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LINKS AND CYCLES II: FAMILIES OF COMMUTATION RELATIONS, PARAMETRIZED BY KNOTS, IN THE MAPPING CLASS GROUPS AND BEYOND

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Abstract. We generate families of commutation relations in various groups, by examining quandle colorings of knots and their quandle 2-cycles. The colorings are via quandles associated to the given groups.

1. Introduction and preliminaries. In this article, we continue and expand an examination, begun in [20] of some relationships between, and applications of, colored links and Dehn quandle cycles. We concentrate on quandle colorings of certain pretzel knots and links, and derive some consequences involving families of commutation relations in various groups, from these colored links. In doing so, we broaden, from the original mapping class groups of the article above, the collection of groups in which these relations hold. Early studies of colorings of links as invariants were made by Fox, see e.g. [5], and the historical developments of coloring-based, and related knot and link invariants, are described in [14]. We concern ourselves mainly with knot colorings by elements of infinite quandles, with a reference back to classical (finite) colorings at the end.

For some groups, e.g. $SL(2,\mathbb{Z}) \cong MCG(\mathbb{T}^2)$, or the mapping class groups of higher genus surfaces, there exist "natural" intrinsic ways of finding an associated quandle, in which certain intersection properties of elements hold. For other groups, such an intrinsic quandle is not apparent, and we use the conjugation quandle (see [8]) of the group, along with somewhat ad hoc, less formally defined "intersection" properties existing among certain elements.

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Quandle preliminaries. A *quandle* Q is a set with two operations, written on the right of the operand, with the following axioms:

Q1) $a\,\overline{a}\rceil = a\lceil\overline{a} = a$ $\quad\forall\, a \in Q$ $\quad$ *"idempotence"*
Q2) $a\lceil\overline{b}\;\overline{b}\rceil = a = a\,\overline{b}\rceil\lceil\overline{b}$ $\quad\forall\, a, b \in S$ $\quad$ *"inverses"*
Q3) $(x\,\overline{a}\rceil)\,\overline{b}\rceil = (x\,\overline{b}\rceil)\,\overline{(a\overline{b}\rceil)}\rceil$ and $(x\lceil\overline{a})\lceil\overline{b} = (x\lceil\overline{b})\lceil\overline{(a\lceil\overline{b})}$ $\forall\, a, b, x \in S$ *"distributivity"*.

The notation $\overline{a}\rceil$ is often written $*\,a$, while $\lceil\overline{a}$ is often written $\bar{*}\,a$, so e.g. $x * a = x\,\overline{a}\rceil$. An algebraic object satisfying only Q2) and Q3) is called a *rack*.

Quandle examples. 1. The *trivial quandle* $\mathbf{T}$, on elements $\{a_1, a_2, a_3, \dots\}$, has trivial operations, $a_i\,\overline{a_j}\rceil = a_i\lceil\overline{a_j} = a_i$ for any $a_i, a_j \in \mathbf{T}$.

2. Any group G gives rise to a *conjugation quandle*, where for $a, g \in G$, we take $a\,\overline{g}\rceil = gag^{-1}$ and $a\lceil\overline{g} = g^{-1}ag$.

3. The set of $(n-1)$-dimensional planes in $\mathbf{R}^n$ forms a *reflection quandle*, where both the operations $\overline{x}\rceil$ and $\lceil\overline{x}$ correspond to reflection in the hyperplane x.

4. The *knot quandle* $Q(K)$ for an oriented knot diagram K, is an ambient isotopy invariant. Elements are labeled arcs of the diagram, and

$$a\,\overline{b}\rceil = c \Leftrightarrow \text{"arc } b \text{ crosses over arc } a \text{ from the right, to produce arc } c\text{"},$$
$$a\lceil\overline{b} = c \Leftrightarrow \text{"arc } b \text{ crosses over arc } a \text{ from the left, to produce arc } c\text{"}.$$

5. Any cyclic group $\mathbb{Z}_n$ may be given the structure of a *dihedral quandle* denoted R_n. For $a, b \in \mathbb{Z}_n$, we define

$$a\,\overline{b}\rceil = 2b - a = a\lceil\overline{b}\,.$$

6. Let M be a free module over a commutative ring R. Suppose $\langle\ ,\ \rangle : M \times M \to R$ is a non-degenerate bilinear form on M, with $\langle \mathbf{x}, \mathbf{x}\rangle = 0$. We get a *symplectic quandle* (see [13] or [17]) structure on M with the operations

$$\mathbf{x}\,\overline{\mathbf{y}}\rceil = \mathbf{x} + \langle \mathbf{x}, \mathbf{y}\rangle \mathbf{y} \quad \text{and} \quad \mathbf{x}\lceil\overline{\mathbf{y}} = \mathbf{x} - \langle \mathbf{x}, \mathbf{y}\rangle \mathbf{y}.$$

7. The *Dehn quandle*, *Dehn*(F) (see e.g. [18]), of an orientable surface F has isotopy classes of circles on F as elements, and action by the mapping class group, $MCG(F)$. For $a \in$ *Dehn*(F),

$$\overline{a}\rceil = \text{right Dehn twist on } a \ (\textit{bracket on right}),$$
$$\lceil\overline{a} = \text{left Dehn twists on } a \ (\textit{bracket on left}).$$

Quandle homology. There is a homology theory for quandles. Details and definitions etc. may be found in [1] or [2]. The related homology for racks was introduced by Fenn, Rourke, and Sanderson. We recall here some basic facts and definitions for quandles.

Let Q be a quandle. Let $C^R_n(Q)$ be the free abelian group generated by n-tuples $(x_1, \dots, x_n)$ of elements of Q. This is the rack chain group. Such a tuple will be referred to as an n-simplex. Note that this differs from the conventional definition where a k-simplex involves $k+1$ terms. We assume $C_0 = 0$.

