single twist-move. Thus we can say that the 'twist-move-unknotting-number' changes by a knot product. A 'non-twist-move-equivalent pair' is changed into a 'twist-move-equivalent pair' by knot product.

THEOREM 7.3. *Let* $k \in \mathbb{N}$. *Let* K (*resp.* J) *be* $(4k+5)$*-dimensional smooth submanifold* $\subset S^{4k+7}$. *Suppose that* K *and* J *differ by a single twist-move and are nonequivalent. Suppose that* K *is equivalent to* $A \otimes^{k+1}$ (*the Hopf link*) *for a* 1*-knot* A.

Then there is a unique equivalence class of simple $(4k+1)$*-knots for* K (*resp.* J) *with the following properties.*

  (i) *There is a representative element* K' *of the above equivalence class for* K *such that* K *is equivalent to* $K' \otimes$ (*the Hopf link*).
  (ii) *There is a representative element* J' *of the above equivalence class for* J *such that* J *is equivalent to* $J' \otimes$ (*the Hopf link*).
  (iii) K' *and* J' *differ by a single twist-move and are nonequivalent.*

NOTE. If $k = 0$ in Theorem 7.3, we have a different result. See Theorem 7.2.

8. Theorems on relations between pass-moves and knot products

THEOREM 8.1. *Suppose that two 1-knots J and K differ by a single pass-move. Let $\mu \in \mathbb{N} \cup \{0\}$. Then the $(4\mu+1)$-submanifolds $\subset S^{4\mu+3}$, $J \otimes^{\mu}$ (the Hopf link) and $K \otimes^{\mu}$ (the Hopf link), differ by a single $(2\mu+1, 2\mu+1)$-pass-move.*

NOTE 8.2. Recall Theorem 2.3.2. Let $n \in \mathbb{N}$ and $\mu \in \mathbb{N} \cup \{0\}$. For any n-dimensional closed oriented submanifold $A \subset S^{n+2}$, $A \otimes^{\mu}$ (the negative Hopf link) $= A \otimes^{2\mu}$ [2].

Compare the above Theorem 8.1 and Note 8.2 with the following Theorem 8.3.

THEOREM 8.3. *Take the same J, K in Theorem 8.1. Then the $(4\mu+3)$-submanifolds $\subset S^{4\mu+5}$ ($\mu \in \mathbb{N} \cup \{0\}$), $J \otimes^{(2\mu+1)}$ [2] and $K \otimes^{(2\mu+1)}$ [2], are not homeomorphic in general and, therefore, are NOT $(2\mu+2, 2\mu+2)$-pass-move-equivalent in general.*

PROBLEM 8.4. In the above Theorem 8.3, of course, if J and K are trivial knots, then the above two $(4\mu+3)$-submanifolds are pass-move-equivalent. What kind of pair, J and K, in Theorem 8.3 satisfies the condition that the above two $(4\mu+3)$-submanifolds are $(2\mu+2, 2\mu+2)$-pass-move-equivalent?

The '$\nu =$ odd' case of Theorem 8.8 and Note to the proof of Theorem 8.8 give partial solutions to Problem 8.4.

THEOREM 8.5. *Let $l \in \mathbb{N}$. Let J, K be simple $(2l+1)$-knots. Suppose that J and K differ by a single $(l+1, l+1)$-pass-move. Let $\mu \in \mathbb{N} \cup \{0\}$. Then the $(2l+4\mu+1)$-submanifolds $\subset S^{2l+4\mu+3}$, $J \otimes^{\mu}$ (the Hopf link) and $K \otimes^{\mu}$ (the Hopf link), differ by a single $(l+2\mu+1, l+2\mu+1)$-pass-move.*

PROBLEM 8.6. If we do NOT suppose that J, K are simple knots in Theorem 8.5, do the above $(2l+4\mu+1)$-submanifolds differ by a single pass-move? Or, are they pass-move-equivalent?

The above problem is really a problem of high dimensional knots. The following one is also such a problem.

PROBLEM 8.7 (A generalization of Problem 8.6).

(1) Suppose that spherical n-knots (resp. n-dimensional closed oriented submanifolds $\subset S^{n+2}$) J and K differ by a single $(p, n+1-p)$-pass-move, where $n \in \mathbb{N}$ and $p \in \mathbb{N}$. Then do the $(n+4\mu+1)$-submanifolds $\subset S^{n+4\mu+3}$, $J \otimes^{\mu}$ (the Hopf link) and $K \otimes^{\mu}$ (the Hopf link), differ by a single pass-move? Or, are they pass-move-equivalent?
(2) How about the case where J and K are even dimensional simple $2m$-knots and where $p = m$? Here, $m \in \mathbb{N}$.
(3) Of course there is a problem in the case of the product with odd times copies of the empty knot [2]. (Note Theorem 8.3 and its proof.)

THEOREM 8.8. *There is a nontrivial 1-knot K which is obtained from the trivial knot by a single pass-move with the following property: Let $\nu \geqq 2$, $\nu \in \mathbb{N}$. $K \otimes^{\nu}$ [2] is the trivial $(2\nu+1)$-knot.*

NOTE 8.9. By Theorem 8.8, we have the following. Let T be the trivial 1-knot. The two 1-knots, K and T, differ by a single pass-move and are nonequivalent. However the

$(2\nu+1)$-dimensional spherical knot, $K \otimes^{\nu} [2]$, is equivalent to the $(2\nu+1)$-dimensional trivial knot, $T \otimes^{\nu} [2]$. Recall $\nu \geqq 2$ and $\nu \in \mathbb{N}$. That is, they differ by ZERO times of pass-moves. Thus we can say that the 'pass-move-unknotting-number' changes by knot products.

THEOREM 8.10. *Let $\mu \in \mathbb{N}$. Let K (resp. J) be a $(4\mu+1)$-submanifold $\subset S^{4\mu+3}$. Let K and J be $(2\mu+1, 2\mu+1)$-pass-move-equivalent. Suppose that K is equivalent to $K' \otimes^{\mu}$ (the Hopf link) for a 1-knot K'. Then there is a 1-knot J' with the properties*:

(i) *J is equivalent to $J' \otimes^{\mu}$ (the Hopf link).*
(ii) *K' and J' are pass-move-equivalent.*

NOTE. See Note under Theorem 8.10 in Section 4.

THEOREM 8.11. *Let $p \in \mathbb{N}$. Let K and J be $(2p+5)$-dimensional smooth submanifolds $\subset S^{2p+7}$. Suppose that K and J differ by a single $(p+3, p+3)$-pass-move and are nonequivalent. Suppose that K is equivalent to*

$$\begin{cases} A \otimes^{p/2+1} (\textit{the Hopf link}) \textit{ for a 1-knot } A & \textit{if } p \textit{ is even} \\ A \otimes (\textit{the Hopf link}) \textit{ for a simple 3-knot } A & \textit{if } p = 1 \ (\textit{and } 2p+5 = 7) \\ A \otimes^{(p-1)/2} (\textit{the Hopf link}) \textit{ for a simple 7-knot } A & \textit{if } p \textit{ is odd and } p \neq 1. \end{cases}$$

Then there is a unique equivalence class of simple $(2p+1)$-knots for K (resp. J) with the following properties:

(i) *There is a representative element K' of the above equivalence class for K such that K is equivalent to $K' \otimes$ (the Hopf link).*
(ii) *There is a representative element J' of the above equivalence class for J such that J is equivalent to $J' \otimes$ (the Hopf link).*
(iii) *K' and J' differ by a single $(p+1, p+1)$-pass-move and are nonequivalent.*

Let P be the 5-twist spun knot of the trefoil knot. Note that P is a 2-knot. Note that Proposition 4.3 of [22] and the last line of Section 7 in page 684 of [22] imply the following.
(1) P is NOT ribbon-move-equivalent to T.
(2) P is NOT (1,2)-pass-move-equivalent to T.

THEOREM 8.12. *Let T be the trivial 2-knot. Let P be as above. Although P is NOT $(1,2)$-pass-move-equivalent to T, we have the following. Let $\nu \geqq 2$ and $\nu \in \mathbb{N}$. The $(2\nu+2)$-submanifold, $P \otimes^{\nu} [2]$, is equivalent to the trivial $(2\nu+2)$-knot, $T \otimes^{\nu} [2]$, and therefore, is $(\nu+1, \nu+2)$-pass-move-equivalent to the trivial knot.*

NOTE 8.13.
(1) Thus we can say that a knot product makes a 'non-pass-move-equivalent pair' into a 'pass-move-equivalent pair'.
(2) Knot products can let us discuss ribbon-moves on 2-knots and high dimensional pass-moves on high dimensional knots on time. This way will make the problem of classification of 2-knots by ribbon-moves easier.

9. Theorems on relations between local move identities of a knot polynomial and knot products. Let (K_+, K_-, K_0) be a crossing-change-triple of 1-links. (See crossing-change-triples in Section 4.) Let $A(K)$ be the Alexander–Conway polynomial of 1-links K. It is well-known that

$$A(K_+) - A(K_-) = (t-1)\cdot A(K_0).$$

Here, we have the following theorems.

THEOREM 9.1. *Let K_+, K_-, K_0 be as above.*

(1) *Let $\mu \in \mathbb{N}\cup\{0\}$. There is a polynomial $\Delta_{2\mu+1}(K_* \otimes^\mu (\text{the Hopf link})) \in \mathbb{Q}[t,t^{-1}]$ whose $\mathbb{Q}[t,t^{-1}]$-balanced class is the $(2\mu+1)$-$\mathbb{Q}[t,t^{-1}]$-Alexander polynomial $A_{2\mu+1}(K_* \otimes^\mu (\text{the Hopf link}))$ $(* = +,-,0)$ such that*

$$\begin{aligned}\Delta_{2\mu+1}(K_+ \otimes^\mu (\text{the Hopf link})) &- \Delta_{2\mu+1}(K_- \otimes^\mu (\text{the Hopf link}))\\ &= (t-1)\cdot \Delta_{2\mu+1}(K_0 \otimes^\mu (\text{the Hopf link})).\end{aligned}$$

(2) *Let $\nu \in \mathbb{N}\cup\{0\}$. There is a polynomial $\Delta_{\nu+1}(K_* \otimes^\nu [2]) \in \mathbb{Q}[t,t^{-1}]$ whose $\mathbb{Q}[t,t^{-1}]$-balanced class is the $(\nu+1)$-$\mathbb{Q}[t,t^{-1}]$-Alexander polynomial $A_{\nu+1}(K_* \otimes^\nu [2])$ $(* = +,-,0)$ such that*

$$\Delta_{\nu+1}(K_+ \otimes^\nu [2]) - \Delta_{\nu+1}(K_- \otimes^\nu [2]) = (t + (-1)^{\nu+1})\cdot \Delta_{\nu+1}(K_0 \otimes^\nu [2]),$$

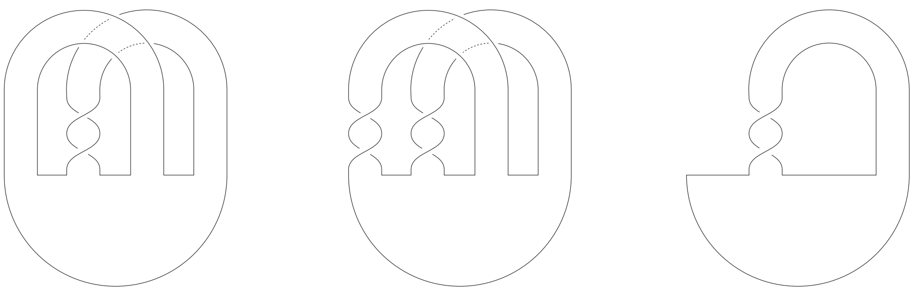
where $[2]$ denotes the empty knot of degree two.

NOTE. After taking knot product, $A(\ )$ is changed into $A_{2\mu+1}(\ )$ (resp. $A_{\nu+1}(\ )$).

By Theorem 2.3.2, Theorem 9.1.(1) follows from Theorem 9.1.(2).

If ν is odd, then $K_+ \otimes^\nu [2]$ is not homeomorphic to $K_- \otimes^\nu [2]$ in general. However Theorem 9.1.(2) is true. We show an example of the '$\nu =$ odd' case of Theorem 9.1.(2).

Let K_+ be the trivial 1-knot, K_- the trefoil knot, and K_0 the Hopf link as shown below. Note that these K_+, K_- and K_0 makes a crossing-change-triple (K_+, K_-, K_0).



K_+ K_- K_0

Take $K_* \otimes [2]$ $(* = +,-,0)$. By [8, 10] and Proposition 5.4 we have the following. We can determine the embedding type of $K_* \otimes [2]$ $(* = +,-,0)$. A Seifert matrix for $K_+ \otimes [2]$ is $\begin{pmatrix} 0 & -1 \\ 0 & -1 \end{pmatrix}$, for $K_- \otimes [2]$ is $\begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$, for $K_0 \otimes [2]$ is (-1).

Hence the 2-Alexander polynomial is the $\mathbb{Q}[t,t^{-1}]$-balanced class of

$$\begin{cases} \det\left\{ t\begin{pmatrix} 0 & -1 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix} \right\} = -t & \text{for } K_+ \otimes [2] \\ \det\left\{ t\begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} \right\} = t^2+t+1 & \text{for } K_- \otimes [2] \\ \det\{t(-1)+(-1)\} = -t-1 & \text{for } K_0 \otimes [2]. \end{cases}$$

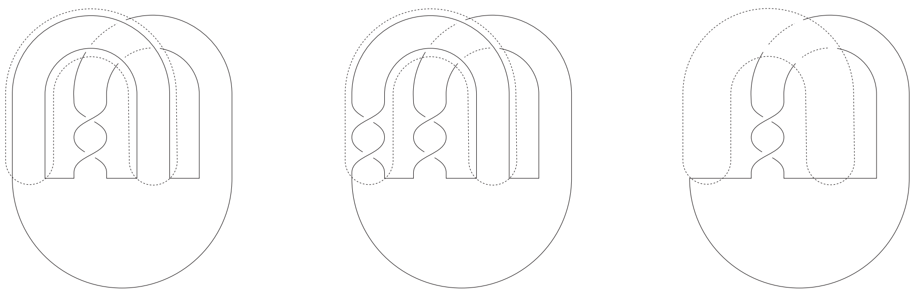
Since $-t-(t^2+t+1) = (t+1)(-t-1)$, the identity in Theorem 9.1.(2) holds for the triple $(K_+\otimes[2], K_-\otimes[2], K_0\otimes[2])$.

Recall the last paragraph before Theorem 9.2 in Section 4.

THEOREM 9.2. *Let $l \in \mathbb{N}\cup\{0\}$. Let K_+ be a $(2l+1)$-dimensional spherical knot $\subset S^{2l+3}$. Let K_-, K_0 be $(2l+1)$-dimensional submanifolds $\subset S^{2l+3}$. Let (K_+, K_-, K_0) be a twist-move-triple. Then there is a polynomial $\Delta_{l+1}(K_*) \in \mathbb{Q}[t,t^{-1}]$ whose $\mathbb{Q}[t,t^{-1}]$-balanced class is the $(l+1)$-$\mathbb{Q}[t,t^{-1}]$-Alexander polynomial $A_{l+1}(K_*)$ $(* = +,-,0)$ and*

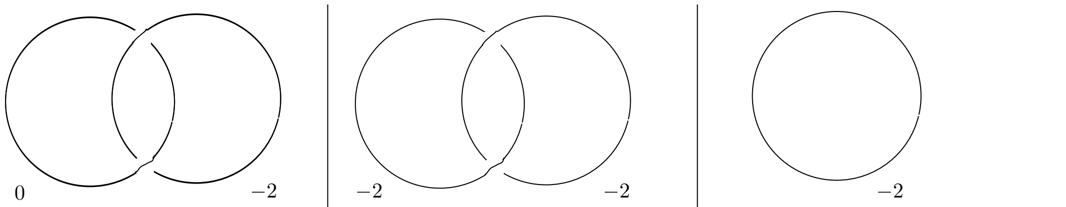
$$\Delta_{l+1}(K_+) - \Delta_{l+1}(K_-) = (t+(-1)^{l+1})\cdot\Delta_{l+1}(K_0).$$

We show an example of Theorem 9.2. Take the same K_* $(*=+,-,0)$ in the examples of Theorem 9.1.(2). Note that (K_+, K_-, K_0) is a twist-move triple in the 3-ball which is represented by the dotted circle in the following figure.



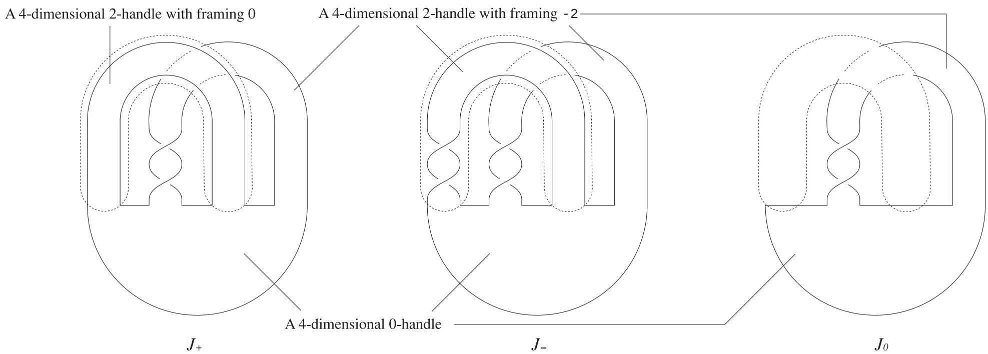
K_+ K_- K_0

Let J_* be $K_*\otimes[2]$ $(*=+,-,0)$. By [8, 10] a Seifert hypersurface for J_* is diffeomorphic to the following 4-manifold.



A Seifert hypersurface for J_+ | A Seifert hypersurface for J_- | A Seifert hypersurface for J_0

(J_+, J_-, J_0) is a twist-move triple in the 5-ball which is represented by the dotted circle in the following figure. (The readers should not mind the twisting or the framing there so much.)



By the calculus in the example of Theorem 9.1.(2), the identity in Theorem 9.2 holds for the triple (J_+, J_-, J_0).

THEOREM 9.3. *Let $\nu \in \mathbb{N} \cup \{0\}$. Let K_+, K_-, K_0 be as in Theorem 9.2. There is a polynomial $\Delta_{l+1+\nu}(K_* \otimes^\nu [2]) \in \mathbb{Q}[t, t^{-1}]$ whose $\mathbb{Q}[t, t^{-1}]$-balanced class is the $(l+1+\nu)$-$\mathbb{Q}[t, t^{-1}]$-Alexander polynomial $A_{l+1+\nu}(K_* \otimes^\nu [2])$ $(* = +, -, 0)$ such that*

$$\Delta_{l+1+\nu}(K_+ \otimes^\nu [2]) - \Delta_{l+1+\nu}(K_- \otimes^\nu [2]) = (t + (-1)^{l+1+\nu}) \cdot \Delta_{l+1+\nu}(K_0 \otimes^\nu [2]).$$

THEOREM 9.4. *Let $k \in \mathbb{N} \cup \{0\}$. Let K_+ be a $(4k+1)$-dimensional spherical knot. Let (K_+, K_-, K_0) be a twist-move-triple. Let $\mu \in \mathbb{N} \cup \{0\}$. Let $bP_{4k+2+4\mu}$ be the bP-subgroup $\subset \Theta^{4k+1+4\mu}$. Suppose that $bP_{4k+2+4\mu}$ is not congruent to the trivial group. Then*

$$\begin{aligned}\mathrm{Arf}(K_+ \otimes^\mu (\textit{the Hopf link})) &- \mathrm{Arf}(K_- \otimes^\mu (\textit{the Hopf link})) \\ &= \left\{|bP_{4k+2+4\mu} \cap I(K_0 \otimes^\mu (\textit{the Hopf link}))| + 1\right\} \bmod 2,\end{aligned}$$

where $I(K_0 \otimes^\mu (\text{the Hopf link}))$ is the inertia group of an oriented smooth manifold which is orientation preserving diffeomorphic to $K_0 \otimes^\mu (\text{the Hopf link})$ and the symbol $|\ |$ denotes the order of a group.

Let K_+ and K_- be n-dimensional spherical knots $\subset S^{n+2}$. Let K_0 be an n-submanifold $\subset S^{n+2}$. Let (K_+, K_-, K_0) be related by a single (p, q)-pass-move in B^{n+2}. Let $p + q = n + 1$. Let $p \neq q$. Recall that we have the following by [24]. (It is quoted in Theorem 6.1 in this paper.)

There is a polynomial $\Delta_p(K_*) \in \mathbb{Q}[t, t^{-1}]$ whose $\mathbb{Q}[t, t^{-1}]$-balanced class is the p-$\mathbb{Q}[t, t^{-1}]$-Alexander polynomial $A_p(K_*)$ for the submanifold K_* $(* = +, -, 0)$ such that

$$\Delta_p(K_+) - \Delta_p(K_-) = (t - 1) \cdot \Delta_p(K_0).$$

THEOREM 9.5. *Let $\nu \in \mathbb{N} \cup \{0\}$. Let K_+, K_-, K_0 be as above. There is a polynomial $\Delta_{p+\nu}(K_* \otimes^\nu [2]) \in \mathbb{Q}[t, t^{-1}]$ whose $\mathbb{Q}[t, t^{-1}]$-balanced class is the $(p+\nu)$-$\mathbb{Q}[t, t^{-1}]$-Alexander polynomial $A_{p+\nu}(K_* \otimes^\nu [2])$ $(* = +, -, 0)$ such that*

$$\Delta_{p+\nu}(K_+ \otimes^\nu [2]) - \Delta_{p+\nu}(K_- \otimes^\nu [2]) = (t + (-1)^{\nu+1}) \cdot \Delta_{p+\nu}(K_0 \otimes^\nu [2]).$$

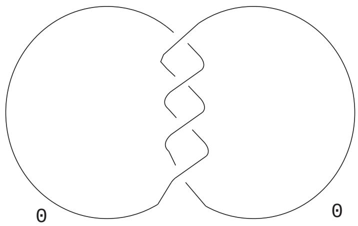
10. A remark on the $\mathbb{Z}[t, t^{-1}]$-case. Some of our results on polynomial invariants could be extended to the case where the word, '$\mathbb{Q}[t, t^{-1}]$-balanced class,' is replaced with the word, '($\mathbb{Z}[t, t^{-1}]$-balanced class of) an element of $\mathbb{Z}[t, t^{-1}]$'. However we must take care of the following Theorem 10.1.