The boundary homomorphism $\partial_n : C_n \to C_{n-1}$ is given by

$$\partial_n(x_1, \dots, x_n) = \sum_{i=2}^{n} (-1)^i \big[(x_1, x_2, \dots, x_{i-1}, \hat{x}_i, x_{i+1}, \dots, x_n) \\ - (x_1 \overline{x_i}\rceil, x_2 \overline{x_i}\rceil, \dots, x_{i-1} \overline{x_i}\rceil, x_{i+1}, \dots, x_n)\big]. \quad (1)$$

The groups $(C^R_*(Q), \partial)$ form a chain complex. Following [1], an n-simplex which has consecutive terms $x_i = x_{i+1}$, where $1 \le i \le n-1$, is degenerate. Let $C^D_n(Q)$ be the n-th degenerate chain group. By axiom Q1), $\partial : C^D_n(Q) \to C^D_{n-1}(Q)$, so the chain complex of degenerate chains forms a subcomplex of the rack chain complex, $(C^R_*(Q), \partial)$. The quandle chain complex is defined as the quotient $C_*(Q) = C^R_*(Q)/C^D_*(Q)$, where degenerate simplices are considered to be 0. By abuse of notation, we use ∂, defined as above, as the boundary operator in the quandle chain complex as well. Throughout what follows, we will assume that the quandle chain groups, $C_n(Q)$, cycle groups $Z_n(Q)$, boundary groups $B_n(Q)$, and homology groups, $H_n(Q)$, are taken with integer, $\mathbb{Z}$, coefficients.

2. Colorings of knots and links. We now give some relevant background information on colorings of knots and links, by elements of quandles, and their relationship to 2-cycles in the homology theory for the quandle in question. For recent related developments on invariants of knotted manifolds via quandle colorings and (co)homology, see [16].

Recalling quandle Example 4 of the previous section, let D be a diagram in the disk D^2, of an oriented knot K. If $A = \{a_1, \dots, a_m\}$ are the arcs of the diagram, the quandle of the diagram, $Q(D)$, is generated by the elements of A. Its relations are given by the *crossing relations* of the form $a_i \overline{a_j}\rceil = a_k$ or $a_i \lceil\overline{a_j} = a_k$, respectively, occurring at the crossings, depending on whether the crossing is respectively positive or negative, as in Fig. 1.

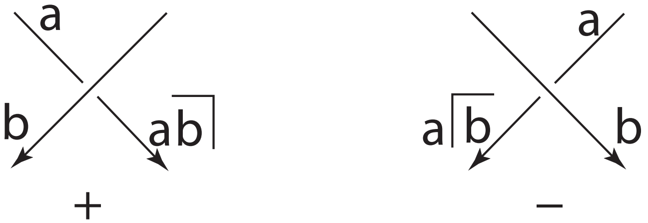

Fig. 1. Crossing conventions

Following e.g. [9], the knot quandle $Q(K)$ is isomorphic to the diagram quandle $Q(D)$, so we work with diagrams throughout this article.

Definition 1. For a knot diagram D, and an arbitrary quandle X, a quandle homomorphism $\phi : Q(D) \to X$ is a *coloring* of D by X. Each arc, a_i, is colored by the element $\phi(a_i) \in X$, where the colors at a crossing obey the crossing relations shown in Fig. 1. A coloring is trivial if only a single color appears.

We can attempt to color a knot or link with elements from any quandle. Some knots admit a coloring by elements of a given quandle. Some do not. For instance, the trefoil admits a coloring by elements of the dihedral quandle R_3, but not by elements of R_5.

[20] showed that the only torus knots of braid index 2, which admitted nontrivial colorings by elements of any Dehn quandle, were the closures of powers of the braid element ${\sigma_1}^3$.

For a quandle Q, a Q-colored, oriented, knot diagram yields a "diagonal" 2-cycle in $H_2(Q)$, whose entries trace along the knot, changing with, and recording, the over and underpasses as follows:

1. Pick a colored arc a_i to start, and move along the knot in the direction given by the orientation.
2. At each successive crossing, write down a quandle 2-simplex (a_i, b_i), where the first entry a_i is the color of the incoming arc, and the second entry b_i is the color of the overcrossing arc.
3. If the crossing is positive, the simplex receives a + sign. If the crossing is negative, the simplex receives a $-$ sign.
4. If the next overcrossing has color b_{i+1}, then the corresponding 2-simplex is (a_{i+1}, b_{i+1}), where

$$\begin{cases} a_{i+1} = a_i \,\overline{b_i}\rceil & \text{if the sign of } (a_i, b_i) \text{ is positive,} \\ a_{i+1} = a_i \lceil\overline{b_i} & \text{if the sign of } (a_i, b_i) \text{ is negative.} \end{cases}$$

A typical resulting 2-chain with entries a_i, b_i looks like

$$(a_1, b_1) \pm (a_2, b_2) \pm \ldots \pm (a_n, b_n).$$

Each entry a_{i+1} results from applying $\overline{b_i}\rceil$ or $\lceil\overline{b_i}$ to a_i. Again, positive simplices correspond to right brackets:

$$(a_i, b_i) \Leftrightarrow a_i \,\overline{b_i}\rceil.$$

Negative simplices correspond to left brackets:

$$-(a_i, b_i) \Leftrightarrow a_i \lceil\overline{b_i}\,.$$

The "anchor" circle a_1 is fixed by the product of successive applications of the b_i's, so e.g. for a product of positive brackets

$$a_1 \,\overline{b_1}\rceil \,\overline{b_2}\rceil \ldots \overline{b_n}\rceil = a_1.$$

This 2-chain is a 2-cycle since the knot is closed and coherently quandle-colored, and the final application of the bracketed version of b_n to a_n, again yields a_1, the color of the initial arc. For a non-trivial knot or link, since the quandle axioms are algebraic equivalents of Reidemeister moves, changing the knot by isotopy yields a new non-trivial coloring of the resulting knot diagram.