Two polynomials $f(t), g(t) \in \mathbb{Z}[t, t^{-1}]$ are said to be $\mathbb{Z}[t, t^{-1}]$-*balanced* if there is an integer n such that $f(t) = \pm t^n \cdot g(t)$.

THEOREM 10.1. *For a natural number n there is a smooth oriented codimension two closed submanifold $K \subset S^{n+2}$ with the following properties.*

(1) *Let $1 \leqq p \leqq n$. Any Seifert hypersurface for K satisfies the condition that, for any p, the p-th Betti number is equal to the $(n+1-p)$-th Betti number.*
(2) *There are Seifert hypersurfaces V and W for K such that, for a nonnegative integer p, the $\mathbb{Z}[t, t^{-1}]$-balanced class of the determinant of a p-Alexander matrix of V and that of W are different although the linear map defined by a $(p-1)$-Alexander matrix associated with V (resp. W) is injective.*

Proof. Let K be a closed oriented smooth 3-dimensional submanifold $\subset S^5$ such that the diffeomorphism type of K is represented by the following framed link: Take the $(2, 2a)$-torus link $\subset S^3$ $(a \in \mathbb{N} - \{1\})$. It is a 2-component 1-link with the linking number a. Suppose that the framing of each component is zero. Then K is diffeomorphic to a homology sphere and, therefore, any Seifert hypersurface for K satisfies condition (1) in Theorem 10.1. The following framed link is the $a = 2$ case of K.



We make two kinds of Seifert hypersurfaces V, W for K as follows.

The first. Regard $\mathbb{R}^5 = \mathbb{R}^4 \times \{t \in \mathbb{R}\}$. Regard the above framed link which represents K as a 4-manifold. This 4-manifold has a handle decomposition

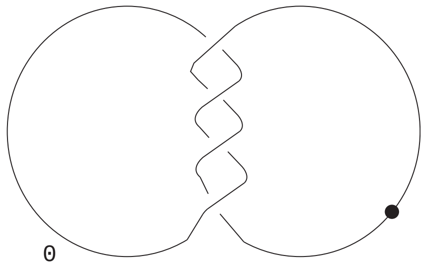
(a 4-dimensional 0-handle) $\cup$ (a 4-dimensional 2-handle) $\cup$ (a 4-dimensional 2-handle),

which is defined by the framed link representation. Suppose that the diffeomorphism type of a Seifert hypersurface V is this 4-manifold. Suppose that V in $\mathbb{R}^5$ satisfies:

(1) The 4-dimensional 0-handle is embedded in $\mathbb{R}^4 \times \{t = 0\}$.
(2) One of the 4-dimensional 2-handles is embedded in $\mathbb{R}^4 \times \{t = 0\}$, call it h^2.
(3) The other of the 4-dimensional 2-handles is embedded in $\mathbb{R}^4 \times \{t \leqq 0\}$. Only the attached part is embedded in $\mathbb{R}^4 \times \{t = 0\}$. We can do this because the framing is zero.

Thus we obtain a Seifert hypersurface V for K. Then we can suppose the following: a positive 2-Seifert matrix associated with V is $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$. The negative 2-Seifert matrix associated with the positive 2-Seifert matrix is $\begin{pmatrix} 0 & 0 \\ -a & 0 \end{pmatrix}$. Hence the 2-Alexander matrix associated with these two matrices is $\begin{pmatrix} 0 & at \\ a & 0 \end{pmatrix}$. Its determinant is $-a^2 t$.
Note that the linear map defined by a 1-Alexander matrix associated with V is injective.

The second. Take the above Seifert hypersurface V. Suppose that a 5-dimensional 3-handle k^3 is embedded in $\mathbb{R}^4 \times \{t \geqq 0\}$. Attach k^3 along the 2-sphere embedded in V which makes the above 4-dimensional 2-handle h^2 into a 4-dimensional 1-handle as shown below by the framed link representation. Suppose that only the attach part is embedded in $\mathbb{R}^4 \times \{t = 0\}$. By this surgery by k^3, V is changed into another Seifert hypersurface W for K. Then the framed link representation of W is as follows: Take the $(2, 2a)$-torus link $\subset S^3$. The framing of one component is zero. The other component is the dot circle (see [13] for the dot circle). Then W is a rational homology ball. The following framed link is the $a = 2$ case of W.



Hence the following holds: the positive 2-Seifert matrix associated with W is 'empty'. The negative 2-Seifert matrix associated with W is 'empty'. Hence the 2-Alexander matrix associated with W is 'empty'. Note that the $\mathbb{Z}[t, t^{-1}]$-balanced class of the determinant of the 2-Alexander matrix 'empty' is that of 1. Note that the linear map defined by a 1-Alexander matrix associated with W is injective. Note that the $\mathbb{Z}[t, t^{-1}]$-balanced class of the determinant of the 2-Alexander matrix 'empty' is NOT that of $a^2 t$. (Recall that we define $a \in \mathbb{N} - \{1\}$.) This completes the proof of Theorem 10.1. ■

11. Proof of theorems in Section 7

Proof of Theorem 7.1. By induction it suffices to prove the $\nu = 1$ case. Take $(2m + 1)$-dimensional oriented closed submanifolds $K_+, K_- \subset S^{2m+3} = \partial B^{2m+4} \subset B^{2m+4}$. Suppose that the $(2m + 1)$-dimensional oriented closed submanifolds, K_+, K_-, differ by only one twist-move in a $(2m + 3)$-ball A trivially embedded in S^{2m+3}. See the left two figures in the upper half of Fig. 3.3.2.

Take a Seifert hypersurface V_+ (resp. V_-) for the $(2m+1)$-dimensional oriented closed submanifold K_+ (resp. K_-) such that the submanifolds, V_+, V_-, differ by only one twist-move in the $(2m + 3)$-ball $A \subset S^{2m+3}$ and that $V_* \cap A$ is the $(2m + 2)$-dimensional $(m + 1)$-handle $h_* (* = +, -)$. See the definition of twist-moves in Section 3.3.

Take the collar neighborhood $A \times [0, 1]$ of B^{2m+4}. Note that $\overline{B^{2m+4} - (A \times [0, 1])}$ and $A \times [0, 1]$ are diffeomorphic to the $(2m + 4)$-ball, where $\overline{Q}$ denotes the closure of Q in B^{2m+4} if $Q \subset B^{2m+4}$. Note that the intersection of the two $(2m + 4)$-balls is $(\partial A \times [0, 1]) \cup (A \times \{1\})$ and is diffeomorphic to the $(2m + 3)$-ball.

Push V_+ (resp. V_-) into B^{2m+4}, fixing $\partial V_+ = K_+$ (resp. $\partial V_- = K_-$), call the submanifold $\subset B^{2m+4}$, V'_+ (resp. V'_-). Suppose that the submanifolds V'_+, V'_- differ only in the $(2m+4)$-ball $A \times [0,1] \subset B^{2m+4}$. Suppose that $V'_* \cap (A \times [0,1])$ is

$$\big(((\partial V_*) \cap A) \times [0, \tfrac{1}{2}]\big) \cup \big((V_* \cap A) \times \{\tfrac{1}{2}\}\big) \quad (* = +, -).$$

Let $B^2 = \{(x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \leqq 1\}$. Then there are smooth maps $f_+ : B^{2m+4} \to B^2$ and $f_- : B^{2m+4} \to B^2$ with the following properties. (Reason: Use the Thom–Pontrjagin construction.)

(i) The submanifold $f_+^{-1}((0,0)) \subset B^{2m+4}$ is V'_+. The submanifold $f_-^{-1}((0,0)) \subset B^{2m+4}$ is V'_-.

(ii) f_+ and f_- coincide on $\overline{B^{2m+4} - (A \times [0,1])}$.

The knot product $K_* \otimes$ (the empty knot [2]) is defined as follows ($* = +, -$). See Section 2.3 and [8, 10] for knot products. Take a smooth map $g : B^2 \to B^2$ such that $(r\cos\theta, r\sin\theta) \mapsto (r\cos 2\theta, r\sin 2\theta)$, where we use the standard polar coordinate. Recall that $g|_{\partial B^2 = S^1} : S^1 \to S^1$ is the empty knot of degree two.

Let $M_* = \{(x,y) \in B^{2m+4} \times B^2 \,|\, f_*(x) - g(y) = (0,0) \in B^2\}$.

The $(2m+3)$-dimensional closed oriented submanifold $\partial M_* \subset \partial(B^{2m+4} \times B^2)$ is the knot product $K_* \otimes [2]$ in the standard $(2m+5)$-sphere. Note that $\partial(B^{2m+4} \times B^2)$ is the standard $(2m+5)$-sphere.

$\partial\big((A \times [0,1]) \times B^2\big) \cap \partial\big(B^{2m+4} \times B^2\big)$ is the $(2m+5)$-ball, call it $\check{A}$.

$\partial\big(\overline{B^{2m+4} - (A \times [0,1])} \times B^2\big) \cap \partial(B^{2m+4} \times B^2)$ is also the $(2m+5)$-ball.

It is $(\partial(B^{2m+4} \times B^2)) - \operatorname{Int} \check{A}$.

By [8, 10] we have the following.

(1) ∂M_* is the double branched covering space of ∂B^{2m+4} along K_*.

(2) A Seifert hypersurface for ∂M_* is the double branched covering space of B^{2m+4} along V'_*.

By (ii) several lines above here, there is a diffeomorphism map

$$\alpha : \partial(B^{2m+4} \times B^2) \to \partial(B^{2m+4} \times B^2)$$

with the following properties:

$$\alpha|_{(\partial(B^{2m+4} \times B^2)) - \operatorname{Int} \check{A}} \text{ is the identity map.} \tag{1}$$

$$\alpha|_{(\partial(B^{2m+4} \times B^2)) - \operatorname{Int} \check{A}}\big((\partial M_+) - \operatorname{Int} \check{A}\big) = (\partial M_-) - \operatorname{Int} \check{A}. \tag{2}$$

$$\begin{aligned} \alpha|_{(\partial(B^{2m+4} \times B^2)) - \operatorname{Int} \check{A}}&\big((\text{the Seifert hypersurface for } \partial M_+) - \operatorname{Int} \check{A}\big) \\ &= (\text{the Seifert hypersurface for } \partial M_-) - \operatorname{Int} \check{A}. \end{aligned} \tag{3}$$

By [8, 10] we have the following.

(1) $\check{A} \cap \partial M_*$ is the double branched covering space of A along $A \cap K_*$ and is $S^{m+1} \times D^{m+2}$. Note that $A \cap K_*$ is $S^m \times D^{m+1}$.

(2) $\check{A} \cap$ (the Seifert hypersurface for ∂M_*) is the double branched covering space of $A \times [0,1]$ along $(A \times [0,1]) \cap V'_*$ and is $D^{m+2} \times D^{m+2}$. Note that $(A \times [0,1]) \cap V'_*$ is $D^{m+1} \times D^{m+1}$.

Hence the intersection of $\check{A} \cap$ (the Seifert hypersurface for ∂M_*), which is $D^{m+2} \times D^{m+2}$, and the standard $(m+4)$-sphere $\partial\check{A}$ is $S^{m+1} \times D^{m+2}$, which is $\check{A} \cap \partial M_*$. Thus we can regard this $D^{m+2} \times D^{m+2}$ as a $(2m+4)$-dimensional $(m+2)$-handle embedded in the standard $(2m+5)$-ball $\check{A}$ which is attached to the standard $(2m+4)$-sphere $\partial \check{A}$.

Since K_+ and K_- differ by only one twist-move, we can suppose that there is a $(m+1)$-Seifert matrix $X_* = (x^*_{i,j})$ for K_* with the following property, where $* = +, -$ (if necessary, change Seifert hypersurface by using embedded handles):

$$\begin{cases} x^+_{11} = 1, & x^-_{11} = 0 \\ x^+_{ij} = x^-_{ij} & \text{if } (i,j) \neq (1,1),\ i,j \leqq \nu,\ i,j \in \mathbb{N}. \end{cases}$$

Here, x^*_{11} is derived from $A \cap V_*$ $(* = +, -)$.

By [8, 10] and Proposition 5.4, we can suppose that the $\nu \times \nu$-matrix $-X_* = (-x^*_{i,j})$ is a $(m+2)$-Seifert matrix for $K_* \otimes [2]$. Here, $-x^*_{11}$ is derived from

$$\check{A} \cap (\text{the Seifert hypersurface for } \partial M_*) \quad (* = +, -).$$

Recall the definition of twist-moves in Section 3. By the above two paragraphs, $K_+ \otimes [2]$ and $K_- \otimes [2]$ differ by a single twist-move. This completes the proof of Theorem 7.1. ∎

Here, we prove the following Proposition 11.1, which is used in the proof of Theorem 9.4 in Section 13.

Suppose that two $(2m+1)$-dimensional oriented closed submanifolds $\subset S^{2m+3}$, K_+ and K_-, differ by a single twist-move as in the proof of Theorem 7.1. By Theorem 7.1, the $(2m+2\nu+1)$-submanifolds $\subset S^{2m+2\nu+3}$, $K_+ \otimes^\nu [2]$ and $K_- \otimes^\nu [2]$, differ by a single twist-move in a $(2m+2\nu+3)$-ball $\check{A}$. Then there is a unique closed oriented $(2m+2\nu+1)$-submanifold $K_0^\otimes \subset S^{2m+2\nu+3}$ such that a triple $(K_+ \otimes^\nu [2], K_- \otimes^\nu [2], K_0^\otimes)$ is related by a single twist-move in $\check{A}$. Note that the equivalence class of the submanifold $K_0^\otimes \subset S^{2m+2\nu+3}$ is determined uniquely. Note that we have the following. Take the Seifert hypersurface for ∂M_* in the $(2m+2\nu+1)$ case of the proof of Theorem 7.1 $(* = +, -)$. Then $\partial\big((\text{the Seifert hypersurface for } \partial M_*) - \operatorname{Int} \check{A}\big)$ in $S^{2m+2\nu+3}$ is $K_0^\otimes \subset S^{2m+2\nu+3}$ $(* = +, -)$. Make K_0 from K_+ as defined in Section 3.3. Then we have the following.

Proposition 11.1. *The $(2m+2\nu+1)$-submanifolds $\subset S^{2m+2\nu+3}$, $K_0^\otimes$ and $K_0 \otimes^\nu$ (the empty knot $[2]$) are equivalent.*

Proof. By induction it suffices to prove the $\nu = 1$ case.

Take V'_+ and V'_- as in the proof of Theorem 7.1. The submanifolds

$$V'_+ \cap \overline{B^{2m+4} - (A \times [0,1])} \text{ in } \overline{B^{2m+4} - (A \times [0,1])}$$
$$\text{and } V'_- \cap \overline{B^{2m+4} - (A \times [0,1])} \text{ in } \overline{B^{2m+4} - (A \times [0,1])}$$

are equivalent. Furthermore, the submanifold

$$\partial(V'_+ \cap \overline{B^{2m+4} - (A \times [0,1])}) \text{ in } \partial(\overline{B^{2m+4} - (A \times [0,1])}) \quad (* = +, -)$$

is equivalent to K_0 in the standard $(2m+3)$-sphere.

Take $M_* \cap \big(\overline{B^{2m+4} - (A \times [0,1])} \times B^2\big)$ $(* = +, -)$. By the construction, the submanifold $\partial(M_* \cap \big(\overline{B^{2m+4} - (A \times [0,1])} \times B^2\big)$ in $\partial\big(\overline{B^{2m+4} - (A \times [0,1])} \times B^2\big)$ $(* = +, -)$ is $K_0^\otimes$ and $K_0 \otimes^\nu [2]$ in the standard $(2m+2\nu+3)$-sphere. ∎

Proof of Theorem 7.2, part (1). Take the 1-knot K in Fig. 11.1.

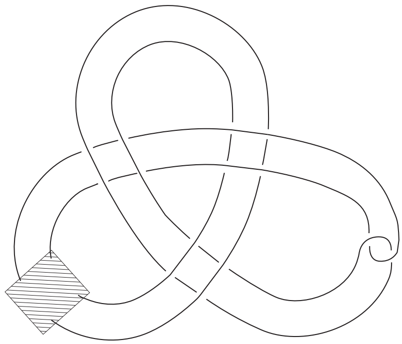


Fig. 11.1. A 1-knot: Twist in the shaded part so that its Seifert matrix is $\left(\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right)$

By [34], K is a nontrivial knot. Note that the unknotting number of K is one.

By [8, 10] and Proposition 5.4, the $(2\nu+1)$-submanifold $K \otimes^\nu [2] \subset S^{2\nu+3}$ has a Seifert hypersurface V with the following conditions.

(i) V has a handle decomposition

$$(\text{a } (2\nu+2)\text{-dimensional 0-handle}) \cup ((2\nu+2)\text{-dimensional } (\nu+1)\text{-handles}),$$

where there may be no $(2\nu+2)$-dimensional $(\nu+1)$-handle.

(ii) A Seifert matrix S associated with V is $\left(\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}-1&-1\\0&0\end{smallmatrix}\right)$.

Note that $S+(-1)^{\nu+1}({}^tS)$ represents the intersection product of $H_{\nu+1}(V;\mathbb{Z})/(\mathrm{Tor})$. Recall $\nu \geqq 2$. By (i), $\mathrm{Tor}\, H_{\nu+1}(V;\mathbb{Z}) \cong 0$. Since the determinant of $S+(-1)^{\nu+1}({}^tS)$ is $+1$ or -1, $\partial V = K \otimes^\nu [2]$ is a homology sphere. By (i), $\pi_1 \partial V = 1$. By [30], ∂V is homeomorphic to the standard sphere. Hence the $(2\nu+1)$-submanifold $K \otimes^\nu [2]$ is a spherical knot. By (i), $K \otimes^\nu [2]$ is a simple knot.

Recall that the trivial $(2\nu+1)$-knot has Seifert matrices, $\left(\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}-1&-1\\0&0\end{smallmatrix}\right)$. By (ii) and [16], $K \otimes^\nu [2]$ is equivalent to the trivial knot. ∎

Note. For the above K the diffeomorphism type of $K \otimes [2]$ is the 3-manifold which is the double branched covering space of S^3 along K (see [8, 10]). It is not homeomorphic to the standard 3-sphere (see [18, 31]).

Proof of Theorem 7.2, part (2). Take the above nontrivial 1-knot K. Take the knot-sum $K \sharp K$. By [6, 27] and the van Kampen's theorem, $K \sharp K$ is nontrivial. By [29], the unknotting number of $K \sharp K$ is two.

Note $(K \sharp K) \otimes^\nu [2]$ is the trivial knot (recall $\nu \geqq 2$).

$K \sharp K$ is an example which we want. ∎

Proof of Theorem 7.2, part (3). The pair of K and $K \sharp K$ is an example which we want. Reason: K is a prime knot and $K \sharp K$ is not. Hence K is not equivalent to $K \sharp K$. $K \otimes^\nu [2]$ and $(K \sharp K) \otimes^\nu [2]$ are trivial knots and hence equivalent spherical knots. ∎

Proof of Theorem 7.2, part (4). Let A be the trefoil knot (and hence a nontrivial 1-knot). By [6, 27] and the van Kampen's theorem, $A \sharp K$ is nontrivial. By [29], the unknotting number of $A \sharp K$ is two. By [8, 10] and [16], $(A \sharp K) \otimes^{\mu}$ (the Hopf link) and $A \otimes^{\mu}$ (the Hopf link) are equivalent and nontrivial. By Theorem 7.1, $A \otimes^{\mu}$ (the Hopf link) is obtained from the trivial knot by a single twist-move. Hence $(A \sharp K) \otimes^{\mu}$ (the Hopf link) is obtained from the trivial knot by a single twist-move. Hence $A \sharp K$ is an example which we want. ∎

Proof of Theorem 7.2, part (5). The pair of A and $A \sharp K$ is an example which we want. ∎

In order to prove Theorem 7.3, we prove some propositions and a theorem.

PROPOSITION 11.2. *Let $l \in \mathbb{N}$. Let K be a simple $(2l+1)$-knot. Then $K \otimes$ (the Hopf link) is a simple knot.*

Proof. By [8, 10] and Proposition 5.4, $K \otimes$ (the Hopf link) satisfies the following.
(1) A Seifert hypersurface has a handle decomposition of one $(2l+6)$-dimensional 0-handle and $(2l+6)$-dimensional $(l+3)$-handles, where there may not be $(2l+6)$-dimensional $(l+3)$-handles.
(2) A $(l+3)$-Seifert matrix Y associated with the above Seifert hypersurface is

$$(-1) \times (\text{a } (l+1)\text{-Seifert matrix of } K).$$

By (2) and the fact that K is PL homeomorphic to the single sphere, $\det(Y - {}^tY)$ is ± 1. By this fact and the above (1) (2) and [30], $K \otimes$ (the Hopf link) is a spherical knot and a simple knot. ∎

PROPOSITION 11.3. *Let $p \in \mathbb{N}$. Suppose that K is a simple $(2p+5)$-knot and that, if $2p+5=7$, the signature of K is a multiple of 16. Then there is a simple $(2p+1)$-knot A with the following properties.*

(i) *K is equivalent to $A \otimes$ (the Hopf link).*
(ii) *If X is a $(p+3)$-Seifert matrix of K, then $-X$ is a $(p+1)$-Seifert matrix of A.*
(iii) *The equivalence class of such a simple knot is unique.*

Proof. Take a $(p+3)$-Seifert matrix X of K. By [16] and Proposition 5.4, there is a simple $(2p+1)$-knot A such that a $(p+1)$-Seifert matrix is the matrix $-X$. By Propositions 5.4 and 11.2, $A \otimes$ (the Hopf link) is a simple $(2p+5)$-knot such that a $(p+2)$-Seifert matrix is the matrix X. By [16], $A \otimes$ (the Hopf link) is equivalent to K. By [16], (iii) holds. ∎

PROPOSITION 11.4. *Let K be a 1-knot. Let $\mu \in \mathbb{N}$. Then $K \otimes^{\mu}$ (the Hopf link) is a $(4\mu+1)$-dimensional simple knot (and hence a spherical knot).*

Proof. By [8, 10] and Proposition 5.4, $K \otimes^{\mu}$ (the Hopf link) satisfies the following.
(1) A Seifert hypersurface has a handle decomposition of one $(4\mu+2)$-dimensional 0-handle and $(4\mu+2)$-dimensional $(2\mu+1)$-handles, where there may not be $(2\mu+1)$-handles.
(2) Let Y be a Seifert matrix of K. A Seifert matrix associated with the above Seifert hypersurface is Y or $-Y$.