Intersection numbers for elements of quandles. In the case of the Dehn quandle of a surface F, whose elements are (isotopy classes of) circles, there is a well-defined notion of intersection number, in particular for circles having intersection number 0 or 1. This is the geometric intersection number of minimally intersecting circles in the specified isotopy classes. For intersection numbers 0 and 1, this notion can be generalized to an algebraic definition for intersection numbers in a general quandle, given in [19]. For elements a, b in a quandle Q, we have

DEFINITION 2.
Intersection 0: $|a \cap b| = 0 \;\Leftrightarrow\; \overline{ab\,}| = a = a|\overline{\,b\,}$. b has no effect on a (and vice versa)
Intersection 1: $|a \cap b| = 1 \;\Leftrightarrow\; \overline{ab\,}| = b|\overline{\,a\,}$, and $|a \cap b| \neq 0$.

Note for any quandle Q, and $a \in Q$, we have $|a \cap a| = 0$. For any quandle Q, and $a, b \in Q$, if $|a \cap b| = 1$ then

$$b\,\overline{a\,}|\,\overline{b\,}| = a \quad \text{and} \quad b\,\overline{a\,\overline{b\,}|}| = b\overline{a\,}|\,\overline{b\,}| = a$$

and similarly for other combinations of brackets. At this point, it is not clear how to define higher intersection numbers for elements of general quandles.

An initial example of a quandle colored knot, depending upon intersection numbers of elements, and its associated 2-cycle, is shown in Fig. 2 below.

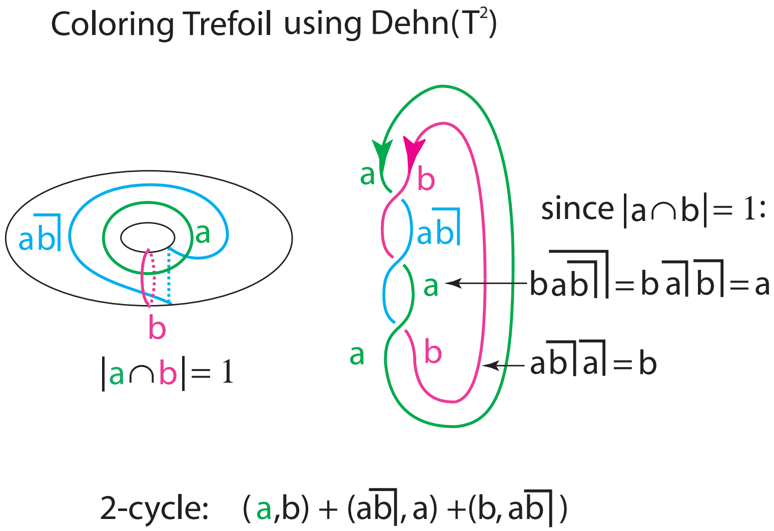

Fig. 2. Dehn quandle colored trefoil

This coloring depends only on the elements in question having intersection number 1. In particular, we can always color coherently oriented (arrows in same direction) triple twists, so that the colors on the left agree, and the colors on the right agree, by using elements a, b of an arbitrary quandle Q, with $|a \cap b| = 1$. See Fig. 3 below.

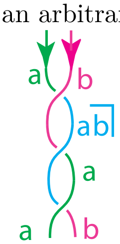

Fig. 3. Coherent triple twist coloring

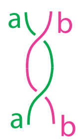

Fig. 4. Double twist coloring

Similarly, we can always color *arbitrarily* oriented double twists using a, b with $|a \cap b| = 0$, since the elements do not interact. See Fig. 4 above. We will make much use of these facts in what is to come.

Pretzel links. Let $r_1, \ldots, r_n \in \mathbb{Z}$.

Definition 3. In a *pretzel link* of the form $P(r_1, \ldots, r_n)$, each r_i represents a column of r_i consecutive twists of two strands, attached together as shown below in Fig. 5. Positive r_i's correspond to positive twists, and negative r_i's correspond to negative twists.

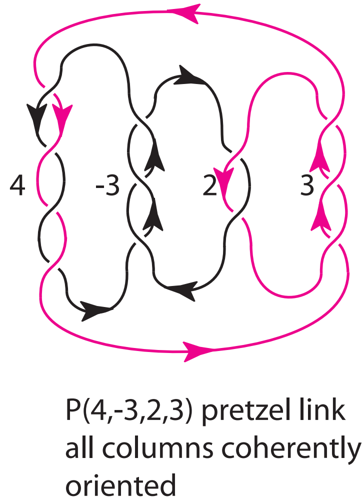

Fig. 5. Pretzel link

By a result of [12], $P(r_1, \ldots, r_n)$ is a *knot* iff either n and all the r_i are odd, or exactly one of the r_i is even.

Some conventions and coloring properties of pretzel knots and links. Here, we will treat pretzel knots and links as somewhat analogous to braids. We think of the column index k for pretzel links $P(r_1, \ldots, r_k)$, as an analog of the braid index, for the braid groups B_k. If we neglect, for a moment, the closure of the pretzel link, which is formed in a standard way in all cases, we can "multiply" the underlying columns of two pretzels P, P', which have the same column index. This is done by placing the columns of P above those of P' (or vice versa) and joining the appropriate strands of r_i and r'_i, for all $1 \leq i \leq k$ (*vertical composition*). This is analogous to braid multiplication in the braid group B_k. We can then form the new resulting pretzel link by taking the closure in the standard way.

There is another obvious operation available to create pretzel links. We can concatenate columns of P and P' *horizontally*, analogously to increasing the braid index. This is clearly additive in the column indices. Denote this operation by the tensor product symbol, $\otimes$. Again, the resulting pretzel link is formed by taking the standard pretzel closure of the resulting collection of columns. In this discussion, we will treat single columns and pairs of columns as allowable objects with which we may work.

For an arbitrary quandle Q, we now look at the interaction of these types of operations on certain Q-colorable pretzel knots and links. To formalize some of the discussion above we have

DEFINITION 4.

1. We say a twist column is *coherently oriented* if the orientation on the strands is in the same direction.
2. Call a twist column 2-*colored* if the left strands at the top and the bottom have the same color $a \in Q$, and the right strands at the top and the bottom have the same color $b \in Q$, with $a \neq b$.