By (2) and the fact that K is diffeomorphic to the single circle, $\det(Y - {}^tY)$ is ± 1. By this fact and the above (1) (2) and [30], $K \otimes^{\mu}$ (the Hopf link) is a spherical knot and a simple knot. ∎

Proposition 11.5. *Let K be a simple 5-knot. Then there is a 1-knot A such that K is equivalent to $A \otimes$ (the Hopf link).*

Proof. Take a 3-Seifert matrix X of K. By using a Seifert surface, there is a 1-knot A such that a Seifert matrix is $-X$. By Propositions 5.4 and 11.4, $A \otimes$ (the Hopf link) is a simple 5-knot such that a 3-Seifert matrix is X. By [16], $A \otimes$ (the Hopf link) is equivalent to K. ∎

Note. The equivalence class of A is not unique. There are countably infinitely many equivalence classes of 1-knots with this property. Reason: Use the nontrivial knot K in the proof of Theorem 7.2. Take knot-sums as many time as we need.

Proposition 11.6. *Let $k \in \mathbb{N}$. Let K be a simple $(4k+1)$-knot. Let J be a $(4k+1)$-submanifold in S^{4k+3} such that J is obtained from K by a single twist-move. Then J is a simple knot.*

Proof. By the definition of twist-moves, there is a $(4k+3)$-ball B trivially embedded in S^{4k+3} in which this twist-move is carried out.

By using the Thom–Pontrjagin construction, we can prove that there is a Seifert hypersurface V_K for K and V_J for J with the following properties.
(1) $V_K \cap B$ (resp. $V_J \cap B$) is a $(4k+2)$-dimensional $(2k+1)$-handle h that is attached to ∂B as explained in the definition of the twist-moves in Section 3.3.
(2) $V_K \cap (S^{4k+3} - \operatorname{Int} B) = V_J \cap (S^{4k+3} - \operatorname{Int} B)$.

We can suppose that the handle h makes an order zero $(2k+1)$-cycle in V_K (resp. V_J). The intersection product between an order zero $(2k+1)$-cycle and itself in a compact oriented $(4k+2)$-manifold is zero. By using this fact, the Meyer–Vietoris exact sequence and the van Kampen's theorem, J is homeomorphic to the standard sphere.

Let $N(J)$ (resp. $N(K)$) be the tubular neighborhood of J (resp. K) in S^{4k+3}. Then we have
$S^{4k+3} - \operatorname{Int} N(K) = (S^{4k+3} - \operatorname{Int} N(K) - \operatorname{Int} B) \cup$(a $(4k+3)$-dimensional $(2k+2)$-handle).
$S^{4k+3} - \operatorname{Int} N(J) = (S^{4k+3} - \operatorname{Int} N(J) - \operatorname{Int} B) \cup$(a $(4k+3)$-dimensional $(2k+2)$-handle).

By the definition of twist-moves, $S^{4k+3} - \operatorname{Int} N(K) - \operatorname{Int} B = S^{4k+3} - \operatorname{Int} N(J) - \operatorname{Int} B$.

Since K is a simple knot, $\pi_1(S^{4k+3} - \operatorname{Int} N(K)) = \mathbb{Z}$. Use the van Kampen's theorem for the above unions. Hence $\pi_1(S^{4k+3} - \operatorname{Int} N(J)) = \mathbb{Z}$.

Let $i \in \mathbb{N}$ and $i \leqq 2k$. There is an i-Seifert matrix X_{iK} (resp. X_{iJ}) for K (resp. J) such that $X_{iK} = X_{iJ}$. (Reason: Use V_K and V_J.) Consider the homology groups, the homotopy groups, and the fundamental group of the infinite cyclic covering space for K (resp. J). Hence J is a simple knot. ∎

We prove the following theorem.

THEOREM 11.7. *Let $n \in \mathbb{N} \cup \{0\}$. Let K be an n-dimensional closed oriented submanifold $\subset S^{n+2}$. Take a map*

$$K \mapsto K \otimes (\textit{the Hopf link})$$

from the set of n-dimensional closed oriented submanifolds $\subset S^{n+2}$ to the set of $(n+4)$-dimensional closed oriented submanifolds $\subset S^{n+6}$. Then we have the following.

(1) *Let K be a simple $(2l+1)$-knot ($l \geqq 2$, $l \in \mathbb{N}$). That is, suppose that the domain of the map is the set of simple $(2l+1)$-knots. Then the image of the map is the set of simple $(2l+5)$-knots. Furthermore the map*

$$\{\textit{simple } (2l+1)\textit{-knots}\} \to \{\textit{simple } (2l+5)\textit{-knots}\} : \qquad K \mapsto K \otimes (\textit{the Hopf link})$$

is a one-to-one map.

(2) *Let K be a simple 3-knot. That is, suppose that the domain of the map is the set of simple 3-knots. Then the image of the map is included in the set of simple 7-knots. Furthermore the map*

$$\{\textit{simple } 3\textit{-knots}\} \to \{\textit{simple } 7\textit{-knots}\} : \qquad K \mapsto K \otimes (\textit{the Hopf link})$$

is injective but not onto.

(3) *Let K be a 1-knot. That is, suppose that the domain of the map is the set of 1-knots. Then the image of the map is the set of simple 5-knots. Furthermore the map*

$$\{1\textit{-knots}\} \to \{\textit{simple } 5\textit{-knots}\} : \qquad K \mapsto K \otimes (\textit{the Hopf link})$$

is onto but not injective. The inverse image of any element by this map is an infinite set.

PROBLEM 11.8. What happens if we define the domain is another set in Theorem 11.7?

Proof of Theorem 11.7. Propositions 11.2 and 11.3 imply part (1).

There is a simple 7-knot with the following property:

$(*)$ The signature is a multiple of 8 but not a multiple of 16.

There is not a simple 3-knot with the above property $(*)$. See [16].

By these facts and Propositions 11.2 and 11.3, part (2) holds.

Part (3) follows from Propositions 11.4 and 11.5, Note to Proposition 11.5, and Theorem 7.2. ■

Proof of Theorem 7.3. By the definition of the twist-move in Section 3.3 there is a $(2k+3)$-Seifert matrix X (resp. Y) for J (resp. K) with the following properties.

(1) X and Y are $c \times c$ matrices for a natural number c.

(2) Let x_{ij} denote (i,j)-element of X. Let y_{ij} denote (i,j)-element of Y. There is a natural number $a \leqq c$ such that

$$\begin{cases} x_{ij} = y_{ij} - 1 & \text{if } (i,j) = (a,a) \\ x_{ij} = y_{ij} & \text{if } (i,j) \neq (a,a). \end{cases}$$

(3) X and Y are not S-equivalent. See [16] for S-equivalent.

Note that we can take Seifert matrices which satisfy the above conditions. If necessary, carry out surgeries on a Seifert hypersurface by handles embedded in S^{4k+7}.

By Theorem 11.7, K is a simple knot. By Proposition 11.6, J is a simple knot.

By Proposition 5.4 and Theorem 11.7, there is a simple $(4k+1)$-knot K' (resp. J') with the following properties.

(i) J is equivalent to $J' \otimes$ (the Hopf link). $-X$ is a $(2k+1)$-Seifert matrix for J'.
(ii) K is equivalent to $K' \otimes$ (the Hopf link). $-Y$ is a $(2k+1)$-Seifert matrix for K'.
(iii) The equivalence class of such a simple knot is unique.

Therefore we can make a $(4k+1)$-dimensional simple knot $\bar{J}$ (resp. $\bar{K}$) with the following properties.

(I) A handle decomposition of a Seifert hypersurface is a set of a $(4k+2)$-dimensional 0-handle and $(4k+2)$-dimensional $(2k+1)$-handles, where there may not be a $(4k+2)$-dimensional $(2k+1)$-handle.
(II) A Seifert matrix associated with this Seifert hypersurface is $-X$ (resp. $-Y$).
(III) $\bar{J}$ and $\bar{K}$ differ by a single twist-move and are nonequivalent.

Reason: Since J (resp. K) is homeomorphic to the standard sphere, we can realize (I), (II) and (III) by using $(2k+1, 2k+1)$-pass-moves and twist-moves.

By [16], a simple $(4k+1)$-dimensional spherical knot $\bar{J}$ (resp. $\bar{K}$) is equivalent to J' (resp. K'). This completes the proof. ∎

Note.
(1) Let $p \in \{0\} \cup \mathbb{N}$. There are countably infinitely many different pairs $([J], [K])$ of different equivalence-classes of $(2p+5)$-dimensional closed oriented submanifolds $\subset S^{2p+7}$ with the following properties: J is a spherical knot. J is obtained from K by a single twist-move. Neither K or J is the product of any $(2p+1)$-dimensional closed oriented submanifold $\subset S^{2p+3}$ and the Hopf link.

Reason: We can prove $\pi_1(S^{2p+7} - K) \cong \pi_1(S^{2p+7} - J)$ in a similar manner to the proof of Proposition 11.6. If K (resp. J) is such a product, then $\pi_1(S^{2p+7} - K)$ (resp. $\pi_1(S^{2p+7} - J)$) is $\mathbb{Z}$ by [8, 10]. It is well-known that there are countably infinitely many spherical $(2p+5)$-knots the fundamental groups of whose complements are not $\mathbb{Z}$.
(2) In the $k = 0$ case in Theorem 7.3 there are countably infinitely many different pairs $([K'], [J'])$ of different equivalence-classes of 1-knots such that K (resp. J) is equivalent to $K' \otimes$ (the Hopf link) (resp. $J' \otimes$ (the Hopf link)) and that K' and J' differ by a single twist-move.

Reason: Refer to the proof of Theorem 7.3 and Note to the proof of Proposition 11.5.