Thus the twist columns in Examples 3, 4 are both 2-colored, and the column in Example 3 is coherently oriented. We consider pretzel link colorings with the following

CONVENTION. The first (leftmost) twist column r_1 will be coherently 2-colored *down*, and we shall label its incoming left strand a and the incoming right strand b, with $a, b \in Q$.

NOTE. For a valid coloring of P, this implies that a appears as the color on the right side, top and bottom, of the last column r_k. Also, since r_1 is *down*, r_k must be *up*.

Here are some initial simple observations about some interactions of pretzel links, Q-colorings, and operations of these types.

PROPOSITION 1. *Let Q be a quandle and let $P = P(r_1, \ldots, r_k)$ represent a pretzel link which admits a valid non-trivial Q-coloring.*

1. *Colorability is preserved by taking "vertical" powers of the underlying twist columns of P, and also by taking "horizontal" tensor powers. That is taking P^n and $\underbrace{P \otimes \ldots \otimes P}_{n}$.*
2. (a) *If P is a Q-colorable pretzel knot, then P^n is a Q-colorable pretzel knot, for n odd.*

   (b) *Also, when $\forall i = 1, \ldots, k$, r_i, k, n are odd, $(P \otimes \ldots \otimes P)^n$ is a Q-colorable pretzel knot.*
3. *Let r_l and r_r be a pair of coherently 2-colored twist columns, with r_l oriented* down *and r_r oriented* up. *We do not require that as integers, $r_l = r_r$. Suppose that the left strands of r_l have color a, as do the right strands of r_r, while the right strands of r_l and the left strands of r_r share the same color, so these latter two pairs of strands can be joined in the standard fashion, for two parallel columns. See Fig. 6 right for an example (in the case shown, $r_l = 3 = r_r$). Let T be the result of joining these sides. Let $P = P(r_1, \ldots, r_k)$ be a Q-colorable pretzel link. By convention, the color on the right of r_k is $a \in Q$. Then $P \otimes T$ is Q-colored.*

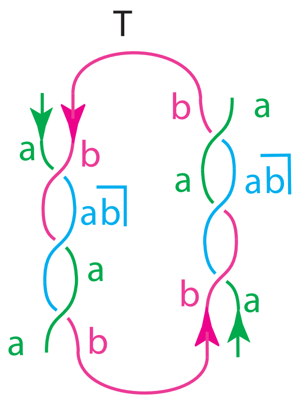

Fig. 6. Colorable tensor factor

Proof. 1. Suppose we have a (nontrivial) coloring of P. We may stack copies of the colored i-th columns of P on top of one another, joining the strands of the column r_i with the strands of the column r_i below it, for all values of i. In each case this gives a valid coloring of the resulting column nr_i. Forming the standard pretzel closure of these new columns gives a valid coloring of P^n. See Fig. 7 below.

By placing a new copy of the columns of P to the right of the previous existing copy, the right strands of column r_k of the initial copy have color a, by convention, as do the left strands of column r_1 of the new copy. The strands of r_k are oriented up, and those of r_1 are oriented down, so the right strands of r_k may be attached in the standard fashion to the left strands of r_1. This process may be iterated n times and the resulting collection of columns may be closed using the standard pretzel closure, giving a new link $P \otimes \ldots \otimes P$ with a valid coloring.

2. Referring to the Kawauchi criteria (below Fig. 5) for $P(r_1, \ldots, r_k)$ to be a pretzel knot, if k and r_i are odd, for all $1 \le i \le k$, and if n is odd, then stacking the columns as in 1. above yields each column nr_i as odd, and k remaining odd, so when the pretzel closure is taken, the result is again a knot. By 1., since the original was colorable, the result is, as well. Again see Fig. 7.

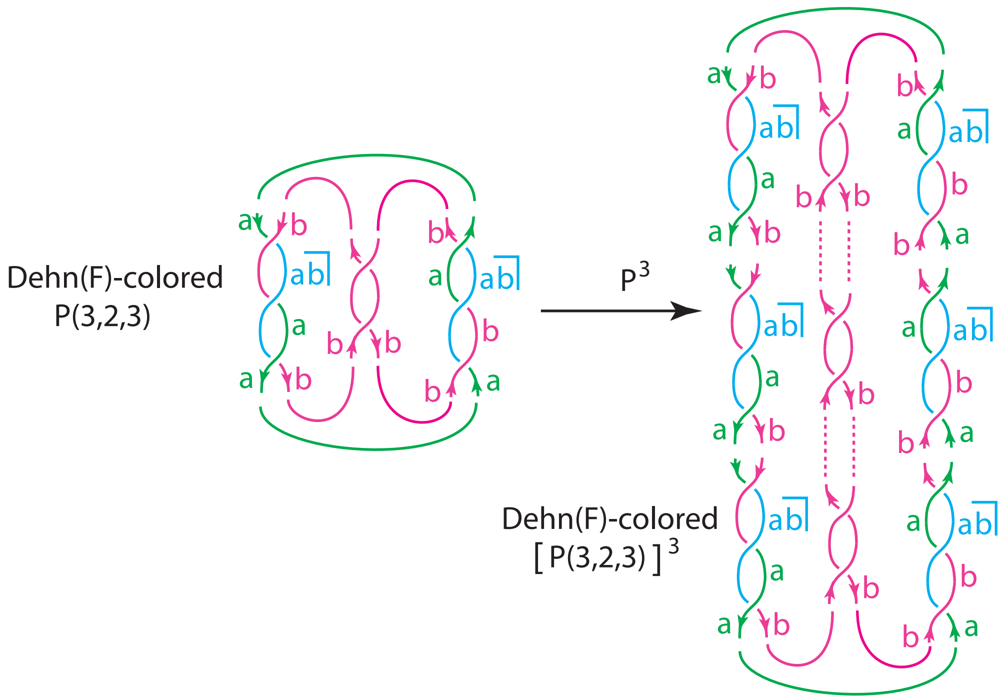

Fig. 7. Q-colorable odd power of pretzel knot

On the other hand, if a colorable P is such that exactly one of the r_i is even, under vertical n-fold composition with n odd, the even column remains even, and the other columns remain odd, and the result is again a colorable knot, as in 1.