12. Proof of theorems in Section 8

Proof of Theorem 8.1. There is a Seifert matrix X (resp. Y) for the 1-knot J (resp. K) with the following properties.
(1) X and Y are $c \times c$ matrices for a natural number c.
(2) Let x_{ij} denote (i,j)-element of X. Let y_{ij} denote (i,j)-element of Y. There are natural numbers $a, b \leqq c$ such that $a \neq b$ and that

$$\begin{cases} x_{ij} = y_{ij} - 1 & \text{if } (i,j) = (a,b) \\ x_{ij} = y_{ij} & \text{if } (i,j) \neq (a,b) \text{ and if } (i,j) \neq (b,a). \end{cases}$$

Recall that the (b,a)-element is determined by the (a,b)-element.

Note that we can take Seifert matrices which satisfy the above conditions. If necessary, carry out surgeries on Seifert hypersurfaces by 3-dimensional 1-handles embedded in S^3.

By Theorem 11.7, $J \otimes^{\mu}$ (the Hopf link) (resp. $K \otimes^{\mu}$ (the Hopf link)) is a spherical knot and is a simple knot.

We can make a $(4\mu+1)$-dimensional spherical knot J' (resp. K') with the following properties.
(1) A handle decomposition of a Seifert hypersurface is a set of a $(4\mu+2)$-dimensional $(2\mu+1)$-handle and $(4\mu+2)$-dimensional $(2\mu+1)$-handles, where there may not be $(4\mu+2)$-dimensional $(2\mu+1)$-handles.
(2) A Seifert matrix associated with the above Seifert hypersurface is $(-1)^{\mu}X$ (resp. $(-1)^{\mu}Y$).
(3) J' is obtained from K' by a single $(2\mu+1, 2\mu+1)$-pass-move.

Reason: Since J (resp. K) is diffeomorphic to the single circle.

By [16], simple $(4\mu+1)$-dimensional spherical knot J' (resp. K') is equivalent to $J \otimes^{\mu}$ (the Hopf link) (resp. $K \otimes^{\mu}$ (the Hopf link)).

By the construction of J' (resp. K'), $J \otimes^{\mu}$ (the Hopf link) and $K \otimes^{\mu}$ (the Hopf link) differ by a single $(2\mu+1, 2\mu+1)$-pass-move. ■

Proof of Theorem 8.3. The pass-move does not change diffeomorphism type of submanifolds. However $J \otimes^{\mu} [2]$ and $K \otimes^{\mu} [2]$ do not have the same homeomorphism type in general by [8, 10]. Example: (The trivial 1-knot)$\otimes[2]$ and (the trefoil knot)$\otimes[2]$. ■

Proof of Theorem 8.5. Note $l \geqq 1$. There is a Seifert matrix X (resp. Y) for the simple $(2l+1)$-knot J (resp. K) with the following properties.
(1) X and Y are $c \times c$ matrices for a natural number c.
(2) Let x_{ij} denote (i,j)-element of X. Let y_{ij} denote (i,j)-element of Y. There are integers $a, b \leqq c$ such that $a \neq b$. We have

$$\begin{cases} x_{ij} = y_{ij} - 1 & \text{if } (i,j) = (a,b) \\ x_{ij} = y_{ij} & \text{if } (i,j) \neq (a,b) \text{ and if } (i,j) \neq (b,a) \end{cases}$$

Recall that the (b,a)-element is determined by the (a,b)-element.

Note that we can take Seifert matrices which satisfy the above conditions. If necessary, carry out surgeries on Seifert hypersurfaces by handles embedded in S^{2l+3}.

By Theorem 11.7, $J \otimes^{\mu}$ (the Hopf link) (resp. $K \otimes^{\mu}$ (the Hopf link)) is a spherical knot and a simple knot.

We can make a $(2l+4\mu+1)$-dimensional spherical knot J' (resp. K') with the following properties.
(1) A handle decomposition of a Seifert hypersurface is a set of a $(2l+4\mu+2)$-dimensional 0-handle and $(2l+4\mu+2)$-dimensional $(l+2\mu+1)$-handles.
(2) A Seifert matrix associated with the above Seifert hypersurface is $(-1)^{\mu}X$ (resp. $(-1)^{\mu}Y$).
(3) J' is obtained from K' by a single $(l+2\mu+1, l+2\mu+1)$-pass-move.

Reason: Since J (resp. K) is homeomorphic to the standard sphere, we can realize (1), (2) and (3) by using $(l+2\mu+1, l+2\mu+1)$-pass-moves.

By [16], simple $(2l+4\mu+1)$-dimensional spherical knot J' (resp. K') is equivalent to $J \otimes^{\mu}$ (the Hopf link) (resp. $K \otimes^{\mu}$ (the Hopf link)).

By the construction of J' (resp. K'), $J \otimes^{\mu}$ (the Hopf link) and $K \otimes^{\mu}$ (the Hopf link) differ by a single $(l+2\mu+1, l+2\mu+1)$-pass-move. ■

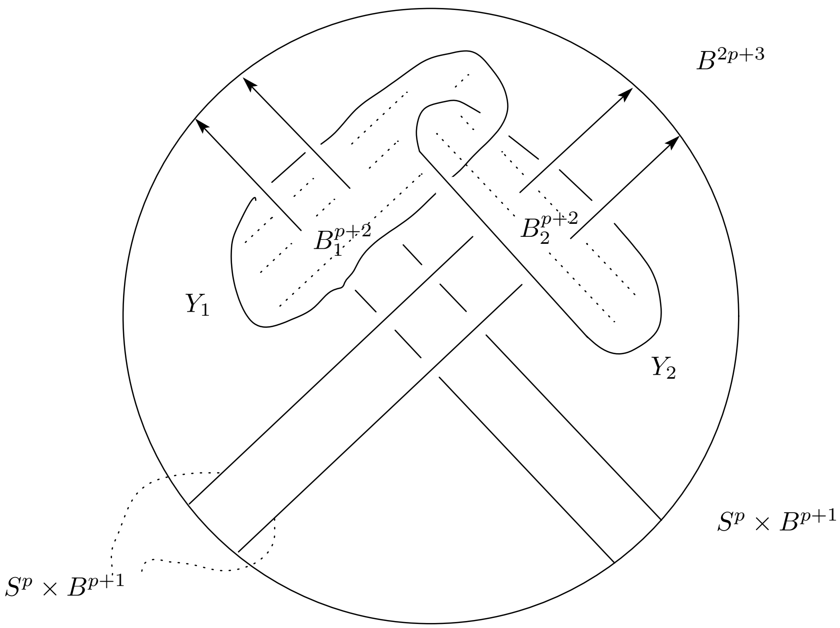
Proof of Theorem 8.8. Take the nontrivial 1-knot K in the figure in the proof of Theorem 7.2. Note that K is obtained from the trivial knot by a single pass-move. ■

NOTE. For the above K, the diffeomorphism type of $K \otimes [2]$ is not homeomorphic to the standard 3-sphere. See Note to the proof of Theorem 7.2.(1).

For the proof of Theorem 8.10 we need a proposition.

PROPOSITION 12.1. *Let $p \in \mathbb{N}$. Let K be a simple $(2p+1)$-knot. Let J be a $(2p+1)$-submanifold in S^{2p+3} such that J is obtained from K by a single $(p+1,p+1)$-pass-move. Then J is a simple knot.*

Proof. By the definition of the $(p+1,p+1)$-pass-move, J is a spherical $(2p+1)$-knot.



See the above figure. We can take two copies of the $(p+1)$-sphere Y_1 and Y_2 in $B^{2p+3} \subset S^{2p+3}$ with the following property. Y_1 (resp. Y_2) is embedded trivially. The linking number of Y_1 and Y_2 is one. Carry out surgeries along two $(p+1)$-spheres Y_1 and Y_2 by two $(2p+3)$-dimensional $(p+2)$-handles with the trivial framing on B^{2p+3}. Then B^{2p+3} becomes the $(2p+3)$-ball again and S^{2p+3} becomes the $(2p+3)$-sphere again. Furthermore the $(p+1,p+1)$-pass-move in the $(2p+3)$-ball B^{2p+3} is done.

Since $p \geqq 1$ holds and K is a simple knot, $\pi_i(S^{2p+3} - N(K)) = \pi_i(S^{2p+3} - N(J))$ for $1 \leqq i \leqq p$). (Use the van Kampen's theorem and the Meyer–Vietoris theorem on the complements and the infinite cyclic covering spaces.) Therefore J is a simple knot. This completes the proof of Proposition 12.1. ■

Proof of Theorem 8.10. By Theorem 11.7, K is a simple knot. By Proposition 12.1, J is a simple knot. By Proposition 11.5 and Theorem 11.7, there is a 1-knot J' to satisfy (i).

By [20], $K' \otimes^{\mu}$ (the Hopf link) and $J' \otimes^{\mu}$ (the Hopf link) have the same Arf invariant.

By [8, 10], a $(2\mu+1)$-Seifert matrix of $K' \otimes^{\mu}$ (the Hopf link) (of $J' \otimes^{\mu}$ (the Hopf link)) is $(\pm 1) \times$ a Seifert matrix of K' (of J'). Hence $\mathrm{Arf}(K') = \mathrm{Arf}(J')$

By Theorem 2.4.1, K' is pass-move-equivalent to J'. Hence J' satisfies (i) and (ii). This completes the proof of Theorem 8.10. ■

Proof of Note to Theorem 8.10. Use the nontrivial knot K in the proof of Theorem 7.2. Take a knot-sum as many times as we need. ■

Proof of Theorem 8.11. By Theorem 11.7, K is a simple knot. By Proposition 12.1, J is a simple knot.

Then there is a $(p+3)$-Seifert matrix X (resp. Y) for J (resp. K) with the following properties.

(1) X and Y are $c \times c$ matrices for a natural number c.

(2) Let x_{ij} denote (i,j)-element of X. Let y_{ij} denote (i,j)-element of Y. There are integers $a, b \leqq c$ such that $a \neq b$. We have

$$\begin{cases} x_{ij} = y_{ij} - 1 & \text{if } (i,j) = (a,b) \\ x_{ij} = y_{ij} & \text{if } (i,j) \neq (a,b) \text{ and if } (i,j) \neq (b,a). \end{cases}$$

Recall that the (b,a)-element is determined by the (a,b)-element.

(3) X and Y are not S-equivalent. See [16] for S-equivalent.

Note that we can take Seifert matrices which satisfy the above conditions. If necessary, carry out surgeries on Seifert hypersurfaces by handles embedded in S^{2p+7}.

By Proposition 11.3 and Theorem 11.7, there are simple $(2p+1)$-knots K' and J' with the following properties.

(i) J is equivalent to $J' \otimes$ (the Hopf link). $-X$ is a $(p+1)$-Seifert matrix for J'.
(ii) K is equivalent to $K' \otimes$ (the Hopf link). $-Y$ is a $(p+1)$-Seifert matrix for K'.
(iii) The equivalence classes of such simple knots are unique.

Therefore we can make a $(2p+1)$-dimensional simple knot $\bar{J}$ (resp. $\bar{K}$) with the following properties.