3. Properly adjoining such a T to an existing colorable $P(r_1, \ldots, r_k)$, with the color of the right strands (oriented up) of r_k agreeing with the color of the left strands of T (oriented down), we see that the remaining free right strands of T are oriented up and have color a agreeing with the left strands of r_1, in the standard pretzel closure of the new object. Thus it is colorable. ∎

Here is a somewhat more specialized result in a context which will be used later.

PROPOSITION 2. *Let P be a Q-colorable k-column pretzel* knot *of the form $P(3, \ldots, 3, 2, 3, \ldots, 3)$ (see the Kawauchi result) where the 2-twist column occurs in position i, with $1 < i < k$. Any such pretzel knot admits a valid coloring with combinations of two colors $a, b \in Q$, such that $|a \cap b| = 1$, and the orientation/coloring conventions concerning r_1 and r_k hold.*

NOTE. In many cases, valid colorings of such knots involving more than two colors are also possible.

Proof. The 2-twist column is r_i. We assume it will either be colored with the single color a, or with the single color b. Following the conventions, since b occurs on the right side of r_1,

- when i is even, we take the color of r_i to be b.
- when i is odd, we take the color of r_i to be a.

We color the triple twist columns between r_1 and r_i with alternating (down/up) 2-colored coherent columns. The scheme above insures the correct colors are adjacent and can be joined, when we reach r_i, moving from left to right. We now consider the coloring of the columns r_j where $j > i$.

With the columns to the left of r_i colored as above, we have the two cases:

1. If $k - i$ is even, take last column, r_k, to be oriented up and colored with a. Working from the right, insert and join the requisite odd number of alternating (up/down) coherent 2-colored columns. We can then use the freely orientable strands in column r_i to rectify any inconsistencies in the orientations of the columns. See Fig. 8 for an example.

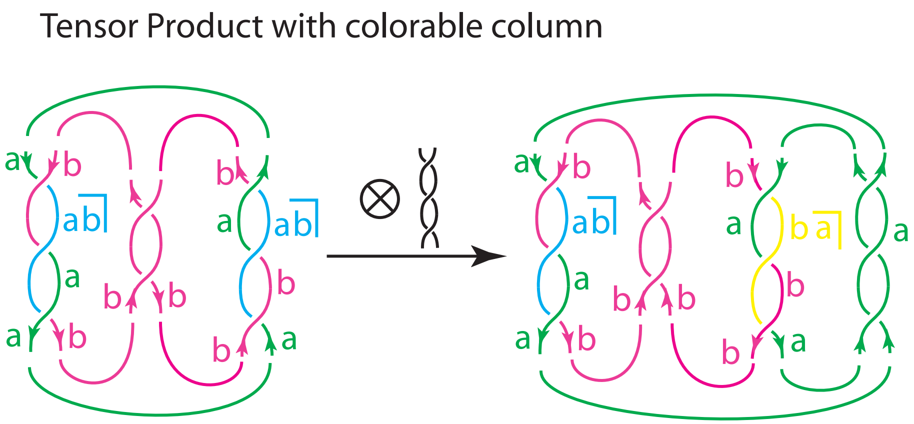

Fig. 8. Coloring transformed knot of form $P(3, \ldots, 3, 2, 3, \ldots, 3)$

2. If $k - i$ is odd, insert and join an odd number of alternating 2-colored columns on the right of r_i, with r_k oriented up. Again we may use the freely orientable strands of r_i to correct any orientation inconsistencies. ∎

3. Commutation relations. In this section, we give a number of versions of the main result, constructing families of commutation relations in certain groups, which are parametrized by quandle colored knots. First we give the result for mapping class groups, and using this as a model, expand to a more general collection of groups. As a corollary we specialize to some particular groups whose associated quandles admit elements with the desired intersection properties.

In general, relations in groups automatically yield relations in the associated quandles. The construction given here allows us to reverse the process under certain circumstances, deriving relations in the more "rigid" groups, from relations (corresponding to quandle 2-cycles) in the "flabbier" quandles.

In what follows, a "*circle*" will denote the isotopy class of a simple closed curve in an orientable surface F. Following the conventions of writing the quandle actions on the right, and hence in the Dehn quandle, Dehn twists on the right of the object they act on, we write homeomorphisms on the right as well.

THEOREM 1. *Let F be an orientable surface, and let a be a non-separating circle in F. Let $Dehn(F)$ be the Dehn quandle of the surface F. There exist families of commutation relations, between arbitrarily long products of Dehn twists, representing elements $\phi \in MCG(F)$, and twists about a. The families are parametrized by certain $Dehn(F)$-colored knots.*

The proof will be constructive. The key lemma is a well known fact about mapping class groups.

LEMMA 1. *For an orientable surface F, and a circle $a \in F$, a mapping class $\phi \in MCG(F)$ commutes with the twists $\overline{a}\rceil, \lceil\overline{a}$ iff ϕ fixes a, i.e. $(a)\phi = a$.*

Proof. We show the proof, a modified version of the one given in [4], for commutation with a right twist. The proof for a left twist is analogous.

$$\begin{aligned} \phi\,\overline{a}\rceil = \overline{a}\rceil\,\phi \quad \text{iff} \quad & \phi\,\overline{a}\rceil\,\phi^{-1} = \overline{a}\rceil \\ \text{iff} \quad & \overline{(a)\phi}\rceil = \overline{a}\rceil \quad \text{by iterated use of quandle axiom Q3)} \\ \text{iff} \quad & (a)\phi = a. \ \blacksquare \end{aligned}$$

Proof of Theorem 1. Let Q here represent the Dehn quandle $Dehn(F)$, of the genus g surface, F. For simplicity here, we will concentrate on working with Q-colored pretzel knots of type $P = P(3, \ldots, 3, 2, 3, \ldots, 3)$. By Lemma 1, a homeomorphism ϕ fixing a circle $a \in F$, commutes with twists on a. Recall the discussion, in Section 2, of the relationship between Q-colorings of knots and 2-cycles in Dehn quandle homology. Q-colored knots with a specified "anchor" circle a, correspond to $Dehn(F)$-homology 2-cycles, having a as the first entry in the first simplex. a is fixed by the ordered product of the second entries. These latter correspond to successive over-crossings.