(I) A handle decomposition of a Seifert hypersurface is a set of a $(2p+2)$-dimensional 0-handle and $(2p+2)$-dimensional $(p+1)$-handles, where there may not be a $(2p+2)$-dimensional $(p+1)$-handle.

(II) A Seifert matrix associated with the above Seifert hypersurface is $-X$ (resp. $-Y$).

(III) $\bar{J}$ and $\bar{K}$ differ by a single $(p+1, p+1)$-pass-move and are nonequivalent.

Reason: Since J (resp. K) is homeomorphic to the standard sphere, we can realize (I), (II) and (III) by using $(p+1, p+1)$-pass-moves.

By [16], simple $(2p+1)$-dimensional spherical knot $\bar{J}$ (resp. $\bar{K}$) is equivalent to J' (resp. K'). This completes the proof. ■

Note. Let $p \in \{0\} \cup \mathbb{N}$. There are countably infinitely many different pairs $([J], [K])$ of different equivalence-classes of $(2p+5)$-dimensional closed oriented submanifolds $\subset S^{2p+7}$ with the following properties: J is a spherical knot. Neither K or J is the product of any $(2p+1)$-dimensional closed oriented submanifold $\subset S^{2p+3}$ and the Hopf link. J is obtained from K by a single $(p+1, p+1)$-pass-move.

Reason: We can prove $\pi_1(S^{2p+7} - K) \cong \pi_1(S^{2p+7} - J)$ in a similar manner of the proof of Proposition 12.1. If K (resp. J) is such a product, then $\pi_1(S^{2p+7} - K)$ (resp. $\pi_1(S^{2p+7} - J)$) is $\mathbb{Z}$ by [8, 10]. It is well-known that there are countably infinitely many spherical $(2p+5)$-knots the fundamental groups of whose complements are not $\mathbb{Z}$.

Proof of Theorem 8.12. The following was proved in [8, 10]. Let V be a Seifert hypersurface for an n-dimensional closed oriented submanifold $K \subset S^{n+2}$. Then there is a Seifert hypersurface W for the $(n+2)$-submanifold $K \otimes [2] \subset S^{n+4}$ such that W is diffeomorphic to $B^{n+3} \cup B^{n+3}$ and that $B^{n+3} \cap B^{n+3}$ is diffeomorphic to $V \times [-1, 1]$.

Note that P has a Seifert hypersurface which is diffeomorphic to the punctured Poincaré sphere.

Therefore, by the above theorem in [8, 10], the $(2\nu+2)$-submanifold $P \otimes^\nu [2]$ has a Seifert hypersurface Q which consists of a $(2\nu+3)$-dimensional 0-handle, $(2\nu+3)$-dimensional $(\nu+1)$-handles, and $(2\nu+3)$-dimensional $(\nu+2)$-handles. Note $\nu \geqq 2$. By the van Kampen's theorem, $\pi_1(\partial Q) = 1$. By using the Meyer–Vietoris exact sequence, Q is a homology $(2\nu+3)$-ball. By the van Kampen's theorem, Q is simply-connected.

By [30], Q is diffeomorphic to the $(2\nu+3)$-ball.

Hence $P \otimes^\nu [2]$ has a Seifert hypersurface which is diffeomorphic to the $(2\nu+3)$-ball.

Hence $P \otimes^\nu [2]$ is equivalent to the trivial knot. ∎

13. Proof of theorems in Section 9

Proof of Theorem 9.1. Part (1) follows from part (2) by Theorem 2.3.2. We prove part (2) of the theorem. By [10], $K_* \otimes^\nu [2]$ has a Seifert hypersurface V_* with a handle decomposition of one $(2\nu+2)$-dimensional 0-handle and $(2\nu+2)$-dimensional $(\nu+1)$-handles, where there may not be $(2\nu+2)$-dimensional $(\nu+1)$-handles $(* = +, -, 0)$. Therefore the ν-Alexander matrix associated with V_* is 'empty'. See Section 5.

By Proposition 5.4, a $(\nu+1)$-positive Seifert matrix $S_{\nu+1}(K_* \otimes^\nu [2])$ and a $(\nu+1)$-negative Seifert matrix $N_{\nu+1}(K_* \otimes^\nu [2])$ are square matrices. By Proposition 5.1, the $(\nu+1)$-$\mathbb{Q}[t, t^{-1}]$-Alexander polynomial of $K_* \otimes^\nu [2]$ is the $\mathbb{Q}[t, t^{-1}]$-balanced class of the determinant of the $(\nu+1)$-Alexander matrix

$$\begin{aligned}
P_{\nu+1}(K_* \otimes^\nu [2]) &= (p^*_{ij}) = t \cdot S_{\nu+1}(K_* \otimes^\nu [2]) - N_{\nu+1}(K_* \otimes^\nu [2]) \\
&= t \cdot S_{\nu+1}(K_* \otimes^\nu [2]) + (-1)^{\nu+1} \cdot {}^tS_{\nu+1}(K_* \otimes^\nu [2]) \\
&\text{(Reason: } S_{\nu+1}(K_* \otimes^\nu [2]) = (-1)^\nu \, {}^tN_{\nu+1}(K_* \otimes^\nu [2]) \text{ by Section 5.)} \\
&= (-1)^{\nu(\nu-1)/2} \, (t \cdot S_1(K_*) + (-1)^{\nu+1} \cdot {}^tS_1(K_*)). \\
&\text{(Reason: } S_{\nu+1}(K_* \otimes^\nu [2]) = (-1)^{\nu(\nu-1)/2} \quad S_1(K_*) \text{ by Proposition 5.4.(7).)}
\end{aligned}$$

Since (K_+, K_-, K_0) is a crossing-change-triple of 1-links, we can suppose that $S_1(K_*) = (s^*_{ij})$ $(* = +, -, 0)$ has the following properties.

(1) $$(s^+_{ij}) \text{ and } (s^-_{ij}) \text{ are } \rho \times \rho \text{ matrices}$$
$$(s^0_{ij}) \text{ is a } (\rho - 1) \times (\rho - 1) \text{ matrix} \quad (\rho \geqq 2,\ \rho \in \mathbb{N})$$

(2) $$s^+_{\rho,\rho} - s^-_{\rho,\rho} = 1$$

(3) $$s^+_{ij} = s^-_{ij} = s^0_{ij} \quad (1 \leqq i \leqq \rho - 1,\ 1 \leqq j \leqq \rho - 1), \qquad s^+_{ij} = s^-_{ij} \quad ((i,j) \neq (\rho, \rho)).$$

Note we can suppose that $\rho - 1 \geqq 1$ by using surgeries of Seifert hypersurfaces by embedded handles, if necessary.

Therefore we can suppose that $P_{\nu+1}(K_* \otimes^\nu [2]) = (p^*_{ij})$ has the following properties $(* = +, -, 0)$.

(1) $$(p^+_{ij}) \text{ and } (p^-_{ij}) \text{ are } \rho \times \rho \text{ matrices}$$
$$(p^0_{ij}) \text{ is a } (\rho - 1) \times (\rho - 1) \text{ matrix} \quad (\rho \geqq 2,\ \rho \in \mathbb{N})$$

(2) $$p^+_{\rho,\rho} - p^-_{\rho,\rho} = c(t + (-1)^{\nu+1}), \quad \text{where } c = \pm 1$$

(3) $$p^+_{ij} = p^-_{ij} = p^0_{ij} \quad (1 \leqq i \leqq \rho - 1,\ 1 \leqq j \leqq \rho - 1), \qquad p^+_{ij} = p^-_{ij}((i,j) \neq (\rho, \rho)).$$

By calculus of determinants,

$$\det P_{\nu+1}(K_+ \otimes^\nu [2]) - \det P_{\nu+1}(K_- \otimes^\nu [2]) = c(t + (-1)^{\nu+1}) \cdot \det P_{\nu+1}(K_0 \otimes^\nu [2]),$$

where $c = \pm 1$. Hence Theorem 9.1.(2) holds. Note that the $\mathbb{Q}[t, t^{-1}]$-Alexander polynomial is a $\mathbb{Q}[t, t^{-1}]$-balanced class. ∎

Proof of Theorem 9.2. The 'l = even' case is proved in [24]. We prove the 'l = odd' case. There is a Seifert hypersurface V_* for K_* $(* = +, -, 0)$ such that (V_+, V_-, V_0) is related by a single twist-move in B^{2l+3}.

Take a positive Seifert matrix $S_\sharp(K_*)$ and a negative Seifert matrix $N_\sharp(K_*)$ for K_* associated with V_* $(* = +, -, 0)$. We can suppose that $S_l(K_+) = S_l(K_-) = S_l(K_0)$ and that $N_l(K_+) = N_l(K_-) = N_l(K_0)$. Since K_+ is a spherical knot, the linear map which is defined by $S_l(K_+) - N_l(K_+)$ is injective. Hence the linear map which is defined by $S_l(K_*) - N_l(K_*)$ is injective $(* = +, -, 0)$. Hence the l-Alexander matrix $t \cdot S_l(K_*) - N_l(K_*)$ associated with V_* is injective. By Proposition 5.1, the $(l+1)$-$\mathbb{Q}[t, t^{-1}]$-Alexander polynomial of K_* is the $\mathbb{Q}[t, t^{-1}]$-balanced class of the determinant of the $(l+1)$-Alexander matrix

$$P_{l+1}(K_*) = t \cdot S_{l+1}(K_*) - N_{l+1}(K_*).$$

Since (K_+, K_-, K_0) is a twist-move-triple, we can suppose that $S_{l+1}(K_*) = (s^*_{ij})$ $(* = +, -, 0)$ has the following property.

(1) $$(s^+_{ij}) \text{ and } (s^-_{ij}) \text{ are } \rho \times \rho \text{ matrices}$$
$$(s^0_{ij}) \text{ is a } (\rho - 1) \times (\rho - 1) \text{ matrix} \quad (\rho \geqq 2,\ \rho \in \mathbb{N})$$

(2) $$s^+_{\rho,\rho} - s^-_{\rho,\rho} = \pm 1$$

(3) $$s^+_{ij} = s^-_{ij} = s^0_{ij} \quad (1 \leqq i \leqq \rho - 1,\ 1 \leqq j \leqq \rho - 1), \qquad s^+_{ij} = s^-_{ij} \quad ((i,j) \neq (\rho, \rho)).$$

Note we can suppose that $\rho - 1 \geqq 1$ by using surgeries of Seifert hypersurfaces by embedded handles, if necessary.

We have

$$P_{l+1}(K_*) = t \cdot S_{l+1}(K_*) - N_{l+1}(K_*) = t \cdot S_{l+1}(K_*) - (-1)^l \cdot {}^tS_{l+1}(K_*).$$

Since l is odd,

$$P_{l+1}(K_*) = t \cdot S_{l+1}(K_*) + {}^tS_{l+1}(K_*).$$

Therefore $P_{l+1}(K_*) = (p^*_{ij})$ satisfies the following ($* = +, -, 0$).

(1) $$(p^+_{ij}) \text{ and } (p^-_{ij}) \text{ are } \rho \times \rho \text{ matrices}$$
$$(p^0_{ij}) \text{ is a } (\rho - 1) \times (\rho - 1) \text{ matrix} \quad (\rho \geqq 2,\ \rho \in \mathbb{N})$$

(2) $$p^+_{\rho,\rho} - p^-_{\rho,\rho} = \pm(t + 1)$$

(3) $p^+_{ij} = p^-_{ij} = p^0_{ij} \quad (1 \leqq i \leqq \rho - 1,\ 1 \leqq j \leqq \rho - 1), \qquad p^+_{ij} = p^-_{ij} \quad ((i, j) \neq (\rho, \rho)).$

By calculus of determinants,

$$\det P_{l+1}(K_+) - \det P_{l+1}(K_-) = \pm(t + 1) \cdot \det P_{l+1}(K_0).$$

Hence Theorem 9.2 holds. Note that the $\mathbb{Q}[t, t^{-1}]$-Alexander polynomial is a $\mathbb{Q}[t, t^{-1}]$-balanced class. ■

Proof of Theorem 9.3. Take V_*, $S_\sharp(K_*)$, $N_\sharp(K_*)$ in the proof of Theorem 9.2. Since K_+ is a spherical knot, we see that $S_l(K_+)$ and $N_l(K_+)$ are square matrix and furthermore we can suppose that the determinant of $S_l(K_+) - N_l(K_+)$ is ± 1. Hence we can suppose that the determinant of $S_l(K_*) - N_l(K_*)$ is ± 1 ($* = +, -, 0$).

By [10], an $(l + \nu)$-Alexander matrix $P_{l+\nu}(K_* \otimes^\nu [2])$ is $\pm(t \cdot S_{l+\nu}(K_*) \pm {}^tN_{l+\nu}(K_*))$. If we let $t = 1$ or $t = -1$, then the determinant of $P_{l+\nu}(K_* \otimes^\nu [2])$ is not zero. Note that $P_{l+\nu}(K_* \otimes^\nu [2])$ is a square matrix because K_+ is a spherical knot. Hence the linear map which is defined by $P_{l+\nu}(K_* \otimes^\nu [2])$ is injective.

Hence the $(l+1+\nu)$-$\mathbb{Q}[t, t^{-1}]$-Alexander polynomial $A_{l+1+\nu}(K_* \otimes^\nu [2])$ is the $\mathbb{Q}[t, t^{-1}]$-balanced class of the determinant of an $(l + 1 + \nu)$-Alexander matrix $P_{l+1+\nu}(K_* \otimes^\nu [2])$.

Proposition 5.4 implies that

$$S_{l+1+\nu}(K_* \otimes^\nu [2]) = (-1)^\xi S_{l+1}(K_*).$$
$$N_{l+1+\nu}(K_* \otimes^\nu [2]) = (-1)^{\xi+\nu} N_{l+1}(K_*).$$

($* = +, -, 0$, ξ is a constant integer.)

Hence we can suppose that

$$\begin{aligned}
&P_{l+1+\nu}(K_* \otimes^\nu [2]) \\
&\quad = t \cdot (-1)^\xi S_{l+1}(K_*) - (-1)^{\xi+\nu} N_{l+1}(K_*) \\
&\quad = (-1)^\xi (t \cdot S_{l+1}(K_*) + (-1)^{\nu+1} N_{l+1}(K_*)) \\
&\quad = (-1)^\xi (t \cdot S_{l+1}(K_*) + (-1)^{\nu+1+l} \cdot {}^tS_{l+1}(K_*)).
\end{aligned}$$

(Reason: $S_{\nu+1}(K_*) = (-1)^l\, {}^tN_{\nu+1}(K_*)$ by Section 5.)

Hence we have the following. Let $P_{l+1+\nu}(K_* \otimes^\nu [2]) = (p^*_{ij})$.

$$\text{(1)} \qquad (p^+_{ij}) \text{ and } (p^-_{ij}) \text{ are } \rho \times \rho \text{ matrices}$$
$$(p^0_{ij}) \text{ is a } (\rho-1)\times(\rho-1) \text{ matrix} \quad (\rho \geqq 2,\ \rho \in \mathbb{N}).$$
$$\text{(2)} \qquad p^+_{\rho,\rho} - p^-_{\rho,\rho} = c(t + (-1)^{l+1+\nu}), \text{ where } c = \pm 1$$
$$\text{(3)} \qquad p^+_{ij} = p^-_{ij} = p^0_{ij} \quad (1 \leqq i \leqq \rho-1,\ 1 \leqq j \leqq \rho-1)$$
$$p^+_{ij} = p^-_{ij} \quad ((i,j) \neq (\rho,\rho)).$$

By calculus of determinants,

$$\det P_{l+1+\nu}(K_+ \otimes^\nu [2]) - \det P_{l+1+\nu}(K_- \otimes^\nu [2]) = c(t + (-1)^{l+1+\nu}) \cdot \det P_{l+1+\nu}(K_0 \otimes^\nu [2]),$$

where $c = \pm 1$. Hence Theorem 9.3 holds. Note that the $(l+1+\nu)$-$\mathbb{Q}[t,t^{-1}]$-Alexander polynomial is a $\mathbb{Q}[t,t^{-1}]$-balanced class. ■

Proof of Theorem 9.4. K_+ is a spherical $(4k+1)$-knot and K_- is obtained from K_+ by a single twist move. Hence $K_* \otimes^\mu$ (the Hopf link) is a spherical $(4k+4\mu+1)$-knot $(* = +, -)$. (K_+, K_-, K_0) is a twist-move-triple. By Proposition 11.1,

$$\big(K_+ \otimes^\mu \text{(the Hopf link)}, K_- \otimes^\mu \text{(the Hopf link)}, K_0 \otimes^\mu \text{(the Hopf link)}\big)$$

is a twist-move-triple. By Theorem 6.3, the proof is completed. ■

Proof of Theorem 9.5. Proposition 5.4 implies

$$S_{p+\nu}(K_* \otimes^\nu [2]) = (-1)^c S_p(K_*)$$
$$N_{p+\nu}(K_* \otimes^\nu [2]) = (-1)^{c+\nu} N_p(K_*),$$

where $* = +, -, 0$ and c is a constant integer.

Since K_+, K_- are spherical knots and (K_+, K_-, K_0) is a (p,q)-pass-move-triple, we have the following. There is a $(p+\nu)$-Alexander matrix

$$P_{p+\nu}(K_* \otimes^\nu [2]) = (-1)^c \big\{t \cdot S_p(K_* \otimes^\nu [2]) - (-1)^\nu \cdot N_p(K_* \otimes^\nu [2])\big\} = (p^*_{ij}),$$

which is a square matrix, with the following property.

$$\text{(1)} \qquad (p^+_{ij}) \text{ and } (p^-_{ij}) \text{ are } \rho \times \rho \text{ matrices}$$
$$(p^0_{ij}) \text{ is a } (\rho-1)\times(\rho-1) \text{ matrix} \quad (\rho \geqq 2,\ \rho \in \mathbb{N})$$
$$\text{(2)} \qquad p^+_{\rho,\rho} - p^-_{\rho,\rho} = (-1)^c\{t + (-1)^{1+\nu}\}$$
$$\text{(3)} \qquad p^+_{ij} = p^-_{ij} = p^0_{ij} \quad (1 \leqq i \leqq \rho-1,\ 1 \leqq j \leqq \rho-1)$$
$$p^+_{ij} = p^-_{ij} \quad ((i,j) \neq (\rho,\rho)).$$

Note we can suppose that $\rho - 1 \geqq 1$ by using surgeries of Seifert hypersurface by embedded handles, if necessary.

By calculus of determinants,

$$\det P_{p+\nu}(K_+ \otimes^\nu [2]) - \det P_{p+\nu}(K_- \otimes^\nu [2]) = (-1)^\zeta \cdot (t + (-1)^{\nu+1}) \cdot \det P_{p+\nu}(K_0 \otimes^\nu [2]), \quad (!)$$

where ζ is a constant integer.

Take $\det P_{p+\nu}(K_* \otimes^\nu [2])$ for each $*$ $(* = +, -, 0)$. Here, there are the following three cases (i), (ii) and (iii).

(i): Suppose that $\det P_{p+\nu}(K_* \otimes^\nu [2]) \neq 0$ for a $*$. Let $p-1 \neq n+1-p$. Then K_+, K_- and K_0 has the same $(p-1)$-Alexander matrix $t \cdot S_{p-1} - N_{p-1}$. The $(p-1)$-Alexander matrix has the following properties. Note that $t \cdot S_{p-1} - N_{p-1}$ is a square matrix. $t \cdot S_{p-1} - N_{p-1}$ is a nonsingular square matrix. Reason: K_+ and K_- are spherical knots. Hence if $t = 1$, $t \cdot S_{p-1} - N_{p-1} = S_{p-1} - N_{p-1}$ is nonsingular.

By [10], a $(p+\nu-1)$-Alexander matrix $P_{p+\nu-1}$ for $K_* \otimes^\nu [2]$ is one of $\pm\{t \cdot S_{p-1} \pm N_{p-1}\}$.
If $t = 1$ or $t = -1$, $P_{p+\nu-1}$ is nonsingular. Hence $P_{p+\nu-1}$ is nonsingular.

Hence the $(p+\mu)$-$\mathbb{Q}[t, t^{-1}]$-Alexander polynomial for $K_* \otimes^\nu [2]$ is $\det P_{p+\mu}(K_* \otimes^\nu [2])$ for the $*$.

(ii): Suppose that $\det P_{p+\nu}(K_* \otimes^\nu [2]) \neq 0$ for a $*$. Let $p-1 = n+1-p$. A $(p+\nu-1)$-Alexander matrix for $K_* \otimes^\nu [2]$ defines an injective map because a $(p-1)$-Alexander matrix for K_* does. See the identity right above Proposition 5.2.

Hence the $(p+\nu)$-$\mathbb{Q}[t, t^{-1}]$-Alexander polynomial for K_* is $\det P_{p+\nu}(K_* \otimes^\nu [2])$ for the $*$.

(iii): Suppose that $\det P_{p+\nu}(K_* \otimes^\nu [2]) = 0$ for a $*$. Then the $(p+\nu)$-$\mathbb{Q}[t, t^{-1}]$-Alexander polynomial for $K_* \otimes^\nu [2]$ is $0 = \det P_{p+\nu}(K_* \otimes^\nu [2])$ for the $*$. Here, note that we do not need to consider whether the linear map defined by a $(p+\nu-1)$-Alexander matrix is injective or not.

By the above (i), (ii), (iii) and the identity (!) several lines above here, the proof is completed. Note that the $(p+\nu)$-$\mathbb{Q}[t, t^{-1}]$-Alexander polynomial is a $\mathbb{Q}[t, t^{-1}]$-balanced class. ∎

14. A problem

PROBLEM 14.1. Suppose that n-dimensional closed oriented submanifolds $K, K' \subset S^{n+2}$ differ by a single twist-move (resp. pass-move). Suppose that m-dimensional closed oriented submanifolds $J, J' \subset S^{n+2}$ differ by a single twist-move (resp. pass-move). Then how do we characterize a relation between $K \otimes J$ and $K' \otimes J'$?

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