As we saw in Figs 3 and 4 and the associated discussion, coherent triple twists can be Q-colored using circles $|a \cap b| = 1$, and arbitrarily oriented double twists can be Q-colored using non-intersecting circles, e.g. $|b \cap b| = 0$.

Using Propositions 1 and 2, we can generate families of Q-colored pretzel knots of type $P(3, \ldots, 3, 2, 3, \ldots, 3)$, and powers of such, which are also Q colored pretzel knots. Each such knot corresponds to a Dehn quandle homology 2-cycle. The homeomorphism ϕ, which is the ordered product of the Dehn twists on the second entries in the 2-simplices comprising each such 2-cycle, must fix the anchor element a, since we indeed have a 2-cycle.

Thus, by Lemma 1, the product of Dehn twists ϕ, must commute with the twists $\overline{a\,}\rceil, \lceil\overline{a\,}$. By repeated use of Propositions 1 and 2, we can generate families of such commutation relations. ∎

Fig. 9 below gives an example of this process, using circles on a genus 2 surface, and the resulting homeomorphism ϕ. The heavy dot denotes the starting point, following the orientation, for reading the entries of the 2-simplices of the 2-cycle. The anchor a is the color of the strand at that initial point.

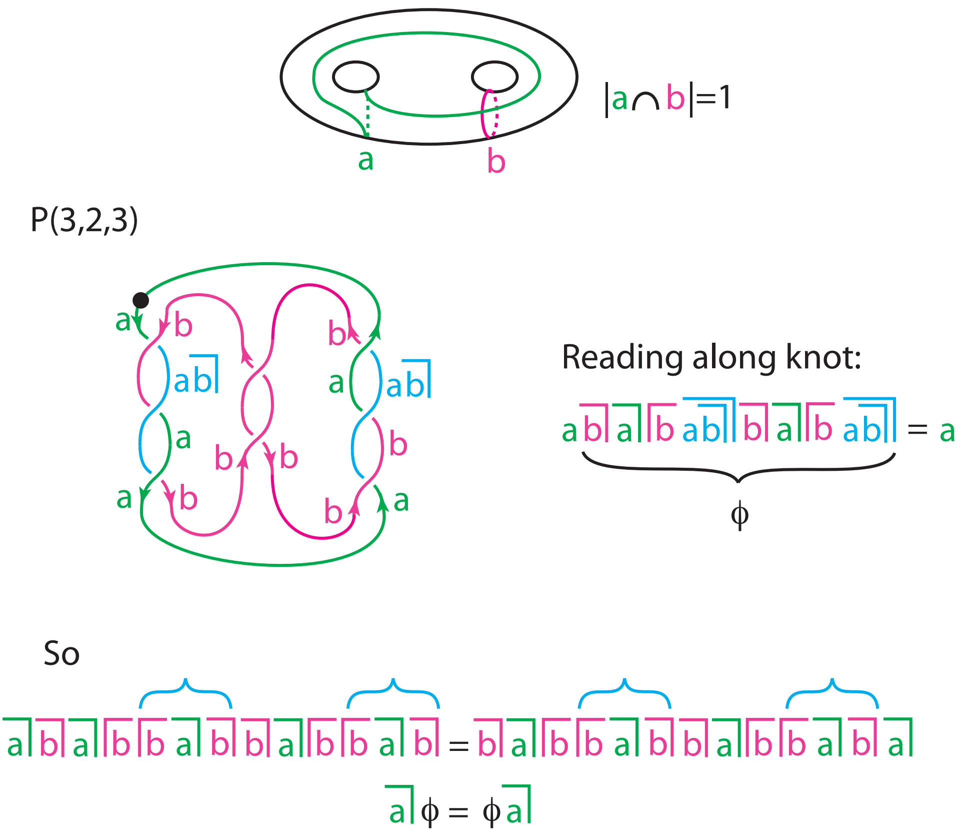

Fig. 9. *Dehn*(F)-colored pretzel knot, homeomorphism ϕ, and commutation relation

We now generalize this result to yield such commutation relations in groups other than the mapping class groups. In certain cases, e.g. the mapping class groups, the quandle action is defined in a natural way, from the action of the group, on the objects of the quandle. In other instances, we use the conjugation quandle associated to a given group. See e.g. [8]. We have the following straightforward translation between behavior of elements in quandles and elements in the related groups.

Definition 5.

1. If $|a \cap b| = 1$ in Q, a, b are *braided* elements in the group: $aba = bab$.
2. If $|a \cap b| = 0$ in Q, a, b are *commuting* elements in the group: $ab = ba$.

We have the following generalization:

Theorem 2. *Let Q_G be a quandle having elements with intersection numbers* 1 *and* 0. *Then the associated group, G, admits families of commutation relations involving arbitrarily long products of elements, with a given fixed element, which are parametrized by Q_G-colored knots.*

Proof. Again, if the group G has a natural quandle structure, we use that. Otherwise we use the conjugation quandle structure on the group. Call the quandle Q_G.

Quandle axiom Q3) is equivalent to the ability to conjugate the quandle operation around. This allows us to maintain the equality in the second line of the proof of Lemma 1. Thus a quandle element fixed by a product commutes with the product when viewed as group element.

In the "translated language" of Definition 5, pretzel knots with double and triple twists admit colorings by intersection number 0 (commuting), and 1 (braided) elements, respectively. Again using the operations of Propositions 1 and 2, we may generate families of such Q_G-colored pretzel knots. More roughly, the "algebra" of such Q_G-colored knots is closed under powers and $\otimes$, and these operations.

Once again, the correspondence between Q_G-homology 2-cycles and such Q_G-colored pretzel knots holds and yields an anchor element, say a, that is fixed by the ordered product, ϕ, of second entries in the 2-simplices. Then we may apply Lemma 1, and the group product ϕ again commutes with a now considered as a group element. ∎

As a corollary, we now specify some groups for which this construction yields such commutation relations. In this context and for later discussion, we recall the following definition

Definition 6. An *Artin group* is a group with a presentation of the form

$$\left\langle x_1, x_2, \ldots, x_n \middle| \langle x_1, x_2\rangle^{m_{1,2}} = \langle x_2, x_1\rangle^{m_{2,1}}, \ldots, \langle x_{n-1}, x_n\rangle^{m_{n-1,n}} = \langle x_n, x_{n-1}\rangle^{m_{n,n-1}} \right\rangle$$

where $m_{i,j} = m_{j,i} \in \{2, 3, 4, \ldots, \infty\}$, and the symbol $\langle x_i, x_j\rangle^{m_{i,j}}$ represents and alternating product of length $m_{i,j}$ of elements x_i and x_j, starting with x_i. So e.g.

$$\langle x_i, x_j\rangle^3 = x_i\, x_j\, x_i$$

and $m_{i,j} = \infty$ means no relation between x_i and x_j exists.

In particular, we have Artin relation descriptions of generators that commute and generators that are braided

- a, b commute $\Leftrightarrow$ $ab = ba$ $\Leftrightarrow$ $\langle a, b\rangle^2 = \langle b, a\rangle^2$ $\Leftrightarrow$ $|a \cap b| = 0$,
- a, b are braided $\Leftrightarrow$ $aba = bab$ $\Leftrightarrow$ $\langle a, b\rangle^3 = \langle b, a\rangle^3$ $\Leftrightarrow$ $|a \cap b| = 1$.

In the following consequence of the above theorem, we list the group in which the commutation relations are found, the quandle in question, and the method of determining intersection number 0 or 1 elements in the quandle.

Corollary 1. *By using the method of construction in Theorems* 1, 2*, the following groups admit such commutation relations*:

1. $MCG(F)$*, genus* $(F) \geq 2$; $Q = Dehn(F)$*, with geometric intersection number of circles.*
2. $SL(2, \mathbb{Z}) = MCG(\mathbb{T}^2)$; Q: *symplectic quandle structure* (*see quandle Example* 6), *with* $Det(M) =$ *symplectic form, giving intersection number on* $\mathbb{T}^2$.
3. $SL(2, \mathbb{R})$; Q: *symplectic quandle with* $Det(M)$ *as symplectic form, giving intersection number.*
4. *Symplectic groups,* $Sp(2n, F)$; Q: *symplectic form gives symplectic quandle structure. It gives algebraic intersection number e.g. of non-zero cycles in* $H_1(F; \mathbb{Z})$.
5. *Braid groups* B_n, $n \geq 3$; Q: *conjugation quandle, where* $|\sigma_i \cap \sigma_{i+1}| = 1 \Rightarrow \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$*, and* $|\sigma_i \cap \sigma_j| = 0 \Rightarrow \sigma_i\sigma_j = \sigma_j\sigma_i$ *for* $|i - j| \neq 1$.
6. *Symmetric groups*; Q: *conjugation quandle structure. For* 2*-cycles,* $|(i, j) \cap (j, k)| = 1$*, and* $|(i, j) \cap (k, l)| = 0$.
7. *Quotients of Artin groups having relations of the form* $aba = bab$ (*braided elements*) (*for intersection number* 0*, any element commutes with itself*) *with* Q: *conjugation quandle structure.*

4. Other results. In this section we give some further results related to the techniques and ideas above. First, we give a more streamlined proof of the main theorem in [20]. In that article the (long) proof was given by a series of geometric and numerical arguments. Here we give a much shorter proof using the following somewhat more sophisticated group theoretic results in the mapping class groups of surfaces, proved by [6] and [7], as given in [4], and rendered in the notation of this article.

Theorem 3. *Let* a *and* b *be two isotopy classes of simple closed curves in a surface* F*. If* (*the intersection number*) $i(a, b) \geq 2$*, then the group generated by* $\overline{a}\rceil$ *and* $\overline{b}\rceil$ *is isomorphic to the free group* F_2*, of rank* 2.

We prove:

Theorem 4. *Let* K *be a torus knot of type* $(2, p)$*, where* p *is odd and* $3 \nmid p$*. Then* K *does not admit any non-trivial colorings by elements of a Dehn quandle,* $Dehn(F_g)$*, for any orientable surface* F_g *of genus* $g \geq 1$*. Alternatively, there do not exist nontrivial quandle homomorphisms* $Q(K) \to Dehn(F_g)$*, for any torus knot* K *of the type described.*

Proof. We have seen (Fig. 3) that coherent triple twists admit valid 2-colorings involving elements $a, b \in Dehn(F)$ with $|a \cap b| = 1$. We have also seen that powers of colorable knots and links, specifically triple twists, remain colorable, as in Propositions 1, 2. Thus, a valid coloring of such a knot K by elements $a, b \in Dehn(F)$, where $|a \cap b| = 1$ could not occur, since the colors at the top left and bottom left, and those at the top right and bottom right would not agree, and a consistently colored closure could not be formed.

Suppose there existed a valid coloring of such a torus knot K, with $|a \cap b| \neq 1$. It would start out (from the top) with two colors a, b. We examine cases.

1. If $|a \cap b| = 0$, then since the 2-strand column, whose standard braid closure forms K, has an odd number of crossings, the colors appearing at the bottom of the braid would be switched. This is because the intersection number 0 curves do not affect one another. Thus the standard braid closure would not receive a valid coloring, as we would need the top left and bottom left to agree, and the top right and bottom right to agree.

2. On the other hand, suppose $|a \cap b| \geq 2$, and a valid coloring using a, b exists. The closure of the colored braid corresponds, as in the previous section, to a Dehn quandle 2-cycle, with anchor, say a. This anchor a is then fixed by the product of the twists about the second entries in successive 2-simplices of the 2-cycle. Thus by Lemma 1, the Dehn twist about a viewed in $MCG(F)$, must commute with the product of bracketed second entries (also viewed as an element of $MCG(F)$). This gives a nontrivial relation in the subgroup of $MCG(F)$ generated by $\overline{a\,}\rceil, \overline{b\,}\rceil$. However, this contradicts Theorem 4 above, which says such a subgroup, generated by elements with intersection number ≥ 2, must be free. So no valid Dehn quandle coloring of a $(2, p)$ torus knot, with $3 \nmid p$, exists. ∎

We now give an example, involving colorings and determinants of knots, which is a consequence of some of the coloring ideas above. For a more in-depth discussion of matrices and determinants as related to homology, and connections among related knot invariants, see [15]. But first a pertinent point about knots colored by the finite quandles R_n.

If the determinant of a knot is equal to 1, the knot has no nontrivial Fox, (R_n) colorings. The determinant of the knot is the order of the first homology group of the double branched cover over S^3, branched over the knot, and Fox n-colorings, modulo trivial colorings, describe the group modulo n. Alternatively, see [11] or [3].

The knot 10_{124}, which may be realized as the three column pretzel knot $P(5, 3, -2)$, shown below in Fig. 10, has coloring matrix

$$M = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 2 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

$Det(M) = -1$, so $Det(K) = |Det(M)| = 1$. Thus, following the comment above, K is not colorable by elements of any R_n, for finite n.

However, the techniques above give the following

Proposition 3. *The pretzel knot $P(5,3,-2)$ is nontrivially colorable by elements of the conjugation quandle Q_G associated to the Artin group*

$$G = \langle a, b, c \mid ababa = babab,\ bcb = cbc,\ ac = ca\rangle.$$

Other pretzel knots and links are colorable in a similar manner, using the conjugation quandles of groups with appropriate Artin relations.

Such a coloring is shown in Fig. 10 below.

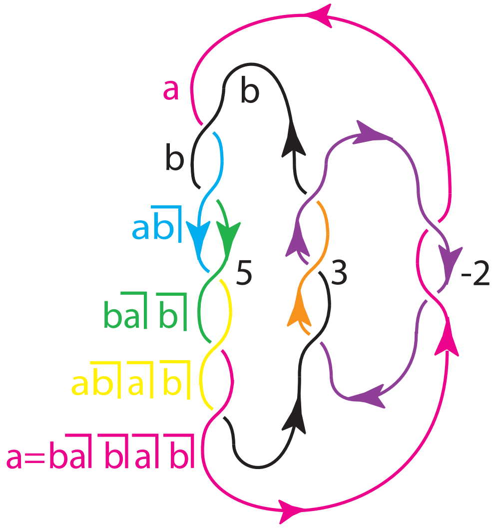

Fig. 10. Colored $P(5,3,-2)$

Proof. We demonstrate the calculation for the first column (5 crossings). The remaining computations follow the examples of colorings of 2-crossing and 3-crossing columns given in Fig. 3 and Fig. 4. We assume the Artin 5-relation in the group, for elements a, b, and recall that the action on two elements x, y in the conjugation quandle is given by $x\,\overline{y|} = y^{-1}xy$ (in the group), when the y-strand crosses over the x-strand, from the right and $y^{-1} = \lceil\overline{y}$ in the group.

1. The **blue** arc receives the color $a\,\overline{b|}$.
2. The **green** arc receives the color $b\ \overline{a\,\overline{b|}|} = b\,\lceil\overline{b\ a|}\ \overline{b|} = b\,\overline{a|}\ \overline{b|}$.
3. The yellow arc receives the color

$$\begin{aligned} a\,\overline{b|}\ \overline{b\,\overline{a|}\,\overline{b|}|} &= a\,\overline{b|}\lceil\overline{b}\ \overline{b\,\overline{a|}|}\ \overline{b|} \\ &= a\,\overline{b\,\overline{a|}|}\ \overline{b|} \\ &= a\lceil\overline{a}\ \overline{b|}\ \overline{a|}\ \overline{b|} \\ &= a\,\overline{b|}\ \overline{a|}\ \overline{b|}. \end{aligned}$$

4. The emerging **magenta** arc receives the color

$$\begin{aligned}
b\,\overline{a}\rvert\,\overline{b}\rvert\,\overline{a\,\overline{b}\rvert\,\overline{a}\rvert\,\overline{b}\rvert}\rvert &= b\,\overline{a}\rvert\,\overline{b}\rvert\,\lceil\overline{b}\;\;\overline{a\,\overline{b}\rvert\,\overline{a}\rvert}\rvert\,\overline{b}\rvert \\
&= b\,\overline{a}\rvert\,\overline{a\,\overline{b}\rvert\,\overline{a}\rvert}\rvert\,\overline{b}\rvert \\
&= b\,\overline{a}\rvert\,\lceil\overline{a}\;\;\overline{a\,\overline{b}\rvert}\rvert\,\overline{a}\rvert\,\overline{b}\rvert \\
&= b\lceil\overline{b}\;\;\overline{a}\rvert\,\overline{b}\rvert\,\overline{a}\rvert\,\overline{b}\rvert \\
&= b\,\overline{a}\rvert\,\overline{b}\rvert\,\overline{a}\rvert\,\overline{b}\rvert \quad \text{Now applying } \overline{\quad}\rvert \text{ as conjugation in } G \\
&= b^{-1}a^{-1}b^{-1}a^{-1}babab \quad \text{Now using Artin 5-relation in } G, \\
&\qquad\qquad\qquad\qquad\qquad \text{replace } babab \text{ with } ababa. \\
&= b^{-1}a^{-1}b^{-1}a^{-1}ababa \\
&= a \text{ as desired,}
\end{aligned}$$

and similarly, using the 5-relation again on the remaining expression, the emerging **black** arc receives the color b. ■

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